On Tensor Powers of Integer Programs

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ABSTRACT

We define a natural product on integer programming problems with nonnegative coefficients. Hypergraph covering problems are a special case of such integer programs, and the product we define is a generalization of the usual hypergraph product. The main theorem of this paper is that the solution to the nth power of an integer program is asymptotically as good as the solution to the same nth power when the variables are not necessarily integral but may be arbitrary nonnegative real numbers.

Key words: integer program, linear program, hypergraph, covering.

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1 Definitions and Notations

The minimization problems we consider here are of the form "Minimize the quantity

$$c_1x_1 + c_2x_2 + \ldots + c_dx_d$$

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1d}x_d &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2d}x_d &\geq b_2 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{md}x_d &\geq b_m \ , \end{aligned}$$

where a_{ij}, b_i, and c_j are fixed nonnegative real numbers and x_j are unknown nonnegative integers." We would lose no generality by throwing out those variables x_j for which $c_j = 0$ and those constraints for which $b_i = 0$, thus making all the b_i and c_j positive. Indeed, if we divide the ith constraint by b_i we also see that no generality is lost by assuming $b_i = 1$ for all i. For the time being, however, we prefer to keep our notation general.

We may write our integer program more compactly as "Minimize c^Tx subject to $Ax \geq b$ with $x \geq 0$," where A is a nonnegative m-by-d matrix, b is a nonnegative column vector of length m, c is a nonnegative column vector of length d, and x ranges over the set of nonnegative integer column vectors of length d. We denote this integer program by the triple $P = (A, b, c)$. Our positivity assumptions on A, b and c imply that feasible solution vectors x exist; the minimum possible value of c^Tx as x ranges over all solution vectors is called the *value* of P, denoted $v(P)$.

Associated with the integer program P is its *linear relaxation*, obtained by dropping the requirement that the entries in the solution vector be integers. We let $v^*(P)$ (the *fractional value* of P) signify the optimum of this relaxed linear program. (Sometimes we will call $v(P)$ the *integer value* of P, to emphasize the distinction between it and $v^*(P)$.) Note that $v^*(P)$ is a real number between 0 and $v(P)$.

Also associated with the minimization program P is the program $P^\perp$, "Maximize $b^T y$ subject to $A^T y \leq c$, with $y \geq 0$." The program $P^\perp$ is called the *dual* of P. The duality theorem asserts that the optimum values to the respective linear relaxations of P and $P^\perp$ are equal; that is, if we extend our definitions of v and v^* to cover maximization programs in the natural way, we have $v^*(P^\perp) = v^*(P)$. However, it is by no means true that $v(P^\perp) = v(P)$; for in general we have

$$0 \leq v(P^\perp) \leq v^*(P^\perp) = v^*(P) \leq v(P),$$

so that if $v^*(P)$ is not an integer there is no chance of the integers $v(P^\perp)$ and $v(P)$ being equal.

Given two minimization programs P and P', there is natural way to define two other programs called their *sum* and *tensor product.* (For wholly analogous constructions in information theory, see pages 65-66 of [6].) Suppose $P = (A, b, c)$, $P' = (A', b', c')$, where A is m-by-d and A' is m'-by-d'. We define $A \oplus A'$ as the $(m + m')$-by-$(d + d')$ matrix

$$\begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix},$$

$b \oplus b'$ as the vector of length $m + m'$ obtained by concatening the vectors b and b', and $c \oplus c'$ as a similar concatenation; we then define the sum $P \oplus P'$ of the programs P and P' to be the program $(A \oplus A', b \oplus b', c \oplus c')$. To define multiplication of programs, it is notationally convenient to allow indices for vectors and matrices to be not just natural numbers but pairs of natural numbers; then the tensor product of A and A' may be defined as the matrix whose $((i,j),(k,l))$th entry is $a_{ik}a'_{jl}$. (If, as is often done, we re-index the product so that the indices are natural numbers, then the matrix $A \otimes A'$ may be depicted as

$$\begin{pmatrix} a_{1,1}A' & a_{1,2}A' & \dots & a_{1,n}A' \\ a_{2,1}A' & a_{2,2}A' & \dots & a_{2,n}A' \\ \dots & \dots & & \dots \\ a_{m,1}A' & a_{m,2}A' & \dots & a_{m,n}A' \end{pmatrix};$$

however, this representation is not necessary for our purposes.) We define $b \otimes b'$ as the column vector of length mm' whose (i,j)th entry is $b_i b'_j$, and $c \otimes c'$ as the column vector of length nn' whose (k,l)th entry is $c_k c'_l$. We conclude by defining the product $P \otimes P'$ of the programs P and P' to be the program $(A \otimes A', b \otimes b', c \otimes c')$.

We leave it to the reader to verify that $\oplus$ and $\otimes$ satisfy the natural commutativity, associativity, and distributivity properties; furthermore, we can define the "empty program" (no variables, no constraints) and the "identity program" (with A the 1-by-1 identity matrix) to serve as identity elements for $\oplus$ and $\otimes$ respectively. We further remark that if one defines $\oplus$ and $\otimes$ for maximization programs in the obvious way, then $(P \oplus P')^\perp = P^\perp \oplus P'^\perp$ and $(P \otimes P')^\perp = P^\perp \otimes P'^\perp$. Lastly, we point out that if P is a minimization program in which some of the entries of the b-vector or c-vector equal 0, there is a canonical program P' obtained by throwing out the corresponding variables and constraints; moreover, the mapping $P \mapsto P'$ preserves v and v^* and commutes with the operations $\oplus$, $\otimes$, and $^\perp$; hence, in the sequel we may without loss of generality assume that b_i and c_j are positive for all i, j, and that $v(P) > 0$.

