Almost-Equivalence of the Germ-Field Markov Property and the Sharp Markov Property of the Brownian Sheet

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Abstract. "Intuitively, the Brownian sheet should be a Markov process if any process is". This feeling is apparently contradicted by the fact that the sharp Markov property fails for triangles, and has led to the widely studied notion of germ-field Markov property for random fields. In this paper, it is shown for the Brownian sheet that in fact, the germ field is equal to the sharp field for almost every curve $y = f(x)$ (when f is drawn at random according to Wiener measure on the set of continuous functions), and thus the two Markov properties coincide in this case. This result follows from sufficient conditions on f for equality of the two fields. In the case that f satisfies a weak regularity assumption (implied by Banach's classical condition (T1)), we give necessary and sufficient conditions on f for equality of the two fields. When f has bounded variation, the condition is that f be singular with respect to Lebesgue-measure.

1. Introduction.

The Brownian sheet has long been known to satisfy the sharp Markov property with respect to finite unions of rectangles (see [W1]; a detailed proof for planar processes with independent increments is given in [Ru]). However, this property fails for the triangle $\{(t_1, t_2) \in \mathbb{R}_+^2 : t_1 + t_2 \leq 1\}$ [W1], leaving the impression that the sharp Markov property is valid only for a very restricted class of sets. In contrast, the weaker germ-field Markov property is valid for all open sets in the plane ([Ro], [Nu]).

Thus, for many sets A (e.g. the triangle), the germ-field $\underline{\underline{G}}(\partial A)$ of the boundary is strictly larger than the sharp field $\underline{\underline{F}}(\partial A)$, whereas for curves Γ which are a finite union of horizontal or vertical segments, $\underline{\underline{F}}(\Gamma) = \underline{\underline{G}}(\Gamma)$. One might think that these were the only curves for which this equality is valid. However, Dalang and Russo [DR] exhibited separation lines containing no vertical or horizontal segment for which the two fields are equal; [DR] also contains a detailed study of the structure of $\underline{\underline{F}}(\Gamma)$ and $\underline{\underline{G}}(\Gamma)$ when Γ is a separation line.

The motivation for the research reported here is to show that, in a sense, the "generic" case is equality of the germ and sharp fields. More precisely, we consider curves Γ which are the graphs of continuous functions $y = \phi(x)$, where $\phi \in C(\mathbb{R}_+, \mathbb{R})$. We show that if ϕ is drawn at random according to Wiener measure on $C(\mathbb{R}_+, \mathbb{R})$, then with probability one, $\underline{\underline{G}}(\Gamma) = \underline{\underline{F}}(\Gamma)$ (see Corollary 4.13).

This result is obtained in several steps. First of all, we show that $\underline{\underline{G}}(\Gamma)$ is also the minimal splitting field $\underline{\underline{M}}(\Gamma)$ (Theorem 2.1), by using a result of Nualart [Nu]. We then give a description of the generators of the minimal splitting field (Theorem 3.8), which corresponds to the "vertical and horizontal shadow" description for domains with smooth boundaries ([W1], [W3], [WZ]). It is then possible to give conditions on ϕ (Assumption 4.5, Remark 4.11) which ensure that $\underline{\underline{F}}(\Gamma) = \underline{\underline{G}}(\Gamma)$ (Theorem 4.9). In particular, if ϕ has almost everywhere an infinite upper-right Dini derivative, then the two fields are equal; this is also the case when ϕ has bounded variation and is singular.

It is more difficult to obtain necessary conditions on ϕ for equality of $\underline{\underline{F}}(\Gamma)$ and $\underline{\underline{G}}(\Gamma)$. We address this question by giving an explicit representation of the closed Gaussian subspace $G(\Gamma)$ spanned by the generators of the germ field (Theorem 5.5), generalizing a result of Dalang and Russo [DR]. Under a regularity assumption on Γ (Assumption 5.7), we give conditions that elements of $G(\Gamma)$ must satisfy in order to be

$\underline{\underline{F}}(\Gamma)$-measurable (Proposition 5.12). This regularity assumption is implied by Banach's condition (T1) that almost every level set of ϕ be finite. Under this assumption, we can then provide necessary and sufficient conditions for equality of $\underline{\underline{F}}(\Gamma)$ and $\underline{\underline{G}}(\Gamma)$ (Theorem 5.14). In particular, if ϕ has bounded variation, then $\underline{\underline{F}}(\Gamma) = \underline{\underline{G}}(\Gamma)$ if and only if ϕ is singular with respect to Lebesgue-measure.

In Section 2 below, we summarize the principal notations and definitions we will be using, and prove equality of the germ and minimal splitting fields.

2. The germ field of a continuous curve is the minimal splitting field.

Throughout this paper, $T = \mathbf{R}_+^2$ will denote the non-negative quadrant in the plane. If $t = (t_1, t_2) \in T$, R_t will denote the set $\{(s_1, s_2) \in T : s_1 \le t_1 \text{ and } s_2 \le t_2\}$. Lebesgue measure on $\mathbf{R}_+^2$ will be denoted dt whereas Lebesgue-measure on $\mathbf{R}$ will be denoted μ_1 (a measure μ_2 will be defined in Section 4).

Let $(\Omega, \underline{\underline{F}}, P)$ be a complete probability space on which a Brownian sheet $(W_t, t \in T)$ is defined. The Brownian sheet can be regarded as the distribution function of a white noise W on T, that is $W_t = W(R_t)$ a.s. (see [W3; chap.3] for a complete definition).

Given $A \subset T$, the *sharp field* $\underline{\underline{H}}(A)$ of A is defined by $\underline{\underline{H}}(A) = \sigma(W_t, t \in A)$, whereas $H(A)$ denotes the closed linear span of $\{W_t, t \in A\}$ (in $L^2(\Omega, F, P)$). The *germ field* $\underline{\underline{G}}(A)$ is defined by

$$\underline{\underline{G}}(A) = \bigcap_{\varepsilon>0} \underline{\underline{H}}(A_\varepsilon),$$

where A_ε is an ε-neighborhood of A. If we set

$$G(A) = \bigcap_{\varepsilon>0} H(A_\varepsilon),$$

then Lemma 3.3 of [M] asserts that $\underline{\underline{G}}(A) = \sigma(G(A))$.

A σ-field $\underline{\underline{A}}$ such that $\underline{\underline{H}}(A)$ and $\underline{\underline{H}}(A^c)$ are conditionally independent given $\underline{\underline{A}}$ is called a *splitting field* for A. The following properties are well-known.

(2.1) $\underline{\underline{H}}(A) \cap \underline{\underline{H}}(A^c) \subset \underline{\underline{A}}$, for any splitting field $\underline{\underline{A}}$ of A ([Mc; Sect.6], [W1]);

(2.2) $\underline{\underline{G}}(\partial A)$ is a splitting field for all open sets (see [Ro; Chap.3 §5] for bounded open sets, [Nu; Th. 3.1] in the general case);

(2.3) $\underline{\underline{H}}(\partial A)$ is a splitting field for A when A is a finite union of rectangles with sides parallel to the coordinate axes [Ru; Th.7.5];

(2.4) $\underline{\underline{H}}(\partial A)$ is *not* a splitting field when A is the triangular region $\{(s_1, s_2) \in T: s_1 + s_2 < 1\}$ ([W1], [W3; p.399]).

Property (2.2) is known as the *germ-field Markov property* of the Brownian sheet. We say that the Brownian sheet has the *sharp Markov property* with respect to $A \subset T$ provided $\underline{\underline{H}}(\partial A)$ is a splitting field for A (see [W2]). Because of (2.4), it has widely been assumed in the literature that the Brownian sheet has the sharp Markov property only with respect to a very restricted class of sets (e.g. those in (2.3)). In fact, as mentioned in the introduction, we will show that the case $\underline{\underline{G}}(\partial A) = \underline{\underline{F}}(\partial A)$ is the "generic" case, at least when ∂A is the graph of a continuous function from $\mathbf{R}_+$ to $\mathbf{R}_+$.

Throughout this paper, we work with a continuous function $\phi: [0, \bar{u}] \to \mathbf{R}$, such that $\bar{u} > 0$, $\phi(\bar{u}) = 0$, and

$$0 \le u < \bar{u} \Rightarrow \phi(u) > 0.$$

The graph Γ of ϕ, defined by $\Gamma = \{(u, \phi(u)) : 0 \le u \le \bar{u}\}$ is a continuous curve, with two-dimensional Lebesgue measure 0, that splits $\mathbf{R}_+^2$ into two disjoint open connected components. The bounded component is

$$D_1 = \{t = (t_1, t_2) \in \mathbf{R}_+^2 : t_1 < \bar{u}, t_2 < \phi(t_1)\},$$

and the unbounded component is

$$\bar{D}_1^c = \{t = (t_1, t_2) \in \mathbf{R}_+^2 : t_1 > \bar{u} \text{ or } (t_1 \le \bar{u} \text{ and } t_2 > \phi(t_1))\}.$$

2.1. **Theorem.** The germ field $\underline{\underline{G}}(\Gamma)$ of Γ is the minimal splitting field for $\underline{\underline{F}}(D_1)$ and $\underline{\underline{F}}(\bar{D}_1^c)$. In particular, $\underline{\underline{G}}(\Gamma) = \underline{\underline{H}}(D_1) \cap \underline{\underline{H}}(\bar{D}_1^c)$.

Proof. This proof is similar to that of [DR; Theorem 3.1], which covers the case where f is non-increasing. It is sufficient to prove that

$$G(\Gamma) = H(D_1) \cap H(\bar{D}_1^c), \tag{2.5}$$

since then, by [M; Lemma 3.3],

$$\underline{\underline{G}}(\Gamma) = \sigma(G(\Gamma)) = \sigma(H(D_1)) \cap \sigma(H(\bar{D}_1^c)) = \underline{\underline{H}}(D_1) \cap \underline{\underline{H}}(\bar{D}_1^c),$$

and the conclusion will follow from (2.1) and (2.2).

Let

$$K = \{h: T \to \mathbf{R}: h(\cdot) = \int_{R_\cdot} g(s)\,ds,\ g \in L^2(T, dt)\}$$

be the reproducing kernel Hilbert space of the Brownian sheet, with the norm

$$\| \int_{R_\cdot} g(s)\,ds \|_K = \| g \|_{L^2(T,dt)} .$$

According to [Nu; Proposition 2.2], the condition below is equivalent to (2.5).

(2.6) For $\Delta \in \{D_1, \bar{D}_1^c\}$, for each $\eta_1 \in K$ with support included in $\bar{\Delta}$, and for $\varepsilon > 0$, there is $\eta_2 \in K$ with support included in Δ such that $\|\eta_1 - \eta_2\|_K < \varepsilon$.

To check (2.6), fix $\eta \in K$ with support in $\bar{D}_1$ (respectively D_1^c), and $g \in L^2(T, dt)$ such that

$$\eta(t) = \int_{R_t} g(s)\,ds, \quad t \in T.$$

For $\alpha > 0$, set $g^\alpha(s_1, s_2) = g(s_1, \alpha s_2)$ and

$$\eta^\alpha(t) = \int_{R_t} g^\alpha(s_1, s_2)\,ds_1\,ds_2, \quad (s_1, s_2) \in T.$$

Note that $\eta^\alpha(t_1, t_2) = \eta(t_1, \alpha t_2)/\alpha$, and for $\alpha > 1$ (resp. $\alpha < 1$) the support of η^α is contained in D_1 (resp. $\bar{D}_1^c$; note that this is due to the fact that Γ contains no vertical segments, and so this proof does not cover the case of [DR; Theorem 3.1], nor vice-versa). So (2.6) holds since $\| g - g^\alpha \|_{L^2}$ converges to 0 as α approaches 1. □

3. The generators of the minimal splitting field.

The generators of the minimal splitting field of a domain with a smooth boundary were determined in [W1], [W3; Th. 3.11] and with a piecewise monotone boundary in [WZ; Prop. 1]. In these cases, the minimal splitting field is generated by the white noise measures of the vertical and horizontal shadows of portions of the boundary, and the proof can be carried out by drawing pictures [W3; Th. 3.11]. The ideas in the smooth case are essentially valid in our setting. However, since we make no regularity assumptions on ϕ or Γ, precise topological definitions and formal proofs are necessary. Let

$$S_2 = \{(t_1, t_2) \in T: \text{there is } (s_1, s_2) \in \Gamma \text{ with } s_1 \ge t_1 \text{ and } s_2 = t_2\}$$

be the "horizontal shadow" of Γ. Note that $S_2 = D_1 \cup \bar{D}_2$, where $D_2 = \mathrm{int}(S_2 \cap \bar{D}_1^c)$.