An easy fact from the next section is that $v^*(P \otimes P') = v^*(P)v^*(P')$; however, an example will show that it is not true in general that $v(P \otimes P') = v(P)v(P')$, and that we must content ourselves with the weaker statement $v(P \otimes P') \leq v(P)v(P')$. If we define $P^{\otimes n}$ as $P \otimes P \otimes \dots \otimes P$ then this implies that $v(P^{\otimes i+j}) \leq v(P^{\otimes i})v(P^{\otimes j})$ for all i, j; by Fekete's lemma [1], we conclude that as n gets large the quantity

$$\sqrt[n]{v(P^{\otimes n})}$$

approaches its infimum, which we call the *asymptotic optimum value* of P.

Theorem 1: *Let P be a program for which all b_i and c_j are equal to 1. If $a_{ij} \leq 1$ for all i, j, then*

$$\sqrt[n]{v(P^{\otimes n})} \to v^*(P).$$

Remark: This theorem cannot be dualized. In other words, $(v((P^{\perp})^{\otimes n}))^{1/n}$ does not approach $v^*(P^{\perp}) = v^*(P)$; see the comment on Shannon capacities at the end of the next section.

The next part of the paper (Section 2) outlines the relationship between integer programming and hypergraph theory and gives the basic results. Sections 3 and 4 contain two quite different proofs of Theorem 1. The first proof is probabilistic and yields the result under a condition slightly weaker than $a_{ij} \leq 1$. The second proof is constructive and employs a greedy algorithm.

2 Background and Preliminary Results

First we briefly recapitulate the discussion of hypergraphs and integer programs contained in [2]. A *hypergraph* $\mathcal{H} = (V, E)$ is a finite vertex set V together with a collection $E \subset 2^V$ of subsets of V, called (hyper)edges. A *cover* of $\mathcal{H}$ is a set of vertices C that intersects every edge of $\mathcal{H}$; that is, for all $e \in E$, $C \cap e \neq \phi$. The *covering number* $\tau(\mathcal{H})$ is the smallest possible cardinality of a cover of $\mathcal{H}$. Suppose $\mathcal{H}$ has d vertices and m edges; then the *incidence matrix* of $\mathcal{H}$ is the m-by-d matrix A with (i, j)th entry equal to 1 if the ith edge contains the jth vertex, and equal to 0 otherwise. If we further let b and c be vectors of length m and d respectively consisting entirely of 1's, and associate with each cover C of $\mathcal{H}$ a d-vector x whose jth entry is 1 or 0 according to whether or not C contains the jth vertex of $\mathcal{H}$, then $\tau(\mathcal{H})$ is seen to equal the value of the integer minimization program (A, b, c). This integer programming viewpoint naturally leads one to consider the relaxed version of the program in which the integrality constraint has been dropped; the value of the relaxed program is called the *fractional covering number* $\tau^*(\mathcal{H})$ of $\mathcal{H}$.

The definitions that appear in part 1 of this paper all correspond to notions that have already been used in the theory of hypergraphs; for example, if P_i is the program that corresponds to the problem of determining $\tau(\mathcal{H}_i)$ $(i = 1, 2)$, then $P_1 \otimes P_2$ corresponds to the problem of determining $\tau(\mathcal{H}_1 \times \mathcal{H}_2)$, where

$$\begin{aligned} V(\mathcal{H}_1 \times \mathcal{H}_2) &= V(\mathcal{H}_1) \times V(\mathcal{H}_2) , \\ E(\mathcal{H}_1 \times \mathcal{H}_2) &= \{e_1 \times e_2 : e_1 \in E(\mathcal{H}_1), e_2 \in E(\mathcal{H}_2)\} . \end{aligned}$$

In [7], McEliece and Posner prove (in different notation) a special case of Theorem 1, namely

$$\lim_{n\to\infty} \sqrt[n]{\tau(\mathcal{H}^n)} = \tau^*(\mathcal{H}) .$$

This amounts to our theorem 1 in the special case that the matrix A consists entirely of 0's and 1's. This analogy suggests the following

Definition : *A program P is a "fuzzy hypergraph covering" (FHC) program if all b_i and c_j are equal to* 1 *and* $0 \le a_{i,j} \le 1$ *for all* i, j. (The terminology arises by analogy with fuzzy sets.)

This paper extends McEliece and Posner's result to general FHC programs.

Remark: Having already assumed $a_{ij}, b_i, c_j \ge 0$, no significant generality is lost in restricting attention to FHC programs. For, it has already been mentioned that we may assume that our program P satisfies $b_i = 1$ for all i and $c_j > 0$ for all j; it is obvious that the normalizations that achieve this do not affect the integral or fractional value of P, and only slightly less obvious that these normalizations commute with $\oplus$ and $\otimes$ in such a way as to preserve the asymptotic optimum of P. It is not immediately clear that the condition $c_j = 1$ for all j is inessential, but such is the case; in proving Theorem 1 in the next section, we first give the argument in the case of FHC programs, and then explain in the last paragraph how to handle the case where the c vector is not all ones.

EXAMPLE

A typical FHC program is the following: "Minimize $x_1 + x_2$ subject to

$$\begin{array}{rcrcl} x_1 & + & \frac{1}{2}x_2 & \geq & 1 \\ \frac{1}{3}x_1 & + & x_2 & \geq & 1 \end{array}$$

with $x_1, x_2 \geq 0$." This program P is associated with the matrix

$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{3} & 1 \end{pmatrix}.$$

Clearly $v(P) = 2$, with optimal solution vectors $x = (1,1)$ and $(0,2)$. To determine $v^*(P)$, note that the feasibility of $x = (3/5, 4/5)$ implies that $v^*(P) \leq 7/5$, while the feasibility of $y = (4/5, 3/5)$ for the dual program $P^\perp$ implies that $v^*(P) = v^*(P^\perp) \geq 7/5$.