An analogous definition of the vertical shadow S_1 of Γ leads to the set $\overline{D}_1$.

3.1. **Proposition.** For $0 \le u \le \bar{u}$, set $S_1^u = \{s \in D_1 : s_1 \le u\}$ and $W_u^1 = W(S_1^u)$. Then

$$W_u^1 \in \underline{\underline{H}}(D_1) \cap \underline{\underline{H}}(\overline{D}_1^c).$$

Proof. Observe that

$$S_1^u = \{s \in T : 0 \le s_1 \le u, 0 \le s_2 < \phi(s_1)\}.$$

For $n \in \mathbb{N}$ and $i = 0, \ldots, n$, set $a_i = iu/n$, and

$$m_i = \min_{a_i \le v \le a_{i+1}} \phi(v), \quad M_i = \max_{a_i \le v \le a_{i+1}} \phi(v),$$

$$s_n^1 = \frac{1}{n}\sum_{i=0}^{n-1} m_i, \quad s_n^2 = \frac{1}{n}\sum_{i=0}^{n-1} M_i,$$

(3.1) $$Y_n^1 = \sum_{i=0}^{n-1}(W_{a_{i+1},m_i} - W_{a_i,m_i}), \quad Y_n^2 = \sum_{i=0}^{n-1}(W_{a_{i+1},M_i} - W_{a_i,M_i}).$$

Then clearly $Y_n^1 \in \underline{\underline{H}}(D_1)$, $Y_n^2 \in \underline{\underline{H}}(\overline{D}_1^c)$, and for $i = 1,2$,

$$E((Y_n^i - W_u^1)^2) = |s_n^i - \int_0^u \phi(v)\,dv| \to 0$$

as $n \to \infty$, since s_n^1 (respectively s_n^2) is the lower (resp. upper) Riemann sum of ϕ and ϕ is continuous. This completes the proof. □

Proposition 3.1 essentially takes care of vertical shadows. To get horizontal shadows, define $p : S_1 \cup S_2 \to [0,\bar{u}]$ by

$$p(t_1,t_2) = \inf\{u \ge t_1 : \phi(u) = t_2\}.$$

This mapping has interesting measurability properties.

3.2. **Lemma.** (a) The function p is Borel.

(b) For any rectangle $\theta = [a,b] \times [c,d]$, contained in D_1 or in D_2, or more generally, for any open subset θ of $D_1 \cup D_2$, $p(\theta)$ is a Borel subset of $D_1 \cup D_2$.

(c) $p(D_1) \cap p(D_2) = \emptyset$.

Proof. (a) For $u, v \in [0,\bar{u}]$, with $u \le v$, set

$$L(u,v) = \max_{u \le x \le v} \phi(x), \quad l(u,v) = \min_{u \le x \le v} \phi(x)$$

(these notations will be used throughout the paper). Observe that for $0 \le v \le \bar{u}$,

(3.2) $\{s \in S_1 \cup S_2 : p(s) \le v\} = \{s \in T : s_1 \le v,\ l(s_1,v) \le s_2 \le L(s_1,v)\}.$

This is clearly a Borel set, since $l(\cdot,v)$ and $L(\cdot,v)$ are continuous functions.

(b) It is sufficient to prove (b) when θ is a closed rectangle in D_1 or D_2. Indeed, any open subset of $D_1 \cup D_2$ is a countable union of such rectangles, and the image of a union is the union of the images. So set

$$\theta = [a,b] \times [c,d], \quad a < b, \quad c < d,$$

and suppose for example that $\theta \subset D_1$ (the case $\theta \subset D_2$ is similar). Set $u_0 = p((b,d))$, $u_1 = p((b,c))$, and

$$\tau(u) = \inf\{v \ge u_0 : l(u_0,v) = l(u_0,u)\}, \quad u_0 \le u \le u_1.$$

Then $l(u_0,\tau(u)) = l(u_0,u)$, $u_0 \le u \le u_1$, and

$$p(\theta) = \{u \in [u_0,u_1] : \tau(u) = u\}.$$

Since $u \mapsto \tau(u)$ is easily seen to be left-continuous, $p(\theta)$ is Borel.

(c) Suppose $s \in D_1$, $t \in D_2$ and $p(s) = p(t)$. Then $s_2 = t_2$ by definition of p, so we suppose for instance that $s_1 < t_1$. Then $\phi(s_1) > s_2$ since $s \in D_1$ and $\phi(t_1) < t_1$ since $t \in D_2$. By the Intermediate Value Theorem, there is $u \in t]s_1,t_1[$ with $\phi(u) = s_2 = t_2$. But then $p(s) \le u < t_1 \le p(t)$, a contradiction. □

3.3. **Proposition.** For $0 \le u \le \bar{u}$, let

$$S_2^u = \{s \in S_1 \cup S_2 : p(s) \le u\}$$

denote the horizontal shadow of $\{(v,\phi(v) : v \le u\}$, and set $W_u^2 = W(S_2^u)$. Then

$$W_u^2 \in \underline{\underline{H}}(D_1) \cap \underline{\underline{H}}(\bar{D}_1^c).$$

Proof. The region S_2^u is characterized in (3.2). In particular, $S_2^u = E_1 \setminus E_2$, where

$$E_1 = \{(s_1,s_2) : 0 \le s_1 \le u, s_2 \le L(s_1,u)\},$$

$$E_2 = \{(s_1,s_2) : 0 \le s_1 \le u,\ S_2 < l(s_1,u)\}.$$

So we only need to show that $W(E_i) \in \underline{\underline{H}}(D_1) \cap \underline{\underline{H}}(\bar{D}_1^c)$, i = 1,2. We only carry out the proof for i = 1, since the case i = 2 is similar.

To show that $W(E_1)$ is $\underline{\underline{H}}(\bar{D}_1^c)$-measurable, we proceed as in the proof of Proposition 3.1, by approximating the area of the region E_1 by upper Riemann sums. The variables which correspond to the Y_n^2 in (3.1) are clearly $\underline{\underline{H}}(\bar{D}_1^c)$-measurable, so $W(E_1)$

is too.

To check that $W(E_1)$ is $\underline{\underline{H}}(D_1)$-measurable, we must take care to use only points in $\overline{D}_1$ in the approximation. This will be achieved using a "horizontal" discretisation instead of a "vertical" one. Set $a = L(u,u) = \phi(u)$, $b = L(0,u)$, $s_1^0 = u$, and for $n \in \mathbb{N}$ and $i = 1,..,n-1$,

$$s_2^i = a + i(b-a)/n,$$
$$s_1^i = \inf\{v \le u : L(v,u) = s_2^i\}$$

(s_1^i exists because $L(\cdot, u)$ is continuous). Now set

$$Y_n^2 = \sum_{i=0}^{n-1} (W_{s_1^i, L(s_1^i, u)} - W_{s_1^{i+1}, L(s_1^i, u)}).$$

Clearly, $E((W(E_1) - Y_n^2)^2) \le u/n$, so the proof will be complete if we show that Y_n^2 is $\underline{\underline{H}}(D_1)$-measurable. This will be the case if we prove that $\phi(s_1^i) = L(s_1^i, u)$, $i = 0, \ldots, n-1$, because in this case we will have

$$(s_1^i, L(s_1^i, u)) \in \overline{D}_1 \quad \text{and} \quad (s_1^{i+1}, L(s_1^i, u)) \in D_1 \, (i \ne n).$$

By definition of s_1^i,

(3.3) $$v \ge s_1^i \Rightarrow L(v,u) \le s_2^i \Rightarrow \phi(v) \le s_2^i,$$

and $v < s_1^i \Rightarrow L(v,u) > s_2^i$. Thus, by definition of $L(\cdot,\cdot)$, for any $\varepsilon > 0$, there is $v^i \in]s_1^i - \varepsilon, s_1^i[$ such that $\phi(v^i) > s_2^i$. But then continuity of ϕ and (3.3) imply $\phi(s_1^i) = s_2^i = L(s_1^i, u)$. This completes the proof. □

The image under the mapping p of Lebesgue-measure on $S_1 \cup S_2$ is a measure μ_2 on the σ-algebra $\underline{\underline{B}}([0,\bar{u}])$ of Borel sets of $[0,\bar{u}]$, defined by

$$\mu_2(I) = \text{Lebesgue-measure of } p^{-1}(I), \; I \in \underline{\underline{B}}([0,\bar{u}]).$$

If we consider that white noise W is an $L^2(\Omega, \underline{\underline{F}}, P)$-valued measure (i.e. a vector measure), its image $\hat{W}$ under p is an orthogonal measure on $[0,\bar{u}]$ with variance μ_2, i.e.

$$I \cap J = \phi, \; I, J \in \underline{\underline{B}}([0,\bar{u}]) \Rightarrow E(\hat{W}(I)\hat{W}(J)) = 0,$$

and for Borel $f \in L^2([0,\bar{u}]), d\mu_2)$,

$$E((\int_{[0,\bar{u}]} f(u)\, d\hat{W}_u)^2) = \int_{[0,\bar{u}]} f^2(u)\, d\mu_2(u).$$

If one wants to avoid images of vector measures, one can simply consider that

$$\int_{[0,\bar{u}]} f(u)\, d\hat{W}_u$$

is a shorthand notation for

$$\int_{S_1 \cup S_2} f(p(t))\, dW_t .$$

3.4. **Proposition.** Suppose $f \in L^2([0,\bar{u}], d\mu_2)$ is Borel. Then

$$\int_{[0,\bar{u}]} f(v)\, d\hat{W}_v \in \underline{\underline{H}}(D_1) \cap \underline{\underline{H}}(\bar{D}_1^c). \tag{3.4}$$

Proof. Observe that if $f = I_{[0,u]}$ ($I_{[0,u]}$ is the indicator function of $[0,u]$), then

$$\int_{[0,\bar{u}]} f(v)\, d\hat{W}_v = W_u^2 ,$$

so the conclusion in this case follows from Proposition 3.3. This implies (3.4) for all f which are a finite linear combination of indicator functions of intervals. Now suppose $(f_n, n \in \mathbb{N})$ is an increasing sequence of uniformly bounded Borel functions that satisfy (3.4), and set $f = \lim_{n \to \infty} f_n$. Then $f_n \to f$ in $L^2([0,\bar{u}]), d\mu_2)$ so

$$\int_{[0,\bar{u}]} f_n\, d\hat{W} \to \int_{[0,\bar{u}]} f\, d\hat{W} \text{ in } L^2(\Omega, \underline{\underline{F}}, P),$$

so f satisfies (3.4). We now apply the Monotone Class Theorem (see [DM; I.21]) to see that (3.4) holds for all bounded Borel functions. If $f \geq 0$ is Borel and $f \in L^2([0,\bar{u}]), d\mu_2)$, then

$$f = \lim_{n \to \infty} \min(f, n) \text{ in } L^2([0,\bar{u}], d\mu_2),$$

so (3.4) again holds in this case. Finally, to conclude for general Borel $f \in L^2([0,\bar{u}], d\mu_2)$, simply decompose f into its positive and negative parts. □

3.5. **Lemma.** Set

$$\underline{\underline{G}}_{in}^1 = \sigma(W(\theta), \theta \subset S_1, \theta \text{ open}),$$

$$\underline{\underline{G}}_{out}^1 = \sigma\Big(\int_{[0,\bar{u}]} f I_{p(D_2)}\, d\hat{W}, f \text{ bounded Borel}\Big)$$

(recall that $p(D_2)$ is a Borel set by Lemma 3.2 (b)). Then $\underline{\underline{G}}_{in}^1$ and $\underline{\underline{G}}_{out}^1$ are independent and $\underline{\underline{H}}(D_1) = \underline{\underline{G}}_{in}^1 \vee \underline{\underline{G}}_{out}^1$.