The tensor square of this program, $P^{\otimes 2}$, has coefficient matrix

$$A^{\otimes 2} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 1 & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{6} & 1 & \frac{1}{2} \\ \frac{1}{9} & \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}$$

and we readily see that $x = (0,1,1,1)$ is a solution vector, so that $v(P^{\otimes 2}) \leq 3$. This illustrates that $v(P^{\otimes 2})$ may be strictly less than $v(P)^2$. Here $v(P^{\otimes 2})^{1/2} = \sqrt{3}$ and in fact by Theorem 1, $v(P^{\otimes n})^{1/n} \downarrow 7/5$.

The following proposition does not make use of the FHC property, but the fuzzy hypergraph point of view may still be helpful to the reader in interpreting the statements and their proofs:

Proposition 1

$$(i) \quad v^*(P \oplus P') = v^*(P) + v^*(P');$$

$$(ii) \quad v(P \oplus P') = v(P) + v(P');$$

$$(iii) \quad v^*(P \otimes P') = v^*(P)v^*(P');$$

$$(iv) \quad v(P \otimes P') \leq v(P)v(P').$$

PROOF: To prove (i) and (ii) note that if x and x' are solution vectors for P and P', then their concatenation is a solution vector for $P \oplus P'$; and conversely, every solution vector for $P \oplus P'$ yields solution vectors to P and P'. To prove (iii) suppose x and x' are optimal solution vectors to the respective linear programs P and P'. Then since

$$(A \otimes A')(x \otimes x') = (Ax) \otimes (A'x') \geq b \otimes b'$$

(note the use of nonnegativity), $x \otimes x'$ is a feasible vector for the product program $P \otimes P'$, with

$$\begin{aligned} (c \otimes c')^T(x \otimes x') &= (c^T x)(c'^T x') \\ &= v^*(P)v^*(P') , \end{aligned}$$

so that $v^*(P \otimes P') \leq v^*(P)v^*(P')$. On the other hand, suppose y and y' are optimal solution vectors to the dual programs $P^\perp$ and $P'^\perp$; then $y \otimes y'$ is a feasible vector for the program $P^\perp \otimes P'^\perp = (P \otimes P')^\perp$ with

$$\begin{aligned} (b \otimes b')^T(y \otimes y') &= (b^T y)(b'^T y') \\ &= v^*(P)v^*(P') , \end{aligned}$$

so that $v^*(P \otimes P') \geq v^*(P)v^*(P')$. (Note that we have applied the duality theorem three times: to P, to P', and to $P \otimes P'$.) We conclude that $v^*(P \otimes P') = v^*(P)v^*(P')$. The proof of (iv) is the same as the first half of the proof of (iii). (We no longer have a duality principle to provide us with the reverse inequality.) □

The preceding proposition gives us an upper bound on $v(P \otimes P')$. The following less obvious result gives us a lower bound:

Proposition 2 $v(P \otimes P') \geq \max\{v^*(P)v(P'), v(P)v^*(P')\}$.

PROOF: Put $P = (A, b, c)$, $P' = (A', b', c')$. By symmetry, it is enough to show that

$$v^*(P) \leq \frac{v(P \otimes P')}{v(P')} .$$

Given an optimal solution vector z to $P \otimes P'$, indexed by pairs (k, l) where $k \leq d$ and $l \leq d'$, define

$$x_k = \frac{1}{v(P')} \sum_l c'_l z_{(k,l)} .$$

We wish to show that x is a feasible solution to P. Fix i and note that

$$\sum_k a_{ik} x_k = \sum_k a_{ik} \frac{1}{v(P')} \sum_l c'_l z_{(k,l)} = \frac{b_i}{v(P')} \sum_l c'_l \sum_k \frac{a_{ik}}{b_i} z_{(k,l)} .$$

Setting

$$y_l = \sum_k \frac{a_{ik}}{b_i} z_{(k,l)} ,$$

we get

$$\sum_k a_{ik} x_k = \frac{b_i}{v(P')} \sum_l c'_l y_l .$$

However, since

$$\sum_l a'_{jl} y_l = \frac{1}{b_i} \sum_{k,l} a_{ik} a'_{jl} z_{(k,l)} = \frac{1}{b_i} ((A \otimes A')(z))_{(i,j)} \geq \frac{1}{b_i} (b \otimes b')_{(i,j)} = b'_j$$

for all j, y is a feasible solution to P' and hence satisfies

$$\sum_l c'_l y_l \geq v(P') .$$

Hence

$$\sum_k a_{ik} x_k \geq \frac{b_i}{v(P')} v(P') = b_i \ ,$$

establishing that x is indeed a feasible solution to P. We conclude that

$$v^*(P) \leq \sum_k c_k x_k = \sum_k c_k \frac{1}{v(P')} \sum_l c'_l z_{(k,l)} = \frac{1}{v(P')} \sum_{k,l} c_k c'_l z_{(k,l)} = \frac{v(P \otimes P')}{v(P')} \ ,$$

which was to be shown. □

The preceding propositions imply that that $\sqrt[n]{v(P^{\otimes n})} \geq \sqrt[n]{v^*(P^{\otimes n})} = v^*(P)$. Our main theorem says that if P is an FHC program, then in fact $\sqrt[n]{v(P^{\otimes n})} \to v^*(P)$ as $n \to \infty$. Our first proof of this fact relies heavily on the ideas of McEliece and Posner, and in particular uses the same sort of probabilistic construction as they did; however, our argument is necessarily more complicated, since optimal solution vectors x will typically need to have entries much larger than 1 in order to satisfy the constraints. In our second proof, we use a greedy construction as in Lovász's proof of the McEliece and Posner theorem [5].

It should be mentioned that the convergence $\sqrt[n]{v(P^{\otimes n})} \to v^*(P)$ does not hold for integer minimization programs in general. As an illustration of this, consider the program "Minimize $x+y+z$ subject to $2x \geq 1$, $2y \geq 1$, $2z \geq 1$ with $x, y, z \geq 0$." Then $v(P^{\otimes n}) = 3^n$ for all n, whereas $v^*(P) = \frac{3}{2}$. Hence we see that in order for convergence to $v^*(P)$ to hold, something like the FHC property is required.

It should also be mentioned that the convergence $\sqrt[n]{v(P^{\otimes n})} \to v^*(P)$ typically does not hold for integer maximization programs, even when all of the a_{ij} are 0's and 1's. For example, consider the problem P of maximizing $x_1 + x_2 + x_3 + x_4 + x_5$ subject to the constraints that $x_1 + x_2$, $x_2 + x_3$, $x_3 + x_4$, $x_4 + x_5$, and $x_5 + x_1$ all be at most 1. Viewed as an integer program, this is equivalent to finding the largest independent set of vertices in the pentagon graph C_5. More generally, the nth power of P is equivalent to finding the largest independent set of vertices in the nth strong power of C_5 (see [3] for graph-product and graph-power terminology). The limit $\sqrt[n]{v(P)}$ is known as the Shannon capacity of the graph C_5 [8]. It has been shown [4] that the Shannon capacity of the graph C_5 is $\sqrt{5}$; on the other hand, $v^*(P)$ is 5/2, since (1/2, 1/2, 1/2, 1/2, 1/2) is a solution to both P and $P^\perp$. This example shows that Theorem 1 does not dualize to a theorem about maximization programs.

3 First Proof of the Main Theorem

The first proof of the theorem requires some notation from the theory of two-player zero-sum games. Treat the matrix A as the payoff matrix in a two-player zero-sum game between Alpha, who names a variable (column of A), and Beta, who names a constraint (row of A), where Alpha tries to maximize the payoff and Beta tries to minimize the payoff; the payoff is a_{ij} when constraint j and variable i are chosen. (To prepare for the multi-indices that are to follow, write a_{ij} as $a(i,j)$.) Alpha has an optimal mixed strategy which chooses each variable $x(j)$ with some probability $u(j)$. The expected value of the payoff under this strategy is called the value of the game and is denoted $\mathcal{S}$ (see [9]). There is a very simple relation between this game and the linear programming problem, namely that if $(\alpha(1), ..., \alpha(d))$ is a feasible solution for the linear program then $u(j) = \alpha(j)/\sum_j \alpha(j)$ gives a strategy for Alpha with a guaranteed payoff of at least $1/\sum_j \alpha(j)$. Moreover, the value $v^*(P)$ of the linear program P is equal to $1/\mathcal{S}$.

To illustrate this, consider the previous example of minimizing $x_1 + x_2$ subject to the constraints $x_1 + (1/2)x_2 \geq 1$ and $(1/3)x_1 + x_2 \geq 1$; $x_1 + x_2$ is minimized by choosing $x_1 = 3/5$ and $x_2 = 4/5$, and $v^*(P) = 7/5$. The best strategy for Alpha in the game with payoff matrix

$$a = \begin{pmatrix} 1 & 1/2 \\ 1/3 & 1 \end{pmatrix}$$

is to choose $u(1) = (3/5)/(7/5) = 3/7$ and $u(2) = 4/7$. Then $\mathcal{S} = 5/7$, which is clear from the fact that the expected payoff against this strategy is 5/7, whether Beta chooses constraint 1 or constraint 2 or on any probabilistic combination of the two. In other words, $\sum_j a(i,j)u(j) = 5/7$ for $i = 1, 2$.

Remark: For ease of exposition, we will assume throughout that this is always true, i.e. assume that all of Beta's strategies are equivalent against Alpha's chosen optimal strategy. There is no loss of generality in doing so, since if this is not the case, there is always a way to make it be true by diminishing some of the $a(i,j)$ without changing the value of the game (in other words, without making the linear programming problem any easier). Informally, this amounts to reigning in the slack in all the constraints where the inequality is strict for the optimal solution vector.

The following condition on the matrix A is certainly satisfied when $a(i,j) \in [0,1]$ for all i, j:

$$\prod_j \left(\frac{a(i,j)}{b_i}\right)^{u(j)} \leq 1 \text{ for some optimal strategy } u \text{ and all } i. \tag{1}$$

Theorem 1′: *Suppose $P = (A, b, c)$ is an integer minimization program in which b_i and c_j are strictly positive for all i and j. If the matrix A satisfies condition (1) , then* $\lim_{n\to\infty} \sqrt[n]{v(P^{\otimes n})} = v^*(P)$.

The proof requires a probabilistic construction. For ease of exposition, first assume that the b and c vectors are all 1's; the last paragraph of the proof handles the case of general positive b and c vectors. Let v_0 be any constant greater than $v^*(P)$ and let $V = \lceil v_0^n \rceil$ for n large (just how large, we will decide later). To determine a set of values for the d^n variables in the n-fold tensor product of P such that the sum of the variables is V, begin with all the variables equal to zero and then select one of them according to a certain probability distribution and increment it. Repeat this V times with the choices being independent and identically distributed. It will be shown that for the correct choice of probability distribution, this procedure has a positive probability (in fact a probability close to 1) of producing a feasible integer vector. The probability distribution is exactly the same as the probability distribution used by McEliece and Posner [7]. That is to say, the probability of choosing the variable $x(j_1, j_2, \ldots, j_n)$ is given by $u(j_1)u(j_2)\cdots u(j_n)$ where u is the optimal strategy for Alpha. The proof that this construction works is, however, more involved than in the paper of McEliece and Posner.