Proof. Observe that for each open set $\theta \subset D^1$ and for each bounded Borel function f,

$$E(W(\theta) \int_{[0,\bar{u}]} f I_{p(D_2)} d\hat{W}) = E(W(\theta) \int_{S^1 \cup S^2} f(p(s)) I_{D^2}(s)\, dW_s)$$

$$= 0$$

since $\theta \cap D^2 = \phi$. Since we are working with Gaussian random variables, $\underline{\underline{G}}^1_{in}$ and $\underline{\underline{G}}^1_{out}$ are independent. To check the second statement of the lemma, it is sufficient to show that if $t \in D_1$, then

$$W_t = W(D_1 \cap \operatorname{int} R_t) + \int_{[0,\bar{u}]} I_{p(\operatorname{int} R_t)} I_{p(D_2)} d\hat{W},$$

where int R_t is the interior of the rectangle R_t. Now for $t \in D_1$,

$$\begin{aligned} W_t - W(D_1 \cap \operatorname{int} R_t) &= W((\operatorname{int} R_t) \backslash (D_1 \cap \operatorname{int} R_t)) \\ &= W(D_2 \cap \operatorname{int} R_t) \\ &= \int_{[0,\bar{u}]} I_{p(\operatorname{int} R_t)} I_{p(D_2)} d\hat{W} \end{aligned}$$

(in the second equality above, we have used the fact that the Lebesgue measure of Γ is zero). □

We now recall the following lemma concerning conditional expectations of Gaussian random variables.

3.6. **Lemma.** Let H^1 and H^2 be two closed subspaces of a Gaussian space H.

(a) Suppose H^1 and H^2 are orthogonal and $Y \in H$ is orthogonal to H^1. Then

$$E(Y \mid \sigma(H^1) \vee \sigma(H^2)) = E(Y \mid \sigma(H^2))$$

(b) Suppose $G \subset H^1 \cap H^2$ is also a closed subspace, such that $pr_G(Y) = pr_{H^2}(Y)$, for all $Y \in H^1$ (pr denotes orthogonal projection). Then $\sigma(G)$ is a splitting field for $\sigma(H^1)$ and $\sigma(H^2)$.

Proof. (a) is a consequence of the fact that conditional expectations of Gaussian random variables are orthogonal projections, and (b) is a standard result (see [C; Lemma 5]). □

3.7. **Lemma.** Suppose $t \in \bar{D}_1^c$. Then

(a) there is a $\underline{\underline{G}}^1_{out}$-measurable variable Z_t such that

$$E(W_t | \underline{\underline{H}}(D_1)) = W(D_1 \cap R_t) + Z_t.$$

(b) $W(D_1 \cap R_t) = W^1_{t_1} - \int I_{p(\{s \in D_1 : s_1 \le t_1, s_2 > t_2\})} d\hat{W}$

Proof. (a) Note that $W_t = Y_1 + Y_2 + Y_3$, where

$$Y_1 = W(R_t \cap D_1), \; Y_2 = W(R_t \cap D_2), \; Y_3 = W(R_t \backslash (D_1 \cup D_2)).$$

Now Y_3 is independent of $\underline{\underline{H}}(D_1)$, since

$$R_s \cap (R_t \backslash (D_1 \cup D_2)) = \emptyset, \quad \forall s \in D_1,$$

and Y_1 is $\underline{\underline{H}}(D_1)$-measurable by Lemma 3.5. So

$$E(W_t | \underline{\underline{H}}(D_1)) = W(D_1 \cap R_t) + E(Y_2 | \underline{\underline{H}}(D_1)).$$

By Lemmas 3.5 and 3.6(a),

$$E(Y_2 | \underline{\underline{H}}(D_1)) = E(Y_2 | \underline{\underline{G}}^1_{in} \vee G^1_{out}) = E(Y_2 | G^1_{out}).$$

This concludes the proof of (a). As for (b), it is an immediate consequence of the definitions of $W^1_{t_1}$ and $\hat{W}$. □

3.8. **Theorem.** Set

$$\underline{\underline{K}}(\Gamma) = \sigma(W^1_u, 0 \le u \le \overline{u}) \vee \sigma(\int_{[0,\overline{u}]} f d\hat{W}, f \text{ bounded Borel}).$$

Then $\underline{\underline{K}}(\Gamma)$ is a splitting field for $\underline{\underline{H}}(D_1)$ and $\underline{\underline{H}}(\overline{D}^c_1)$ and $\underline{\underline{K}}(\Gamma) = \underline{\underline{H}}(D_1) \cap \underline{\underline{H}}(\overline{D}^c_1)$ (in particular, $\underline{\underline{K}}(\Gamma)$ is the minimal splitting field).

Proof. Proposition 3.1 and 3.4 imply that

$$\underline{\underline{K}}(\Gamma) \subset \underline{\underline{H}}(D_1) \cap \underline{\underline{H}}(\overline{D}^c_1).$$

To get the converse inclusion, observe that by Lemmas 3.5 and 3.7,

$$E(W_t | \underline{\underline{H}}(D_1)) \in \underline{\underline{K}}(\Gamma), \quad \forall t \in \overline{D}^c_1,$$

so by the above inclusion,

$$E(W_t | \underline{\underline{H}}(D_1)) = E(W_t | \underline{\underline{K}}(\Gamma)), \quad \forall t \in \overline{D}^c_1.$$

By Lemma 3.6(b), this means that $\underline{\underline{K}}(\Gamma)$ is a splitting field for $\underline{\underline{H}}(D_1)$ and $\underline{\underline{H}}(\bar{D}_1^c)$. By (2.1), we get the desired equality. □

4. Sufficient conditions for equality of the germ and sharp fields.

In this section, we will need the notion of stochastic line integrals for monotone curves. These notions were introduced in [CW; §4] in a more general context.

Suppose $0 \le a < b$ and $f: [a,b] \to \mathbf{R}_+$ is monotone and continuous. Let $\Gamma^f = \{(u, f(u)): a \le u \le b\}$ be the graph of f, which is a continuous monotone curve. Set

$$W_u^f = W_{u,f(u)}, \quad W_u^{f,1} = W(S_1(\Gamma_u^f)), \quad W_u^{f,2} = W(S_2(\Gamma_u^f)),$$

where $\Gamma_u^f = \{(v, f(v)): a \le v \le u\}$ and $S_1(A)$ (resp. $S_2(A)$) denotes the horizontal (resp. vertical) shadow of the set A (see beginning of Section 3). Then $(W_u^{f,i}$, $a \le u \le b)$ is a continuous martingale, i = 1,2, with quadratic variation

$$< W^{f,1} >_u = \int_a^u f(v)\,dv, \quad < W^{f,2} >_u = \int_{f(a)}^{f(u)} v\,df(v).$$

When f is non-decreasing, $(W_u^b$, $a \le u \le b)$ is a continuous martingale, and

$$< W^f >_u = uf(u) - af(a).$$

In this case, if $h \in L^2([a,b], d(uf(u)))$, the *stochastic line integral* of h along Γ^f is by definition

$$\int_{\Gamma^f} h\,\partial W = \int_a^b h(u)\,dW_u^f.$$

When f is non-increasing, and $h \in L^2(f(u)\,du) \cap L^2(u\,df(u))$ is Borel, the *stochastic line integral* of h along Γ^f is defined by

$$\int_{\Gamma^f} h\,\partial W = \int_a^b h(a)\,dW_u^{f,1} = \int_a^b h(u)\,dW_u^{f,2}.$$

Recall that if f is non-decreasing on [a,b], the *inverse* of f is the function $f^{-1}(v) = \sup\{u: f(u) \le v\}$, whereas if f is non-increasing, the inverse of f is $f^{-1}(v) = \sup\{u: f(u) \ge v\}$. We also set $\hat{f}^{-1}(v) = \inf\{u: f(u) \ge v\}$ (resp. $\hat{f}^{-1}(v) = \inf\{u: f(u) \le v\}$ when f is non-decreasing (resp. non-increasing).

4.1. **Lemma.** Suppose $f: [a,b] \to \mathbf{R}_+$ is monotone. Then

$$\int_{\Gamma^f} h\,\partial W = \int_{\mathbf{R}_+^2} (h(S_1)\,I_{S_1(\Gamma^f)}(s) + h(f^{-1}(s_2))\,I_{S_2(\Gamma^f)}(s))\,dW_s$$

for all h for which the left-hand side is defined.

Proof. The equality is a simple consequence of the definition of line-integral when h is the indicator function of an interval. The general case follows by a standard Monotone Class argument. □

4.2. **Lemma.** Fix $u_0 \le \bar{u}$, and set $f_1(u) = L(u, u_0)$, $u \le u_0$, and $f_2(u) = L(u_0, u)$, $u_0 \le u \le \bar{u}$. Then for $i = 1, 2$,

(a) $f_i(f_i^{-1}(s_2)) = \phi(s_2)$, and $f_i(\hat{f}_i^{-1}(s_2)) = \phi(s_2)$, when $(0, s_2) \in S_2(\Gamma^{f_i})$.

(b) Set $B_i = \{u : \phi(u) = f_i(u)\}$. Then B_i^c is an open set, so we can write

$$B_i^c = \bigcup_k]u_i^k, v_i^k[,$$

where $u_i^k < v_i^k$ and the union is countable and disjoint. Then

$$f_i(u_i^k) = \phi(u_i^k) = \phi(v_i^k) = f(u_i^k).$$

In particular,

$$\int_{B_i^c} d f_i(u) = 0.$$

Proof. We only carry out the proof for B_1, so we drop the index 1: $f(u) = L(u, u_0)$.

(a) By definition, $s_1 = f^{-1}(s_2)$ satisfies $f(s_1 + \delta) < f(s_1)$, $\forall \delta > 0$. Thus

$$f(s_1) > \max_{s_1+\delta<u<u_0} \phi(u).$$

Now assume $f(s_1) > \phi(s_1)$. We show that this leads to a contradiction. Indeed, there would be $\varepsilon > 0$ and $\delta > 0$ such that

$$s_1 \le u \le s_1 + \delta \Rightarrow \phi(u) < f(s_1) - \varepsilon.$$

But then

$$f(s_1) = \max_{s_1 \le u \le u_0} \phi(u) = \max(\max_{s_1 \le u \le s_1+\delta} \phi(u), \max_{s_1+\delta \le u \le u_0} \phi(u)) < f(s_1),$$

which is the desired contradiction.

(b) Note that u^k and v^k belong to B, so $\phi(u^k) = f(u^k)$ and $\phi(v^k) = f(v^k)$. It remains to check that $f(u^k) = f(v^k)$. Observe that $\phi(u) < f(u)$, $u^k < u < v^k$, by definition of B. Now suppose $f(v^k) < f(u^k)$. Since f is continuous, there is $w^k \in]u^k, v^k[$ such that $f(u^k) > f(w^k) > f(v^k)$. But then

$$\max_{w^k \le u \le u_0} \phi(u) > \max_{v^k < u < u_0} \phi(u).$$

But since ϕ is continuous, there would be $\tilde{u} \in [w^k, v^k]$ such that $\phi(\tilde{u}) = f(w^k)$. Now

$$\phi(\tilde{u}) \le f(\tilde{u}) \le f(w^k) = \phi(\tilde{u}),$$

so $f(\tilde{u}) = \phi(\tilde{u})$, a contradiction since $\tilde{u} \in B^c$.