PROOF OF THEOREM 1′: Choose a vector at random according to the scheme described in the previous paragraph. The random vector will be feasible if for every constraint $C(i_1, \ldots, i_n)$ the sum of the coefficients of the $v_0{}^n$ randomly chosen variables in that constraint is at least 1. For each variable $x = x(j_1, \ldots, j_n)$, the coefficient in the constraint $C = C(i_1, \ldots, i_n)$ is just the product

$$\prod_{k=1}^{n} a(i_k, j_k).$$

Notice that the value of this product depends only on the number of times each pair (i, j) occurs in the list of (i_k, j_k) and with that in mind, define the *type* of the variable-and-constraint pair (x, C) to be the matrix Z where $Z(i, j)$ is $1/n$ times the number of times the pair (i, j) occurs in the list of (i_k, j_k). Also define

$$r(Z) = \prod_{i,j} a(i, j)^{Z(i,j)}$$

so that the coefficient of x in C is just $r(Z)^n$. Note that r(Z) is a weighted geometric mean of the entries of A.

The proof will proceed by finding for each constraint C a matrix Z for which with high probability at least $r(Z)^{-n}$ variables x are chosen such that (x, C) is of type Z. In other words, the sum of the coefficients in C of the randomly chosen variables sum to at least 1 even if you ignore all but those variables x for which (x, C) is of one particular type. (This is less surprising than it might at first seem, since the number of types is polynomial in n, whereas all other quantities are growing exponentially; hence, in restricting to those x for which (x, C) is of a certain type, we aren't losing an exponentially significant contribution.)

Fix a particular constraint $C = C(i_1, \ldots, i_n)$ and define its type β to be the vector of length m such that $n\beta(i)$ is equal to the number of times i appears in the list of the i_k. Define the m-by-d matrix

$$Z(i,j) = a(i,j)\beta(i)u(j)/\mathcal{S}.$$

Note that

$$\sum_j Z(i,j) = \beta(i)\left(\sum_j u(j)a(i,j)/\mathcal{S}\right) = \beta(i), \tag{2}$$

$$P(\tilde{Z}) \geq Q(n)\mathcal{S}/\tilde{r}^n \tag{3}$$

by the remarks made in the first paragraph of this section. Now define an approximation $\tilde{Z}$ to Z that satisfies

(i) $Z(i,j) = 0$ implies $\tilde{Z}(i,j) = 0$;

(ii) $n\tilde{Z}(i,j)$ is an integer;

(iii) $|Z(i,j) - \tilde{Z}(i,j)| < 1/n$; and

(iv) $\sum_j \tilde{Z}(i,j) = \sum_j Z(i,j) = \beta(i)$.

Define

$$\tilde{r} = \prod_{i,j} a(i,j)^{\tilde{Z}(i,j)}.$$

The reason for conditions (i) and (iii) is so that calculations involving $\tilde{Z}$ can be approximated by calculations involving Z. The reason for conditions (ii) and (iv) is so that $\tilde{Z}$ will actually be the type of a variable pair (x, C) for some x.

The immediate object is to estimate the number of variables x of the V that are chosen (with repetition) for which the pair (x, C) is of the type $\tilde{Z}$, and show that this number is very likely to be at least $\tilde{r}^{-n}$. Each time a variable $x = x(j_1, \ldots, j_n)$ is chosen, the chance that (x, C) is of the type $\tilde{Z}$ is just the chance that for each i, the values of the j_k for which $i_k = i$ form the multiset that has $n\tilde{Z}(i,1)$ ones, $n\tilde{Z}(i,2)$ twos, and so on. Denote this probability by $P(\tilde{Z})$. Then

$$P(\tilde{Z}) = \prod_i \left[\text{multi}\left(n\beta(i); n\tilde{Z}(i,1), \ldots, n\tilde{Z}(i,m)\right) \prod_j u(j)^{n\tilde{Z}(i,j)} \right],$$

where multi$(x; y_1, y_2, \ldots)$ denotes the multinomial coefficient with x on top and $y_1, y_2, \ldots$ on the bottom. Evaluate these multinomial coefficients by using the approximation

$$x! \approx x^x e^{-x} \text{ where } 0^0 \stackrel{def}{=} 1.$$

Since Stirling's formula is only off by a constant factor, this cruder approximation is off by at most a factor of *const* $x^{1/2}$ and is exact for $x = 0$. (Here and throughout, *const* denotes a constant, possibly different each time.) Since all the arguments are integers from 0 to n, with no error for the 0 values, the total error in estimating $P(\tilde{Z})$ will be a factor of at most a constant times $n^{m(d+1)/2}$; and hence, at most a constant times n^{md}. Thus, after all the e^x and n^x factors cancel, we obtain

$$P(\tilde{Z}) \geq const\ n^{-md} \left[\prod_i \beta(i)^{\beta(i)} \prod_{i,j} \tilde{Z}(i,j)^{-\tilde{Z}(i,j)} \prod_{i,j} u(j)^{\tilde{Z}(i,j)} \right]^n. \qquad (4)$$

Note that if we replace $\tilde{Z}$ by Z everywhere in the bracketed expression, it becomes

$$\prod_i \beta(i)^{\beta(i)} \prod_{i,j} Z(i,j)^{-Z(i,j)} \prod_{i,j} u(j)^{Z(i,j)}$$

$$
\begin{aligned}
&= \prod_{i,j} (u(j)\beta(i)/Z(i,j))^{Z(i,j)} \\
&= \prod_{i,j} (\mathcal{S}/a(i,j))^{Z(i,j)} \\
&= \mathcal{S} \prod_{i,j} a(i,j)^{-Z(i,j)} \\
&= \mathcal{S}/r
\end{aligned}
$$

where the first equality follows from (2); we proceed to rewrite (4) in terms of $\mathcal{S}/r$. Specifically, we will approximate (4) by a version with Z replacing $\tilde{Z}$, thereby introducing an additional error factor of the form *const* $(1-\delta(n))^n$ with $\delta(n) \to 0$ as $n \to \infty$. By property (iii) in the definition of $\tilde{Z}$,

$$\frac{\tilde{Z}(i,j)^{\tilde{Z}(i,j)}}{Z(i,j)^{Z(i,j)}} \geq \inf_{|x-y|<1/n} \frac{x^x}{y^y} \stackrel{def}{=} 1-\theta(n)$$

since the function $x\ln(x)$ (with $0 \ln 0 \stackrel{def}{=} 0$) is uniformly continuous on $[0,1]$, $\theta(n) \to 0$ as $n \to \infty$. Thus

$$\frac{\prod_{i,j} \tilde{Z}(i,j)^{\tilde{Z}(i,j)}}{\prod_{i,j} Z(i,j)^{Z(i,j)}} \geq 1-\delta(n) \stackrel{def}{=} (1-\theta(n))^{md} .$$

Also note that

$$\frac{u(j)^{\tilde{Z}(i,j)}}{u(j)^{Z(i,j)}} = u(j)^{\tilde{Z}(i,j)-Z(i,j)}$$

which is $\geq u(j)^{1/n}$ when $u(j) > 0$ and is $= 1$ when $u(j) = 0$; either way, the fraction is at least ${u_{\min}}^{1/n}$, where $u_{\min}$ is the minimum of the positive entries of u. Thus

$$\frac{\prod_{i,j} u(j)^{\tilde{Z}(i,j)}}{\prod_{i,j} u(j)^{Z(i,j)}} \geq \left({u_{\min}}^{1/n}\right)^{md}$$

which, when raised to the nth power, gets absorbed into the constant factor. We conclude that

$$P(\tilde{Z}) \geq const\ n^{-md}(1-\delta(n))^n \left[\prod_i \beta(i)^{\beta(i)} \prod_{i,j} Z(i,j)^{-Z(i,j)} \prod_{i,j} u(j)^{Z(i,j)}\right]^n$$

$$= \quad const\ n^{-md}(1-\delta(n))^n(\mathcal{S}/r)^n \ . \tag{5}$$

The other estimate of this sort that we will need is a bound on $\tilde{r}$ in terms of r. Take $\eta > 0$ with $\eta \le a(i,j)$ and $\eta \le 1/a(i,j)$ for all $a(i,j) \ne 0$. Then

$$\frac{a(i,j)^{\tilde{Z}(i,j)}}{a(i,j)^{Z(i,j)}} = a(i,j)^{\tilde{Z}(i,j)-Z(i,j)} \ge \eta^{1/n}$$

for all i,j, and

$$\begin{aligned} \tilde{r} &= \prod_{i,j} a(i,j)^{\tilde{Z}(i,j)} \\ &\ge \prod_{i,j} \eta^{1/n} \prod_{i,j} a(i,j)^{Z(i,j)} \\ &= const^{1/n}\ r \ . \end{aligned}$$

Then from equation 5 it follows that

$$P(\tilde{Z}) \ge Q(n)(\mathcal{S}/\tilde{r})^n \tag{6}$$

where $Q(n) = const\ (1-\delta(n))^n n^{-md}$. (What will end up being important about $Q(n)$ is that $\sqrt[n]{Q(n)} \to 1$.) That is, if one chooses a variable $x = x(j_1, ... j_n)$ at random with probability $u(j_1)\cdots u(j_n)$, the probability that (x, C) is of type $\tilde{Z}$ is at least $Q(n)(\mathcal{S}/\tilde{r})^n$. Also, recall that if (x, C) is of type $\tilde{Z}$ then the coefficient of x in C is $\tilde{r}$.

The last step in the proof of Theorem 1 is an application of the following lemma.

Lemma 3 *Let a, b, c, ϵ be positive real numbers with $ab/(1+\epsilon) \ge c \ge 1 \ge b$. Consider a family $\{X_i\}$ of at least a^n i.i.d. Bernoulli random variables with $\mathbf{P}(X_i = 1) \ge b^n$. Then there is some positive constant δ and some positive integer N for which $\mathbf{P}(\sum X_i < c^n) < e^{-e^{\delta n}}$ whenever $n > N$. Furthermore, N and δ can be chosen to depend only on ϵ.*

Assuming the lemma for the moment, the rest of the proof of Theorem 1 is as follows. We have selected $V = \lceil {v_0}^n \rceil$ variables for some $v_0 > v^*(P) = 1/\mathcal{S}$. Then for any fixed constraint, we have chosen a matrix $\tilde{Z}$ and its associated value $\tilde{r}$ so that each variable chosen by our random scheme has coefficient at least $\tilde{r}^n$ with probability at least $P(\tilde{Z})$. Let $a = v_0$, $b = P(\tilde{Z})^{1/n}$ and $c = 1/\tilde{r}$. Let X_i be the Bernoulli random variable that equals 1 if the ith variable chosen has the property that (x, C) is of type $\tilde{Z}$ and equals 0 otherwise. Then equation (6) implies that the first inequality in the hypothesis of Lemma 3 is satisfied with any $\epsilon < v_0 S - 1$ for sufficiently large n. The second inequality is guaranteed by the hypothesis on $a(i,j)$ and u in Theorem 1 and the last is always true. The conclusion of the lemma is that the probability of there being enough variables of type $\tilde{Z}$ to satisfy the constraint (namely $\tilde{r}^{-n}$ of them) is at least $1 - e^{e^{-\delta n}}$. This is true uniformly over all constraint types for sufficiently large n, and since there are only exponentially many constraints C, the sum of the failure probabilities over all constraints goes to zero as n goes to infinity. In particular, the constraints are all satisfied with nonzero probability for n sufficiently large, and that proves the theorem.