Finally, the last statement holds since

$$\int_{B^c} df(u) = \sum_k (f(v^k) - f(u^k)) = 0.$$

□

4.3. **Lemma.** Fix $a < b < u_0 < \bar{u}$ and set $f_1(u) = L(u, u_0)$, $u \le u_0$. Suppose $B \in \underline{\underline{B}}([a,b])$ satisfies $B \subset B_1$ (defined in Lemma 4.2(b)). Then

$$\int_{\Gamma^{f_1}} I_B \partial W \in \underline{\underline{H}}(\Gamma)$$

(recall that $\Gamma = \{(u, \phi(u))\}$, $\Gamma^{f_1} = \{(u, f_1(u))\}$).

Proof. By definition of B_1, we have $\phi(u) = f_1(u)$, $\forall u \in B$. Since the non-negative measure $f_1(u)\,du - u\,df_1(u)$ is outer regular, for each $\varepsilon > 0$ there is an open set U such that

(4.1) $$B \subset U \text{ and } \int_U (f_1(u)\,du - u\,df_1(u)) \le \int_B (f_1(u)\,du - u\,df_1(u)) + \varepsilon.$$

Now

$$U = \bigcup_i I_i, \text{ where } I_i =]u_i, v_i[, \text{ and } I_i \cap I_j = \emptyset, i \ne j.$$

If $B \cap I_i = \emptyset$, we simply can remove that I_i from the union, without affecting (4.1). So we assume $B \cap I_i \ne \emptyset$, for all i. Set

$$a_i = \inf(B \cap I_i), \quad b_i = \sup(B \cap I_i),$$

and

$$V = \bigcup_i [a_i, b_i].$$

Then

(4.2) $$B \subset V \text{ and } \int_V (f_1(u)\,du - u\,df_1(u)) < \int_B (f_1(u)\,du - u\,df_1(u)) + \varepsilon.$$

Furthermore,

$$Y = \int_{\Gamma^{f_1}} I_V \partial W = \sum_i (W^{f,1}_{b_i, f(b_i)} - W^{f,1}_{a_i, f(a_i)}) - \sum_i (W^{f,2}_{b_i, f(b_i)} - W^{f,2}_{a_i, f(a_i)})$$

$$= \sum_i (W_{b_i, f(b_i)} - W_{a_i, f(a_i)}),$$

so $Y \in \underline{\underline{H}}(\Gamma)$ since $f(b_i) = \phi(b_i)$ and $f(a_i) = \phi(a_i)$ (because $a_i, b_i \in B_1$). Finally, by Lemma 4.1 and (4.2),

$$\| \int_{\Gamma^{f_1}} I_V \partial W - \int_{\Gamma^{f_1}} I_B \partial W \|_{L^2} \leq \varepsilon.$$

Since ε is arbitrary, this completes the proof. □

4.4. **Lemma.** Fix $\bar{u} \geq b > a > u_0$ and set $f_2(u) = L(u_0, u)$, $u \geq u_0$. Suppose $B \in \underline{\underline{B}}([a,b])$ satisfies $B \subset B_2$ (defined in Lemma 4.2). Then

$$\int_{\Gamma^{f_2}} I_B \partial W \in \underline{\underline{H}}(\Gamma).$$

Proof. The proof is similar to that of Lemma 4.3, using the measure $d(uf(u))$. Details are left to the reader. □

The following condition will turn out to be sufficient for the equality of the germ and sharp fields of Γ.

4.5. **Assumption.** There is a countable dense set Q on $[0, \bar{u}]$ such that for $u_0 \in Q$, the maps $u \mapsto L(u, u_0)$ ana $u_0 \mapsto L(u_0, u)$ are singular (with respect to Lebesgue measure).

4.6. **Lemma.** Under Assumption 4.5, the maps $u \mapsto L(u, u_0)$ are singular for all $u \in [0, \bar{u}]$.

Proof. Fix $u_0 \in [0, \bar{u}]$. To begin with, if $L(0, u_0) = \phi(u_0)$, then $L(\cdot, u_0)$ is constant on $[0, u_0]$, hence singular. So assume $L(0, u_0) > \phi(u_0)$. Now set

$$v_0 = \sup\{u \leq u_0 : L(u, u_0) > \phi(u_0)\}.$$

If $v_0 < u_0$, then $L(\cdot, u_0)$ is constant on $[v_0, u_0]$, so it suffices to check that $L(\cdot, u_0)$ is singular on $[0, v_0]$. We may thus assume in addition that $L(u, u_0) > \phi(u_0)$, for all $u < u_0$. Then fix $\delta > 0$, and let $u_1 \in [u_0 - \delta, u_0[$ be such that $\phi(u_1) = L(u_0 - \delta, u_0)$. Now fix $d_0 \in [u_1, u_0] \cap Q$, where Q is given in 4.5. We claim that

$$(4.3) \qquad L(u, d_0) = L(u, u_0), \quad \forall u < u_0 - \delta.$$

Indeed, for $u < u_0 - \delta$, $L(u, d_0) \leq L(u, u_0)$ by definition. So suppose

$L(u,d_0) < L(u,u_0)$. Now $L(u,d_0) \geq \phi(u_1) = L(u_0 - \delta, u_0)$, so if $v \in [0,u_0]$ is such that $\phi(v) = L(u,u_0)$, then $u \leq v < u_0 - \delta$. Thus $L(u,d_0) \geq \phi(v) = L(u,u_0)$, a contradiction.

By Assumption 4.5 and (4.3), $L(\cdot,u_0)$ is singular on $[0,u_0 - \delta]$, for all $\delta > 0$, implying $L(\cdot,u_0)$ is singular on $[0,u_0]$. The proof for $L(u_0,\cdot)$ is similar and is omitted. □

4.7. **Lemma.** Fix u_0, $a,b \in [0,u_0]$ such that $a < u_0 < b$ and $L(a,u_0) = \phi(u_0) = L(u_0,b)$, and define $f_1 : [a,u_0] \to \mathbf{R}_+$ and $f_2 : [u_0,b] \to \mathbf{R}_+$ by $f_1(u) = L(a,u)$ and $f_2(u) = L(u,b)$. Then under Assumption 4.5, $W(S_1(\Gamma^{f_i})) \in \underline{\underline{H}}(\Gamma)$ and $W(S_2(\Gamma^{f_i})) \in \underline{\underline{H}}(\Gamma)$, $i = 1,2$.

Proof. We only indicate the proof in the case $i = 1$. Let A_1 be a set of Lebesgue measure 1 in $[a,u_0]$ such that

$$\int_{A_1} df_1(u) = 0 .$$

Then $A_2 = B_1 \cap A_1^c$ satisfies the hypothesis of Lemma 4.3, so

$$\int_{\Gamma^{f_1}} I_{A^1} \partial W = W_{u_0, L(a,u_0)} - W_{a, L(a,a)} - \int_{\Gamma^{f_1}} I_{A_2} \partial W$$

$$\in \underline{\underline{H}}(\Gamma)$$

since $L(a,u_0) = \phi(u_0)$ and $L(a,a) = \phi(a)$. By the change of variables formula of [DM2; chap. VI 2, (55.1)],

$$\int_{f_1(a)}^{f_1(u_0)} I_{A_1}(f_1^{-1}(s_2))\, ds_2 = \int_{[a,u_0]} I_{A_1}(u)\, df_1(u) = 0 ,$$

so $(s_1,s_2) \mapsto I_{A_i}(f_1^{-1}(s_2))$ is a.s. zero on $S_2(\Gamma^{f_1})$ for $i = 1$, and a.s. one for $i = 2$. But then Lemma 4.1 implies that

$$\int_{\Gamma^{f_1}} I_{A_1} \partial W = \int_{\mathbf{R}_+^2} I_{A_1}(s_1)\, I_{S_1(\Gamma^{f_1})}(s)\, dW_s$$

$$= W(S_1(\Gamma^f))$$

since $(s_1,s_2) \mapsto I_{A_1}(s_1)$ is a.s. the constant function 1 on $S_1(\Gamma^{f_1})$. Similarly,

$$\int_{\Gamma^{f_1}} I_{B_2} \partial W = W(S_2(\Gamma^f)) .$$

This completes the proof. □

4.8. **Proposition.** Under Assumption 4.5, $W^i_{u_0} \in \underline{\underline{H}}(\Gamma)$, $0 \le u_0 \le \bar{u}$, $i = 1,2$.

Proof. We begin with the case $i = 1$. Fix $n \in \mathbb{N}$, and set $x^n_j = j u_0/n$, $j = 0, \ldots, n$. Define $g^n : [0, u_0] \to \mathbb{R}$ by

$$g^n(u) = L(x^n_j, x^n_{j+1}) \text{ if } x^n_j < u \le x^n_{j+1}.$$

Note that there is $v^n_j \in [x^n_j, x^n_{j+1}]$ such that $g^n(u) = \phi(v^n_j)$, $u \in]x^n_j, x^n_{j+1}]$.

The map g^n is piecewise monotone, though not continuous. Let $0 = j^n_0 < j^n_1 < \cdots < j^n_m = n$ be such that g^n is monotone on $[x_{j^n_l}, x_{j^n_{l+1}}]$ but is not monotone on $[x_{j^n_l}, x_{j^n_{l+1}+1}]$, for each l (it is important to take intervals closed on the left). Then if g^n is non-decreasing on $[x_{j^n_l}, x_{j^n_{l+1}}]$, it will be non-increasing on $[x_{j^n_{l+1}}, x_{j^n_{l+2}}]$. So we assume without loss of generality that

$$g^n \text{ is non-decreasing on } [x_{j^n_0}, x_{j^n_1}]. \tag{4.4}$$

We then define $h^n : [0, u_0] \to \mathbb{R}$ by

$$h^n(u) = L(v_{j^n_0}, u) \text{ if } v_{j^n_0} \le u \le v_{j^n_{l+1}} \text{ and } l \text{ is even},$$

$$h^n(u) = L(u, v_{j^n_{l+1}}) \text{ if } v_{j^n_l} \le u \le v_{j^n_{l+1}} \text{ and } l \text{ is odd},$$

$$h^n(u) = L(v_{j^n_m}, u) \text{ if } v_{j^n_m} \le u \le u_0.$$

Now h^n is piecewise monotone, and singular by Assumption 4.5, with intervals of monotonicity $[0, v^n_{j_1}]$, $[v^n_{j_1}, v^n_{j_2}]$, ..., $[v_{j^n_{m-1}}, u_0]$ and $\phi \le h^n \le g^n$. Since the area of the vertical shadow of Γ^{g^n} decreases to the area of $D^1_{u_0}$, the same is true for the area of $S_1(\Gamma^{h^n})$. But then the random variables $W(S_1(\Gamma^{h^n}))$, which are $\underline{\underline{H}}(\Gamma)$-measurable by Lemma 4.7 and Assumption 4.5, converge to $W(D^1_{u_0})$, completing the proof for $i = 1$.

To see that $W^2_{u_0} \in \underline{\underline{H}}(\Gamma)$, set $k^n(u) = \min\{h^n(v) : u \le v \le u_0\}$. Then k^n is non-decreasing, and singular since h^n is (see proof of Example 4.10(a) below). Furthermore, it is not difficult to see that

$$\lim_{n\to\infty} k^n(u) = l(u, u_0),$$

and so

$$\lim_{n\to\infty} W(S_2(\Gamma^{k^n})) = W(S_2(\Gamma^{f_2})) \text{ in } L^2(\Omega, \underline{\underline{F}}, P),$$

where $f_2(u) = l(u, u_0)$. So the proof will be complete if we show that $W(S_2(\Gamma^{k^n})) \in$

$\underline{\underline{H}}(\Gamma)$. Since h^n is piecewise monotone, k^n will coincide with h^n on finitely many intervals of the form $[v_{j_l^n}, w_l^n]$, where l will be even (by (4.4)) and

$$w_l^n = \sup\{u : v_{j_l^n} \le u \le u_0 \text{ and } h^n(v) = k^n(v),\ \forall v \in [v_{j_l^n}, u]\}$$
$$= \inf\{u \ge v_{j_l^n} : L(v_{j_l^n}, u) = \min_{\substack{\lambda > l \\ \lambda \text{ even}}} \phi(v_{j_\lambda^n})\}.$$

By Lemma 4.2 (a), we will have

$$k^n(w_l^n) = h^n(w_l^n) = L(v_{j_l^n}, w_l^n) = \phi(w_l^n),$$

so by Lemma 4.7, $W(S_2(\Gamma^{f_l})) \in \underline{\underline{H}}(\Gamma)$, where $f_l : [v_{j_l^n}, w_l^n] \to \mathbf{R}$ is defined by $f_l(u) = h^n(u)$, $v_{j_l^n} \le u \le w_l^n$. But then, since $S_2(\Gamma^{k^n})$ is the disjoint union of the $S_2(\Gamma^{f_l})$, we get the desired result. □

4.9. **Theorem.** Under Assumption 4.5, $\underline{\underline{H}}(\Gamma) = \underline{\underline{G}}(\Gamma)$.