THE CASE WHERE THE b OR c VECTORS ARE NOT ALL ONES: Recall that by changing a_{ij} to a_{ij}/b_i, the b vector may be assumed to be all ones. So assume the c vector is not all ones. Suppose $(\alpha(1), \ldots, \alpha(d))$ is an optimal solution to the program. Then letting $u(j) = \alpha(j)/\sum_j \alpha_j$ gives a strategy with a cost $c^T u$ that achieves a payoff of $1/\sum_j \alpha_j$. Ignore the cost for the moment and use the same randomized algorithm as before to choose $(\epsilon + \sum_j \alpha_j)^n$ variables to increment, where ϵ is a new, arbitrarily small, positive number. With high probability, the constraints are now satisfied. The expected total cost of the variables chosen is $(\epsilon + \sum_j \alpha_j)^n (c^T u)^n$, so the probability that the cost exceeds $(2\epsilon + \sum_j \alpha_j)^n (c^T u)^n$ goes to zero as n goes to infinity and is therefore eventually less than one. But $c^T u = c^T \alpha / \sum_j \alpha_j$, so for large enough n there are feasible integer vectors with cost at most $(\epsilon + c^T \alpha)^n$ for arbitrarily small ϵ and that proves the theorem. □

PROOF OF LEMMA 3: This is a standard large deviation estimate, but in order to get δ depending only on ϵ the usual moment estimate will be redone from scratch. The following fact can easily be seen by looking at chords of the graph of $\ln(1-x)$ near $x = 0$.

$$\frac{\ln(1-zu)}{z\ln(1-u)} \to 1 \text{ uniformly over } z \in [0,1] \text{ as } u \downarrow 0. \tag{7}$$

(The expression is taken to be 1 when z or u is 0.) Letting t be a free parameter, the moment calculation is

$$\begin{aligned} \mathbf{P}(\sum X_i < c^n) &= \mathbf{P}(e^{\sum -X_i} > e^{-c^n}) \\ &\leq \mathbf{E}e^{t(-\sum X_i)} \,/\, e^{tc^n} \\ &\leq (\mathbf{E}e^{-tX_1})^{a^n}/e^{tc^n} \\ &\leq (1 - b^n(1-e^{-t}))^{a^n}/e^{tc^n}. \end{aligned}$$

Using equation (7) with $z = b^n$ and $u = 1 - e^{-t}$ it is apparent that for any $\gamma \in (0,1)$, it is possible to pick t, hence u, sufficiently small and positive, so that the following inequality holds for any b:

$$1 - b^n(1-e^{-t}) \leq \left([1-(1-e^{-t})]^{b^n}\right)^{\gamma} = e^{-t\gamma b^n}.$$

Then

$$\mathbf{P}(\sum X_i < c^n) \leq e^{-t(\gamma a^n b^n - c^n)}. \tag{8}$$

Fix any γ such that $\gamma ab > c$. The right hand side of equation (8) increases when b and c are decreased by the same factor, and also when b is decreased, so assume without loss of generality that $ab/(1+\epsilon) = c = 1$. Then the exponent in the right hand side grows like $(1+\epsilon)^n$, so for any $\delta \in (0, \ln(1+\epsilon))$, there is an N for which the left hand side of equation (8) is bounded by $e^{-e^{n\delta}}$ whenever $n > N$. It is clear that δ and N can be chosen to depend only on ϵ.

4 Second Proof of the Main Theorem

Our first proof of the main theorem made delicate use of the structure of the nth power of an integer program. In contrast, the proof presented in this section is based on very general lemmas about FHC programs, and only at the very end does the notion of a product of integer programs make an appearance. Even then, we appeal only to the most basic facts about $P^{\otimes n}$ — namely, that $v(P^{\otimes n}) = v(P)^n$, and that the number of constraints in $P^{\otimes n}$ can be bounded by an exponential function.

Say that an integer program $P = (A, b, c)$ is of *semi-FHC type* if all of the entries of A are in $[0, 1]$ and all of the entries of c are 1's (the entries of b may be arbitrary real numbers). Given an integer program $P = (A, b, c)$ of semi-FHC type, let $S(P)$ denote the sum of the entries of b and $D(P)$ denote the maximum column-sum of A. When P is of FHC type, $S(P)$ is the number of constraints of P, which we also denote by m. (The notation "$D(P)$" originates from the fact that in the case that P is a hypergraph-covering program, $D(P)$ coincides with the maximum degree of the hypergraph, i.e., the maximum number of edges sharing a vertex.)

Our argument begins with the observation that $v^*(P) \geq S(P)/D(P)$ for any semi-FHC program P. To see this, let y be the vector of length m all of whose components equal $1/D(P)$. Since every row-sum of A^T is at most $D(P)$, all components of the vector $A^T y$ are less than or equal to 1. Hence y is a feasible solution to the dual program $P^\perp$, whence $v^*(P) = v^*(P^\perp) \geq b^T y = S(P)/D(P)$.

In particular, suppose P is an FHC program and Q is a semi-FHC program such that every feasible solution to P is also feasible for Q. Then $S(Q)/D(Q) \leq v^*(Q) \leq v^*(P)$. This upper bound on the ratio $S(Q)/D(Q)$ is the key ingredient in the proof of the following fact.

Lemma 4 *If $P = (A, b, c)$ is an FHC program with m constraints and d unknowns, then* *there exists a d-component nonnegative integer vector x^* such that the sum of the entries of x^* is at most $2\lceil v^*(P)\ln 10\rceil$ and at least $\frac{1}{4}$ of the entries of the column vector Ax^* exceed 1.*

PROOF: Let $N = \lceil v^*(P)\ln 10\rceil$; we define a sequence of semi-FHC programs $P^{(0)} = P$, $P^{(1)}$, $P^{(2)}$, ..., $P^{(N)}$ and d-component vectors $u^{(0)} = 0$, $u^{(1)}$, $u^{(2)}$, ..., $u^{(N)}$ in the following iterative way. We assume that $P^{(k)}$ has already been defined, and wish to define $P^{(k+1)}$. Take j (more properly speaking, j_k) such that the jth column-sum of $A^{(k)}$ is as large as possible (i.e., is equal to $D(P^{(k)})$), and let $u^{(k+1)}$ be the vector obtained from $u^{(k)}$ by incrementing the jth component by 1. Let $b^{(k+1)}$ equal $b^{(k)}$ minus the jth column of $A^{(k)}$. Let $c^{(k+1)} = c =$ the all-1's vector. Lastly, to define $A^{(k+1)}$, call a row of $A^{(k)}$ *satisfied* if the corresponding entry of $b^{(k+1)}$ is negative; replace all the entries in all the satisfied rows of $A^{(k)}$ by 0's and call the resulting matrix $A^{(k+1)}$.