Proof. By Proposition 4.8 and Theorem 3.8, we have $\underline{\underline{H}}(\Gamma) \supset \underline{\underline{H}}(D_1) \cap \underline{\underline{H}}(\bar{D}_1^c)$. The converse inclusion is also clear since $W_{u,\phi(u)} \in \underline{\underline{H}}(D_1) \cap \underline{\underline{H}}(\bar{D}_1^c)$, $0 \le u \le \bar{u}$. So the conclusion follows from Theorem 2.1. □

4.10. **Examples.**

(a) If $\phi : [0, \bar{u}] \to \mathbf{R}_+$ has bounded variation and is singular with respect to Lebesgue measure, then Assumption 4.5 is satisfied (the converse is also true: see Proposition 6.1).

(b) If $\phi : [0, \bar{u}] \to \mathbf{R}_+$ is such that any one of its four Dini derivatives is $+\infty$ a.s. for Lebesgue-measure, then Assumption 4.5 is satisfied. For example,

$$g(u) = \limsup_{h \downarrow 0} \frac{\phi(u+h) - \phi(u)}{h} = +\infty \text{ a.s.}$$

implies Assumption 4.5.

Proof. (a) We only check that $u \mapsto f(u) = L(u_0, u)$ is singular. In fact, we prove the stronger statement

$$\int_B df(u) \le \int_B d|\phi|(u), \quad B \in \underline{\underline{B}}([u_0, \bar{u}]). \tag{4.5}$$

Indeed, assume to begin with that $B =]a, b]$, $a < b$. Set

$$v_1 = \sup\{u : f(u) = f(a)\}, \quad v_2 = \inf\{u : f(u) = f(b)\}.$$

If $v_2 \le v_1$, (4.5) is clearly satisfied. So assume $v_1 < v_2$. Then $f(v_1) < f(v_2)$ and by Lemma 4.2 (a), $f(v_1) = \phi(v_1)$ and $f(v_2) = \phi(v_2)$. Thus

$$\int_{]a,b]} df(u) = f(v_2) - f(v_1) = \phi(v_2) - \phi(v_1) \le \int_{]a,b]} d\,|\phi|\,(u),$$

and so (4.3) holds in this special case and also if B is a countable union of intervals. The general case follows since any Borel set can be approximated from above in $df(u) + d\,|\phi|(u)$-measure by a countable union of intervals.

(b) Fix $u_0 \in [0, \bar{u}]$. For any $u \le u_0$, such that $g(u) = +\infty$, we have $L(u, u_0) > \phi(u)$. So by Lemma 4.2 (b),

$$\int_{\{u : g(u) = +\infty\} \cap [0, u_0]} L(du, u_0) = 0.$$

But since by hypothesis, $\{v : g(v) = \infty\} \cap [0, u_0]$ has Lebesgue-measure u_0, $u \mapsto L(u_0, u)$ is singular.

We now check that $u \mapsto L(u_0, u)$ is singular. Indeed, $u \mapsto L(u_0, u)$ has a Lebesgue-decomposition

$$L(u_0, u) = \int_{u_0}^{u} h(v)\, dv + \nu(u),$$

where $\nu(\cdot)$ is a non-decreasing singular function and h is non-negative and integrable on $[u_0, u]$. Thus

$$\frac{\partial}{\partial u} L(u_0, u) = h(u) \text{ a.s.} \tag{4.6}$$

with respect to Lebesgue-measure. However, at every point u in the support of $v \mapsto L(u_0, v)$, $L(u_0, u) = \phi(u)$, so for du-almost all u in the support, $h(u) = g(u) = +\infty$. Since h is integrable the support of $v \mapsto L(u_0, v)$ must be a Lebesgue-null set, concluding the proof. □

4.11. **Remark.** Theorem 4.9 remains valid if Assumption 4.5 is replaced by an analogous assumption on the functions $l(\cdot, u_0)$ and $l(u_0, \cdot)$, $0 \le u_0 < \bar{u}$.

- Theorem 4.9 also remains valid if Assumption 4.5 is replaced by either of the following two assumptions:

$$L(\cdot, u_0) \text{ and } l(\cdot, u_0) \text{ are singular}, \quad \forall u_0 \in D$$

or

$$L(u_0, \cdot) \text{ and } l(u_0, \cdot) \text{ are singular}, \quad \forall u_0 \in D$$

(recall that D is a countable dense set). The important feature in each of these assumptions is that we have both non-increasing and non-decreasing functions in each case.

- Assumption 4.5 does not imply that $u \mapsto l(0,u)$ is singular, as shown by the example below.

4.12. **Example.** We are going to define a function $\phi : [0,1] \to \mathbb{R}_+$ which is continuous, satisfies Assumption 4.5, but for which $l(0,\cdot)$ is absolutely continuous with respect to Lebesgue-measure. Let $f : [0,1] \to \mathbb{R}$ be a continuous function which has the following properties:

- $f(0) = f(1) = 0$
- $f(u) > 0,\ 0 < u < 1$
- $\limsup_{h \to 0} \dfrac{f(u+h) - f(u)}{h} = +\infty$, for almost all $u \in [0,1]$
- $\max\{f(u) : 0 \le u \le 1\} = 1.$

Let C be a Cantor set, obtained by removing from $[0,1]$ at the k^{th} step 2^{k-1} open intervals $I_{k,1}, \ldots, I_{k,2^{k-1}}$ of length 4^{-k}. Then C has Lebesgue measure 1/2. Let $I_{k,j} =]u_{k,j}\, v_{k,j}[$. We then set

$$\phi_1(u) = 2^{-k/2} f((u - u_{k,j})4^k), \quad u_{k,j} \le u \le v_{k,j},$$
$$\phi_1(u) = 0 \qquad , \quad u \in C.$$

Observe that ϕ_1 is continuous on $[0,1]$, and for all $u \in C$,

$$\limsup_{h \downarrow 0} \frac{\phi_1(u+h) - \phi_1(u)}{h} = +\infty.$$

Finally, we set

$$\phi(u) = 1/2 - \int_0^u I_C(v)\,dv + \phi_1(x).$$

It is then clear that ϕ satisfies the condition of Example 4.10 (b), and so Assumption 4.5 is valid. On the other hand, it is easy to see that

$$l(0,u) = \frac{1}{2} - \int_0^u I_C(v)\,dv,$$

so $l(0,\cdot)$ is absolutely continuous with respect to Lebesgue-measure. □

4.13. **Corollary.** Let $C = \{f : \mathbb{R}_+ \to \mathbb{R} : f(0) = 1$ and f is continuous$\}$, and let B denote Wiener measure on C. For $f \in C$, set

$$\zeta(f) = \inf\{u > 0 : f(u) = 0\} \quad \text{if } \{ \quad \} \neq \emptyset,$$

$$\zeta(f) = +\infty \qquad \text{otherwise}.$$

Let $\phi(f)$ be the restriction of f to $[0, \zeta(f)]$. Then

$$B\{f \in C : \zeta(f) < +\infty \text{ and } \underline{\underline{H}}(\Gamma^{\phi(f)}) = \underline{\underline{G}}(\Gamma^{\phi(f)})\} = 1.$$

Proof. Immediate consequence of Example 4.10 (b) and Theorem 4.9. □

5. Some necessary conditions for equality of the germ and sharp fields.

In Theorem 3.8, we determined the generators of the minimal splitting field for $\underline{\underline{H}}(D_1)$ and $\underline{\underline{H}}(\bar{D}_1^c)$. By Theorem 2.1, these are the generators of the germ field. In this section, we are going to give an explicit integral representation for the closed Gaussian linear subspace spanned by these generators. By Theorem 2.1, this subspace is $G(\Gamma)$.

These results generalize those of [DR; Th. 3.8], where such a representation is given when ϕ is decreasing. Also Rozanov [R; Chap. 3; Sect. 3.5] has given an implicit representation of $G(\Gamma)$ in terms of the solution to a generalized partial differential equation, when Γ is the boundary of any bounded open set. Our representation can thus be viewed as the solution to Rozanov's equation when Γ is the graph of a continuous function ϕ.

Once we have a good description of $G(\Gamma)$ at our disposal, it becomes possible, under a rather weak regularity assumption on ϕ, to give conditions under which an element of $G(\Gamma)$ does *not* belong to $H(\Gamma)$. Though we do not give a complete description of $H(\Gamma)$ (and such a description is unnecessary for our purposes), this approach leads to necessary and sufficient conditions on ϕ for equality of the germ and sharp fields (provided ϕ satisfies Assumption 5.6 below). In particular, when ϕ has bounded variation, Corollary 5.15 gives a complete answer to the question: " when are $\underline{\underline{F}}(\Gamma)$ and $\underline{\underline{G}}(\Gamma)$ equal?"

If $f_1, f_2 : [0, \bar{u}] \to \mathbb{R}$ are two Borel functions, we define three Borel functions $J_1(f_1, f_2)$, $J_2(f_1, f_2)$ and $J(f_1, f_2)$ on T by

$$J_1(f_1, f_2)(t) = (f_1(t_1) + f_2(p(t)))\, I_{D_1}(t)$$

$$J_2(f_1, f_2)(t) = f_2(p(t))\, I_{D_2}(t)$$

$$J(f_1, f_2)(t) = J_1(f_1, f_2)(t) + J_2(f_1, f_2)(t).$$

5.1. **Lemma.** (a) Suppose $J(f_1, f_2) \in L^2(T, dt)$. Then $J_i(f_1, f_2) \in L^2(T, dt)$, $i = 1,2$. In particular, $f_2 I_{p(D_2)} \in L^2([0, \bar{u}], d\mu_2)$.

(b) Suppose $(f_1^n, n \in \mathbb{N})$ and $(f_2^n, n \in \mathbb{N})$ are sequences of Borel functions such that $J(f_1^n, f_2^n) \in L^2(T, dt)$ for all $n \in \mathbb{N}$ and

$$g = \lim_{n\to\infty} J(f_1^n, f_2^n)$$

exists in $L^2(T, dt)$. Then

$$g_i = \lim_{n\to\infty} J_i(f_1^n, f_2^n)$$

exists in $L^2(T, dt)$, $i = 1,2$. In addition, there is a Borel function $f_2 : [0, \bar{u}] \to \mathbb{R}$ such that

$$f_2 I_{p(D_2)} = \lim_{n\to\infty} f_2^n I_{p(D_2)} \quad \text{in} \quad L^2([0, \bar{u}], d\mu_2)$$

and $g_2(t) = f_2(p(t))$ dt-a.s. (Note: the question of convergence of $f_2^n I_{p(D_1)}$ is addressed in Lemma 5.2).