Note that under this scheme, if we fix i between 1 and m and look at the ith entries of the successive vectors $b^{(0)}, b^{(1)}, b^{(2)}, ..., b^{(N)}$, we see a sequence of numbers that decreases by at most 1 at each stage until a negative term appears, at which point the sequence is constant (since the corresponding row of A gets "zeroed out"). Hence all the entries of all the b-vectors lie in the interval $[-1, 1]$.

Also note that a feasible solution for $P^{(k)}$ remains feasible for $P^{(k+1)}$, since the only change made in passing from the former to the latter is that certain constraints have been relaxed (some of the entries in the b-vector have decreased) while other constraints have been effectively dropped (some of the rows of the A-matrix have been zeroed out). Therefore any feasible solution for $P^{(0)} = P$ is feasible for each $P^{(k)}$, so that $S(P^{(k)})/D(P^{(k)}) \leq v^*(P)$. Hence

$$\begin{aligned} S(P^{(k+1)})/S(P^{(k)}) &= \frac{S(P^{(k)}) - D(P^{(k)})}{S(P^{(k)})} \\ &= 1 - \frac{D(P^{(k)})}{S(P^{(k)})} \end{aligned}$$

$$\leq \; 1 - \frac{1}{v^*(P)}$$

for all k between 0 and $N-1$. Multiplying these N inequalities together, we obtain

$$\begin{aligned} S(P^{(N)})/S(P) &\leq \left(1 - \frac{1}{v^*(P)}\right)^N \\ &\leq \left(1 - \frac{1}{v^*(P)}\right)^{v^*(P)\ln 10} \\ &\leq \left(\frac{1}{e}\right)^{\ln 10} \\ &= \frac{1}{10} \,. \end{aligned}$$

We have shown that the sum of the entries of $b^{(N)}$ (all of which lie between -1 and 1) is at most 1/10 the sum of the entries of $b^{(0)} = b =$ the all-1's vector. This means that at least a quarter of its entries are less than 1/2 (since otherwise the sum of the entries of $b^{(N)}$ would be at least $\frac{3}{4}\left(\frac{1}{2}\right) + \frac{1}{4}(-1) = \frac{1}{8} > \frac{1}{10}$, a contradiction). On the other hand, all of the entries of $b^{(0)}$ were 1's, so at least a quarter of the entries of $b^{(0)} - b^{(N)}$ exceed 1/2; since $b^{(N)} \geq b^{(0)} - Au^{(N)}$, we have $Au^{(N)} \geq b^{(0)} - b^{(N)}$, so that at least a quarter of the entries of $Au^{(N)}$ exceed 1/2. Setting $x^* = 2u^{(N)}$, we obtain a vector whose entries sum to $2N$, with the property that at least a quarter of the entries of Ax^* exceed 1. □

Lemma 5 *If P is an FHC program with $S(P) > 1$ constraints, then*

$$v(P) \leq 100\, v^*(P) \ln S(P) \,.$$

PROOF: Let x^* be the vector of Lemma 4 with entries summing to $2\lceil v^*(P)\ln 10\rceil$ and with the property that at least $\frac{1}{4}$ of the entries x_j^* of x^* exceed 1. Let P' be the integer program obtained from P by dropping all the constraints that correspond to these values of j. Note that any feasible solution to P', if added to x^*, yields a feasible solution to P; hence

$$v(P) \leq v(P') + 2\lceil v^*(P)\ln 10\rceil \,.$$

Note that P' has at most $\frac{3}{4}$ as many constraints as P. Hence, iterating this reduction process K times, where

$$K = \lceil \log_{4/3} S(P) \rceil > \log_{4/3} S(P) ,$$

we obtain a program Q such that

$$v(P) \leq v(Q) + 2K \lceil v^*(P) \ln 10 \rceil .$$

But the number of constraints in Q is at most

$$\left(\frac{3}{4}\right)^K S(P) < \left(\frac{3}{4}\right)^{\log_{4/3} S(P)} S(P) = 1;$$

that is, Q is the empty program, with no constraints and with value 0. Hence

$$\begin{aligned} v(P) &\leq 2K \lceil v^*(P) \ln 10 \rceil \\ &= 2 \left\lceil \frac{\ln S(P)}{\ln 4/3} \right\rceil \lceil v^*(P) \ln 10 \rceil \\ &\leq 100 v^*(P) \ln S(P) , \end{aligned}$$

as claimed. □

COROLLARY (Theorem 1): If P is an FHC program, then $\sqrt[n]{v(P^{\otimes n})} \to v^*(P)$.

PROOF: Let m be the number of constraints of P; we may suppose $m > 1$ (since the result is trivial for $m = 1$). Since $P^{\otimes n}$ has only m^n constraints, by Lemma 5 we have

$$\begin{aligned} v(P^{\otimes n}) &\leq 100\, v^*(P^{\otimes n}) \ln m^n \\ &= 100\, n \ln m \,(v^*(P))^n . \end{aligned}$$

Since $\sqrt[n]{100\, n \ln m} \to 1$ as $n \to \infty$, the theorem follows. □

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