Proof. (a) Let $\|\cdot\|$ denote the norm in $L^2(T, dt)$. Since $J_1(f_1, f_2)(t) J_2(f_1, f_2)(t) = 0$, $\forall t \in T$, we have

$$\|J(f_1, f_2)\|^2 = \|J_1(f_1, f_2)\|^2 + \|J_2(f_1, f_2)\|^2. \tag{5.1}$$

This implies the first statement in (a). The second is a consequence of the relation

$$\int_{[0,\bar{u}]} (f_2(u))^2 I_{p(D_2)}(u)\, d\mu_2(u) = \int_{\mathbb{R}_+^2} (J_2(f_1, f_2)(t))^2\, dt. \tag{5.2}$$

(b) If $(J(f_1^n, f_2^n), n \in \mathbb{N})$ is a Cauchy sequency in $L^2(T, dt)$, then (5.1) implies that $(J_i(f_1^n, f_2^n), n \in \mathbb{N})$ is also, $i = 1,2$. This yields the first statement of (b). To get the second statement, observe by (5.2) that $(f_2^n I_{p(D_2)}, n \in \mathbb{N})$ converges in $L^2([0, \bar{u}]), d\mu_2)$. Choose a Borel function f_2 which is a μ_2-version of this limit. Then

$$\begin{aligned} \|g_2 - (f_2 \circ p) I_{D_2}\| &\le \|g - J_2(f_1^n, f_2^n)\| \\ &\quad + \|J_2(f_1^n, f_2^n) - (f_2^n \circ p) I_{D_2}\| + \|(f^n \circ p) I_{D_2} - (f_2 \circ p) I_{D_2}\| \\ &= \|g - J_2(f_1^n, f_2^n)\| + \|f_2^n I_{p(D_2)} - f_2\|_{L^2(d\mu_2)} \\ &\to 0 \end{aligned}$$

as $n \to \infty$. □

Set

$$L_1 = \bigcap_{\varepsilon>0} L^2([0,\bar{u}-\varepsilon], d\mu_1], \quad L_2 = \bigcap_{\varepsilon>0} L^2([0,\bar{u}-\varepsilon], d\mu_2).$$

If $g \in L^2(T,dt)$ satisfies $g = J_1(f_1,f_2)$ for some couple $(f_1,f_2) \in L_1 \times L_2$, we say that g is *determined* by (f_1,f_2). If $k \in \mathbb{R}$, and we set

$$f_1^k = f_1 - k, \quad f_2^k = (f_2 + k) I_{p(D_1)} + f_2 I_{p(D_1^c)},$$

then (f_1^k, f_2^k) also determines g.

Now fix $s^0 = (s_1^0, s_2^0) \in D_1$ such that $R_{s^0} \subset D_1$ (recall that $R_{s^0} = \{s \in \mathbb{R}_+^2 : s_1 \le s_1^0$ and $s_2 \le s_2^0\}$). For any $g \in L^2(T,dt)$ which is determined by some (f_1,f_2), we may (by adding an appropriate constant to f_1) choose $(\tilde{f}_1, \tilde{f}_2)$ that determines g and such that

$$\int_0^{s_1^0} \tilde{f}_1(u)\,du = 0 \tag{5.3}$$

Finally, for $\varepsilon > 0$, set $\tau_\varepsilon = \inf\{u : \phi(u) = \varepsilon\}$ and $D_\varepsilon^1 = \{(s_1,s_2) \in D^1 : s_1 < \tau^\varepsilon\}$.

5.2. **Lemma.**

(a) If $(f_1,f_2) \in L_1 \times L_2$, then for each $\varepsilon > 0$, $t \mapsto f_1(t_1) I_{D_\varepsilon^1}$ and $t \mapsto (f_2 \circ p)(t) I_{[0,\tau_\varepsilon]}(p(t))$ belong to $L^2(T,dt)$.

(b) If $J_1(f_1,f_2) \in L^2(T,dt)$, then $(f_1,f_2) \in L_1 \times L_2$.

(c) Suppose $g \in L^2(T,dt)$ is determined by $(f_1,f_2) \in L_1 \times L_2$ where f_1 satisfies (5.3). Then for each $\varepsilon > 0$, there are constants K_1^ε and K_2^ε such that

$$\| f_1 I_{[0,\tau_\varepsilon]} \|_{L^2([0,\bar{u}], d\mu_1)} \le K_1^\varepsilon \|g\|,$$

$$\| f_2 I_{[0,\tau_\varepsilon]} \|_{L^2([0,\bar{u}], d\mu_2)} \le K_2^\varepsilon \|g\|.$$

(d) Suppose $(g^n, n \in \mathbb{N})$ is a sequence in $L^2(\mathbb{R}_+^2, dt)$ converging to g. If g^n is determined by $(f_1^n, f_2^n) \in L_1 \times L_2$, where f_1^n satisfies (5.3), then there is $(f_1,f_2) \in L_1 \times L_2$ that determines g, and for each $\varepsilon > 0$,

$$f_1 I_{[0,\tau_\varepsilon]} = \lim_{n\to\infty} f_1^n I_{[0,\tau_\varepsilon]} \text{ in } L^2([0,\bar{u}], d\mu_1),$$

$$f_2 I_{p(D_1)\cap[0,\tau_\varepsilon]} = \lim_{n\to\infty} f_2^n I_{p(D_1)\cap[0,\tau_\varepsilon]} \text{ in } L^2([0,\bar{u}], d\mu_2),$$

Proof. (a) Observe that

$$\int_T (f_1(t_1))^2 I_{D_\varepsilon^1}(t)\,dt = \int_0^{\tau_\varepsilon} (f_1(u))^2 \phi(u)\,du \le L(0,\bar{u}) \int_0^{\tau_\varepsilon} (f_1(u))^2\,d\mu_1,$$

and

$$\int_T ((f_2 \circ p)(t))^2 I_{[0,\tau_\varepsilon]}(p(t))\,dt = \int_0^{\tau_\varepsilon} (f_2(u))^2 d\mu_2(u),$$

so the conclusion follows by definition of L_1 and L_2 and the fact that $\tau_\varepsilon < \bar{u}$.

(b) Suppose $J_1(f_1, f_2) \in L^2(T, dt)$. Then $J_1(f_1, f_2) \in L^2(R_{s^0}, dt)$. Since $p(t_1, t_2) = p(0, t_2)$, for all $t \in R_{s^0}$, we get

$$\int_0^{s_1^0} dt_1 \int_0^{s_2^0} dt_2 (f_1(t_1) + f_2(p(0, t_2)))^2 < \infty.$$

By Fubini's theorem, this implies that for almost-all t_1,

$$t_2 \mapsto f_1(t_1) + f_2(p(0, t_2))$$

belongs to $L^2([0, s_2^0], dt_2)$. But then $t_2 \mapsto f_2(p(0, t_2))$ also belongs to $L^2([0, s_2^0], dt_2)$, which, when $\varepsilon < s_2^0$, implies in particular that

$$\int_0^{\tau_\varepsilon} dt_1 \int_0^{\varepsilon} dt_2 (f_2(p(t)))^2 = \tau_\varepsilon \int_0^{\varepsilon} dt_2 (f_2(p(0, t_2)))^2 < +\infty.$$

But then $t \mapsto f_2(p(t))$ belongs to $L^2([0, \tau_\varepsilon] \times [0, \varepsilon], dt)$, and since

$$J_1(f_1, f_2)(t) = (f_1(t_1) + f_2(p(t)) I_{D^1}(t), \tag{5.4}$$

we get that

$$\int_0^{\tau_\varepsilon} dt_1 \int_0^{\varepsilon} dt_2 (f_1(t_1))^2 < +\infty. \tag{5.5}$$

But then

$$\varepsilon \int_0^{\tau_\varepsilon} dt_1 (f_1(t_1))^2 < +\infty,$$

so $f_1 \in L^2([0, \tau_\varepsilon], d\mu_1)$. Since ε is arbitrary, $f_1 \in L_1$.

On the other hand, (5.5) implies

$$\int_0^{\tau_\varepsilon} dt_1 \int_0^{\phi(t_1)} dt_2 (f_1(t_1))^2 \le \frac{L(0, \bar{u})}{\varepsilon} \int_0^{\tau_\varepsilon} dt_1 \int_0^{\varepsilon} dt_2 (f_1(t_1))^2 < +\infty,$$

so $t \mapsto f_1(t_1) I_{D^1_{\tau_\varepsilon}}$ belongs to $L^2(T, dt)$. But then (5.4) implies that $t \mapsto f_2(p(t)) I_{D^1_{\tau_\varepsilon}}$ belongs to $L^2(T, dt)$. By the definition of μ_2, we find that $f_2 \in L^2([0, \tau_\varepsilon], d\mu_2)$. Since ε is arbitrary, $f_2 \in L_2$, concluding the proof of (b).

(c) Set $\tilde{f}_1(t) = f_1(t_1)$, $t = (t_1, t_2)$, and fix $r > 0$ such that $r < s_2^0$. We have

$$
\begin{aligned}
+\infty > \|g\|^2 &\geq \int_0^{s_1^0} dt_1 \int_r^{s_2^0} dt_2\, g^2(t)\, dt \\
(5.6) \qquad &= (s_2^0 - r) \int_0^{s_1^0} (f_1(t_1))^2 dt_1 + 2 \int_0^{s_1^0} dt_1\, f_1(t_1) \int_r^{s_2^0} dt_2\, f_2(p(t)) + s_1^0 \int_r^{s_2^0} (f_2(s(t)))^2 dt_2
\end{aligned}
$$

(we have used the fact that on R_{s^0}, $f_2 \circ p$ is a function of t_2 only. The use of $r > 0$ is necessary to write the double product as an iterated integral). By (5.3), the second term in the last equality above is zero, and so, letting r go to zero in (5.6), we get

$$
(5.7) \qquad \|\tilde{f}_1 I_{R_{s^0}}\| \leq \|g\|, \quad \|(f_2 \circ p) I_{R_{s^0}}\| \leq \|g\|.
$$

We are now going to use several times the fact that $g = \tilde{f}_1 + f_2 \circ p$ on D^1, and $\tilde{f}_1$ only depends on t_1 and $f_2 \circ p$ only depends on $p(t)$. We have

$$
\begin{aligned}
\int_{R_{s^0}} (f_2(p(t)))^2 dt &= \int_0^{s_2^0} dt_2 \int_0^{s_1^0} dt_1\, (f_2(p(t)))^2 \\
&= \int_0^{s_2^0} dt_2\, (f_2(p(t)))^2 s_1^0 \\
&\geq \frac{s_1^0}{\bar{u}} \int_0^{s_2^0} dt_2\, (f_2(s(t)))^2 \int_0^{p(0,t_2)} dt_1 \\
&= \frac{s_1^0}{\bar{u}} \int_{p^{-1}(p(R_{s^0}))} (f_2(p(t)))^2 dt,
\end{aligned}
$$

so by (5.7),

$$
\|(f_2 \circ p) I_{p^{-1}(p(R_{s^0}))}\| \leq \frac{\bar{u}}{s_1^0} \|g\|.
$$

For $0 < \varepsilon < s_2^0$, this inequality implies

$$
\begin{aligned}
\|\tilde{f}_1 I_{[0,\tau_\varepsilon]\times[0,\varepsilon]}\| &\leq \|\tilde{f}_1 I_{p^{-1}(p(R_{s^0}))}\| = \|(g - (f_2 \circ p)) I_{p^{-1}(p(R_{s^0}))}\| \\
&\leq (1 + \frac{\bar{u}}{s_1^0}) \|g\|.
\end{aligned}
$$

Now since $\tilde{f}_1$ depends only on t_1, we get

$$
\|\tilde{f}_1 I_{D_\varepsilon^1}\| \leq \frac{L(0,\bar{u})}{\varepsilon} \|\tilde{f}_1 I_{[0,\tau_\varepsilon]\times[0,\varepsilon]}\|,
$$

so for some constant $\tilde{K}_1^\varepsilon$, (5.8) $$\|\tilde{f}_1 I_{D_{\varepsilon}^1}\| \le \tilde{K}_1^{\varepsilon} \|g\|.$$

If we now set $K_2^{\varepsilon} = 1 + \tilde{K}_1^{\varepsilon}$, we get

(5.9) $$\|(f_2 \circ p) I_{D_{\varepsilon}^1}\| \le K_2^{\varepsilon} \|g\|.$$

Finally, observe that (5.8) implies

(5.10) $$\|f_1 I_{[0,\tau_\varepsilon]}\|_{L^2([0,\bar{u}],d\mu_1)} \le \frac{1}{\varepsilon} \|\tilde{f}_1 I_{D_{\varepsilon}^1}\| \le K_1^{\varepsilon} \|g\|$$

for an appropriate constant K_1^{ε}, and (5.9) implies

(5.11) $$\|f_2 I_{p(D_1)\cap[0,\tau_\varepsilon]}\|_{L^2([0,\bar{u}],d\mu_2)} \le \|(f_2 \circ p) I_{D_{\varepsilon}^1}\| \le K_2^{\varepsilon} \|g\|.$$

(d) If $\varepsilon \le \varepsilon'$, then $\tau_\varepsilon \ge \tau_{\varepsilon'}$. Furthermore, $\lim_{\varepsilon\to 0} \tau_\varepsilon = \bar{u}$. By (5.10), $(f_1^n I_{[0,\tau_{1/k}]}, n \in \mathbb{N})$ converges in $L^2([0,\bar{u}],d\mu_1)$. Fix $k_0 \in \mathbb{N}$ large enough, and let h_1^k be a Borel function which is a version of the limit of this sequence, and define $f_1 : [0,\bar{u}] \to \mathbb{R}$ by

$$f_1(u) = \begin{cases} h_1^{k_0}(u) & \text{if } u \le k_0, \\ h_1^{k+1}(u) & \text{if } \tau_{1/k} < u \le \tau_{1/(k+1)}, \\ 0 & \text{if } u = \bar{u}. \end{cases}$$

Then f_1 is Borel, and

$$f_1 I_{[0,\tau_{1/k}]} = f_1^k I_{[0,\tau_{1/k}]} \quad \mu_1\text{-a.s.}$$

In a similar fashion, we get from inequality (5.11) a Borel function $f_2 : [0,\bar{u}] \to \mathbb{R}$ such that for each $k \ge k_0$,

$$f_2 I_{p(D_1)\cap[0,\tau_{1/k}]} = \lim_{n\to\infty} f_2^n I_{p(D_1)\cap[0,\tau_{1/k}]} \quad \text{in } L^2([0,\bar{u}],d\mu_2).$$

It only remains to be shown that (f_1, f_2) determines g (note that f_1 satisfies (5.3)). This follows from the equality

$$\|(g - J_1(f_1,f_2)) I_{D_{\varepsilon}^1\cap([0,\tau_\varepsilon]\times[\varepsilon,\infty[)}\| = 0, \quad \forall\varepsilon > 0,$$

which is valid by the definition of f_1 and f_2. □

5.3. **Remark.** It is not possible to remove the $I_{D_{\varepsilon}^1}$ or the $I_{[0,\tau_\varepsilon]}$ in the statements of Lemma 5.2, as the following example shows. Consider the continuous function $\phi : [0,1] \to \mathbb{R}_+$ which is obtained by modifying the function $\tilde{\phi}(u) = 1 - u$ on each interval $[1 - 2^{-n+1}, 1 - 2^{-n}]$ as follows: ϕ is linear on each of the intervals

$$I_n^1 = [1 - 2^{-n+1}, 1 - 7\,2^{-n+1}/8],$$
$$I_n^2 = [1 - 7\,2^{-n+1}/8, 1 - 3\,2^{-n+1}/4],$$
$$I_n^3 = [1 - 3\,2^{-n+1}/4, 1 - 2^{-n}],$$

and

$$\phi(1 - 7\,2^{-n+1}/8) = 2^{-n/2}, \phi(1 - 3\,2^{-n+1}/4) = 2^{-n+1}.$$

(See Figure 1). Now set $a_n = 2^n/n$ and define f_1, f_2: $[0,\bar{u}] \to \mathbf{R}$ by

$$f_1(u) = a_n, \quad f_2(u) = 0 \text{ if } u \in I_n^1,$$
$$f_1(u) = a_n, \quad f_2(u) = -a_n \text{ if } u \in I_n^2,$$
$$f_1(u) = 0, \quad f_2(u) = 0 \text{ if } u \in I_n^3.$$

Observe that $(f_1, f_2) \in L_1 \times L_2$, since f_i is bounded on $[0,\tau_\varepsilon]$, $\varepsilon > 0$, $i = 1,2$, and

$$J_1(f_1, f_2) = \sum_{n\in\mathbf{N}} a_n I_{(I_n^1 \cup I_n^2)\times[0,2^{-n+1}]},$$

so

$$\int_{\mathbf{R}_+^2} (J_1(f_1,f_2)(t))^2 dt = \sum_{n\in\mathbf{N}} \frac{2^{2n}}{n^2} 2^{-n-1} 2^{-n+1} = \sum_{n\in\mathbf{N}} \frac{1}{n^2} < +\infty.$$

On the other hand,

$$\int_{D_1} (f_1(t_1))^2 dt_1\, dt_2 = \sum_{n\in\mathbf{N}} a_n^2 (2^{-2n} + \frac{1}{2} 2^{-n-1} 2^{-n/2})$$
$$= \sum_{n\in\mathbf{N}} (\frac{1}{n^2} + \frac{2^{n/2-2}}{n^2})$$
$$= +\infty.$$

Similarly,

$$\int_{D_1} (f_2(p(t)))^2 dt_1\, dt_2 = +\infty.$$

□

5.4. **Lemma.** Set $\underline{\underline{L}} = \{(f_1,f_2) \in L_1 \times L_2 : J(f_1,f_2) \in L^2(T,dt)\}$. For $(f_1,f_2) \in \underline{\underline{L}}$, set

$$I(f_1,f_2) = \int_T J(f_1,f_2)(t)\, dW_t.$$

(a) If $f_1 \in L^2([0,\bar{u}], \phi(u)\,du)$ and $f_2 \in L^2([0,\bar{u}], d\mu_2)$ are Borel, then $(f_1,f_2) \in \underline{\underline{L}}$ and

$$I(f_1, f_2) = \int_0^{\bar{u}} f_1(u)\, dW_u^1 + \int_{[0,\bar{u}]} f_2(u)\, d\hat{W}_u .$$

(b) For all $(f_1, f_2) \in \underline{\underline{L}}$, $I(f_1, f_2) \in G(\Gamma)$.

Proof. Under the hypothesis of (a), we have

$$\int_0^1 f_1(u)\, dW_u^1 = \int_{D^1} f_1(t_1)\, dW_{t_1, t_2}$$

and

$$\int_{[0,\bar{u}]} f_2(u)\, d\hat{W}_u = \int_{D_1 \cup D_2} f_2(p(t))\, dW_t$$

$$= \int_{D_1} f_2(p(t))\, dW_t + \int_{D_2} f_2(p(t))\, dW_t ,$$

so by definition of $J(f_1, f_2)$, the conclusion of (a) holds.

To see (b), observe that by Lemma 5.1(a), $J_i(f_1, f_2) \in L^2(T, dt)$, $i = 1,2$. Since

$$\int_T J_2(f_1, f_2)(t)\, dW_t = \int_{[0,\bar{u}]} f_2(u)\, I_{p(D_2)}(u)\, d\hat{W}_u \in G(\Gamma)$$

by Proposition 3.4, we only need to show that

$$I_1(f_1, f_2) = \int_{\mathbb{R}_+^2} J_1(f_1, f_2)(t)\, dW_t \in G(\Gamma) .$$

Set $f_i^\varepsilon = f_i I_{[0,\tau_\varepsilon]}$, $i = 1,2$, $\varepsilon > 0$. By Lemma 5.2 (c),

$$Y_1^\varepsilon = \int_0^{\bar{u}} f_1^\varepsilon(u)\, dW_u^1 \text{ and } Y_2^\varepsilon = \int_{[0,\bar{u}]} f_2^\varepsilon(u)\, d\hat{W}_u$$

are well-defined. These two variables belong to $G(\Gamma)$ by Theorem 3.8. Now

$$\| J_1(f_1, f_2) - J_1(f_1^\varepsilon, f_2^\varepsilon) \| \le \| J_1(f_1, f_2)\, I_{(D_1^\varepsilon)^c} \| + \| (f_2 \circ p)\, I_{[0,\tau_\varepsilon] \times [0,\varepsilon]} \| .$$

The first term on the right-hand side tends to zero as $\varepsilon \to 0$, and

$$\| (f_2 \circ p)\, I_{[0,\tau_\varepsilon] \times [0,\varepsilon]} \|^2 = \tau_\varepsilon \int_0^\varepsilon (f_2 \circ p(0, t_2))^2\, dt_2$$

$$\le \frac{\bar{u}}{s_1^0} \int_{[0, s_1^0] \times [0,\varepsilon]} (f_2 \circ p(t))^2\, dt$$

$$\le \frac{\bar{u}}{s_0^1} \int_{[0, s_1^0] \times [0,\varepsilon]} g^2(t)\, dt$$

(the last inequality uses the same type of argument as in the proof of (5.6) and (5.7)). thus

$$I_1(f_1,f_2) = \lim_{\varepsilon\to 0}\int_T J_1(f_1^\varepsilon,f_2^\varepsilon)(t)\,dW_t = \lim_{\varepsilon\to 0}(Y_1^\varepsilon + Y_2^\varepsilon)$$

$$\in G(\Gamma).$$

□

The following theorem gives an explicit representation for $G(\Gamma)$.

5.5 **Theorem.** (a) $G(\Gamma) = \{I(f_1,f_2) : (f_1,f_2) \in \underline{\underline{L}}\}$

(b) If (f_1,f_2), $(\tilde{f}_1,\tilde{f}_2) \in \underline{\underline{L}}$, then $I(f_1,f_2) = I(\tilde{f}_1,\tilde{f}_2)$ if and only if there is $k \in \mathbf{R}$ such that

$$f_1 = \tilde{f}_1 + k \quad \mu_1\text{-a.s.}$$

$$f_2 I_{p(D_1)} = (\tilde{f}_2 - k) I_{p(D_1)} \quad \mu_2\text{-a.s.}$$

$$f_2 I_{p(D_2)} = \tilde{f}_2 I_{p(D_2)} \quad \mu_2\text{-a.s.}$$

Proof. To prove (a), it is sufficient by Proposition 5.4 (b) to prove that $G(\Gamma) \subset \tilde{H}$, where $\tilde{H} = \{I(f_1,f_2) : (f_1,f_2) \in \underline{\underline{L}}\}$. Since $\tilde{H}$ is a linear subspace containing the generators of $G(\Gamma)$, this inclusion will hold provided $\tilde{H}$ is closed in $L^2(\Omega,\underline{\underline{F}},P)$. So suppose $(Y_n, n \in \mathbf{N})$ is a sequence in $\tilde{H}$ converging in $L^2(\Omega,\underline{\underline{F}},P)$ to Y, and $Y_n = I(f_1^n,f_2^n)$, where $(f_1^n,f_2^n) \in \underline{\underline{L}}$, f_1^n satisfies (6.3), for all $n \in \mathbf{N}$. We must show that $Y = I(f_1,f_2)$ for some $(f_1,f_2) \in \underline{\underline{L}}$. Now convergence of Y_n implies convergence in $L^2(T,dt)$ of $(J(f_1^n,f_2^n), n \in \mathbf{N})$, to $g \in L^2(T,dt)$, say. By Lemma 5.1 (b), this implies $g = g^1 + g^2$, where g^i is the limit in $L^2(T,dt)$ of $(J_i(f_1^n,f_2^n), n \in \mathbf{N})$, and $g^2(t) = h_2(p(t))\, I_{D_2}(t)$, for some Borel $h_2 : [0,\bar{u}] \to \mathbf{R}$.

By Lemma 5.2 (d), $g^1 = J_1(\tilde{f}_1,\tilde{f}_2)$ for some couple $(\tilde{f}_1,\tilde{f}_2)$. We now set

$$f_1 = \tilde{f}_1, \quad f_2 = \tilde{f}_2 I_{p(D_1)} + h_2 I_{p(D_2)}.$$

Clearly, $g = J(f_1,f_2)$.

(b) Suppose $I(f_1,f_2) = I(\tilde{f}_1,\tilde{f}_2)$. Choose $k, k' \in \mathbf{R}$ so that f_1^k and $\tilde{f}_1^{k'}$ both satisfy (5.3). Then $I(f_1^k - \tilde{f}_1^{k'}, f_2^k - \tilde{f}_2^{k'}) = 0$. So we only need to show that if $(f_1,f_2) \in \underline{\underline{L}}$ satisfies (5.3) and $I(f_1,f_2) = 0$, then $f_1 = 0$ μ_1-a.s., $f_2 = 0$ μ_2-a.s.

To begin with, equality (5.1) implies that

$$J_1(f_1, f_2)(t) = 0 \, dt\text{–a.s. and } f_2(p(t)) I_{D_2}(t) = 0 \text{ dt–a.s.}$$

This last is equivalent to $f_2 I_{p(D_2)} = 0$ μ_2-a.s. On the other hand, Lemma 5.2(c) implies $f_1 = 0$ $d\mu_1$–a.s. and $f_2 I_{p(D_1)} = 0$ $d\mu_2$-a.s. Since

$$\mu_2([0, \bar{u}] \setminus (p(D_1) \cup p(D_2)) = 0,$$

the conclusion follows. □

We are now going to give properties of the elements of $G(\Gamma)$ which also belong to $H(\Gamma)$. We will do this only under a (weak) regularity assumption on ϕ. These properties will lead to necessary and sufficient conditions for equality of the germ and sharp fields of Γ.

Let us introduce some notation. Set $\psi_0(u) = u$, and for $k \geq 1$, set

$$\psi_k(u) = \begin{cases} \inf\{v > \psi_{k-1}(u) : \phi(v) = \phi(u)\} \text{ if } \{ \ \} \neq \phi, \\ +\infty \qquad \text{otherwise}, \end{cases}$$

and $\tau(u) = \sup\{k \geq 0 : \psi_k(u) < +\infty\}$. We have the following elementary results.

5.6. **Lemma.**

(a) $\psi_k(u) < \psi_{k+1}(u) < +\infty \Rightarrow \psi_{k+1}(u) \in p(D_1) \cup p(D_2)$.

(b) $\psi_{k+1}(u) = \psi_k \circ \psi_1(u) = \psi_1 \circ \psi_k(u)$.

(c) $\tau(u) < +\infty \Rightarrow \tau(\psi_{k+1}(u)) = \tau(\psi_k(u)) - 1, 0 \leq k < \tau(u)$.

(d) ψ_k is Borel, $k \geq 0$.

(e) τ is Borel.

Proof. (a) Observe that when $\psi_k(u) < \psi_{k+1} < +\infty$, we have either

$$\psi_k(u) < v < \psi_{k+1}(u) \Rightarrow \phi(v) < \phi(u)$$

or

$$\psi_k(u) < v < \psi_{k+1}(u) \Rightarrow \phi(u) > \phi(u)$$

In the first case, $\psi_{k+1}(u) \in p(D_2)$, and in the second $\psi_{k+1}(u) \in p(D_1)$.

(b) and (c) follow clearly from the definitions.

(d) By (b), it suffices to show that ψ_1 is Borel. Fix $u_0 \in [0, \bar{u}]$. For $u \leq u_0$, set

$$k^1(u) = \sup\{v \leq u_0 : l(v, u_0) = l(u, u_0)\},$$

$$k^2(u) = \sup\{v \le u_0 : L(v,u_0) = L(u,u_0)\}.$$

Then k^1 and k^2 are right-continuous, hence Borel, and

$$\{u \in [0,\bar{u}] : \psi_1(u) \ge u_0\} = \{u \le u_0 : k^1(u) = u\} \cup \{u \le u_0 : k^2(u) = u\}.$$

This implies that ψ_1 is Borel.

Property (e) is a direct consequence of (d). □

Throughout the remainder of this section, we shall be working under the following assumption on ϕ.

5.7. **Assumption.** $\tau < \infty$ μ_2-a.s.

This assumption is implied by the following (weak) regularity assumption on ϕ.

5.8. **Proposition.** Suppose that *Banach's condition (T1)* holds, that is, for μ_1-almost all $x \in \mathbb{R}_+$,

$$E_x = \{u \in [0,\bar{u}] : \phi(u) = x\}$$

is finite (see [S; Chap. IX. §6]). Then Assumption 5.7 holds.

Proof. The proof relies on the following fact: let $(O^k, k \in \mathbb{N})$ be a non-increasing sequence of subsets of $\mathbb{R}_+$, and set

$$O = \bigcap_{k \in \mathbb{N}} O^k.$$

Then

$$p((D_1 \cup D_2) \cap (\mathbb{R}_+ \times O)) = \bigcap_{k \in \mathbb{N}} p((D_1 \cup D_2) \cap (\mathbb{R}_+ \times O^k)).$$

Indeed, "$\subset$" is obvious. To see "$\supset$", suppose $u \in [0,\bar{u}]$ satisfies

$$\forall k \in \mathbb{N}, \exists\, t^k \in (D_1 \cup D_2) \cap (\mathbb{R}_+ \times O^k) : p(t^k) = u.$$

By definition of p, there is $t_2 \in \mathbb{R}_+$ such that $t_2^k = t_2$, for all $k \in \mathbb{N}$. Thus

$$(t_1^1, t_2) \in (D_1 \cup D_2) \cap (\mathbb{R}_+ \times O), \text{ and } p(t_1^1, t_2) = u$$

yielding the desired conclusion.

Now to prove the lemma, let $N \subset \mathbb{R}_+$ be a μ_1-null set such that E_x is finite when $x \notin N$, and let $(O^k, k \in \mathbb{N})$ be a non-increasing sequence of open sets containing N such that $\mu_1(O \setminus N) = 0$, where

$$\mathbf{O} = \bigcap_{k \in \mathbb{N}} \mathbf{O}^k .$$

By the fact proved above and Lemma 3.2 (b),

$$R = p((D_1 \cup D_2) \cap (\mathbb{R}_+ \times \mathbf{O}))$$

is a Borel set, and clearly,

$$p^{-1}(R) = (D_1 \cup D_2) \cap (\mathbb{R}_+ \times \mathbf{O}) .$$

Observe that if $u \in p(D_1 \cup D_2) \setminus R$, then $\tau(u)$ is finite. Since $\mu_2((p(D_1 \cup D_2))^c) = 0$ and

$$\mu_2(R) = \int_{\mathbb{R}_+^2} I_{(D_1 \cup D_2) \cap (\mathbb{R}_+ \times \mathbf{O})}(t)\, dt = 0 ,$$

the proof is complete. □

Proposition 5.8 gives a large class of functions for which Assumption 5.7 holds. Indeed, necessary and sufficient conditions for Banach's condition (T1) to hold are given in [S; Chap. IX. Th. (6.2)]. In particular, (T1) holds if ϕ has bounded variation [S; Chap. IX. Th. (6.3)]. Condition (T1) is also satisfied by functions which are smooth except at finitely many points $u_1, \ldots, u_k$, in neighborhoods of which they resemble $c + (u - u_i) \sin(1/(u - u_i))$ or some other oscillating function. Finally, observe that $\tau(u) = +\infty$ if for some $\varepsilon > 0$, ϕ is constant on $[u, u + \varepsilon]$. However, in this case, $\mu_2([u, u+\varepsilon]) = 0$, so the presence of such horizontal segments on Γ does not affect Assumption 5.7.

An element u_0 of $[0, \bar{u}]$ is a *strict local maximum* (respectively *minimum*) of ϕ provided there is $\delta > 0$ such that $|u - u_0| < \delta$ implies $\phi(u) < \phi(u_0)$ (resp. $\phi(u) > \phi(u_0)$). A *strict local extremum* is a point which is either a strict local maximum or minimum. A *point of increase* of ϕ is an element $u_0 \in [0, \bar{u}]$ such that for some $\varepsilon > 0$,

$$0 < \delta < \varepsilon \;\Rightarrow\; L(u_0 - \varepsilon, u_0 - \delta) < \phi(u_0) < l(u_0 + \delta, u_0 + \varepsilon) .$$

A *point of decrease* is defined analogously.

5.9. **Lemma.** The following statements are implied by Assumption 5.7.

(a) Almost every $u \in [0, \bar{u}]$ (with respect to μ_2) is a point of increase or decrease.

(b) For μ_2-almost every $u \in [0, \bar{u}]$, $\psi_k(u)$ is a point of increase or decrease, for all $0 \le k \le \tau(u)$.

Proof. (a) Under Assumption 5.7, we have

$$\mu_2((p(D_1) \cup p(D_2)) \cap \{\tau < +\infty\}) = \mu_2([0,\bar{u}]).$$

Fix $u \in (p(D_1) \cup p(D_2)) \cap \{\tau < +\infty\}$. Assume to begin with that $u \in p(D_1)$. Then there is $\varepsilon_1 > 0$ such that

$$0 < \delta < \varepsilon_1 \Rightarrow l(u - \varepsilon_1, u - \delta) > \phi(u). \tag{5.12}$$

Now since $\tau(u) < +\infty$, either there is $\varepsilon_2 > 0$ such that

$$0 < \delta < \varepsilon_2 \Rightarrow l(u + \delta, u + \varepsilon_2) > \phi(u) \tag{5.13}$$

or

$$0 < \delta < \varepsilon_2 \Rightarrow L(u + \delta, u + \varepsilon_2) < \phi(u). \tag{5.14}$$

If (5.12) and (5.13) happen, u is a strict local minimum. Since a continuous function has at most countably many strict local minima (see e.g. [S; Chap IX. Th. (1.1)]) and μ_2 is diffuse, (5.14) holds for μ_2-almost all u in $p(D_1)$, i.e. u is a point of decrease μ_2-a.s.

A similar reasoning shows that if $u \in p(D_2) \cap \{\tau < +\infty\}$, then u is μ_2-a.s. a point of increase.

(b) We continue the reasoning begun in part (a). Set $Q = (p(D_1) \cup p(D_2)) \cap \{u : u \text{ is a strict local extremum}\}$. Then Q is a countable set, and so $\phi(Q)$ is too. In particular, $[0,\bar{u}] \times \phi(Q)$ has two-dimensional Lebesgue-measure zero. Now

$$\{u \in (p(D_1) \cup p(D_2)) \cap \{\tau < +\infty\} : \psi_k(u) \text{ is not a point of increase or decrease, for some } k \le \tau(u)\}$$

is contained in $p([0,\bar{u}] \times \phi(Q))$. Since

$$\mu_2\{p(t) : t_2 \in \phi(Q)\} \le \int_{[0,\bar{u}]\times\phi(Q)} dt = 0,$$

the lemma is proved. □

5.10. **Lemma.** (a) Suppose $N \subset p(D_1) \cup p(D_2)$. Then

$$\mu_2(N) = 0 \iff \phi(N) \text{ is a } \mu_1\text{–null set}.$$

(b) Let $N \subset p(D_1) \cup p(D_2)$ be a μ_2-null set. Then

$$\mu_2\{u \in [0,\bar{u}] : \text{there is } k \in \mathbb{N} \text{ s.t. } \psi_k(u) \in N\} = 0.$$

Proof. (a) By definition,

$$\mu_2(N) = \int_{D_1 \cup D_2} I_{p^{-1}(N)}(t)\,dt = \int_0^{L(0,\bar{u})} dt_2 \int_0^{k(t_2)} dt_1\, I_{p^{-1}(N)}(t_1,t_2),$$

where $k(t_2) = \inf\{u : L(u,\bar{u}) = t_2\}$. Now since $N \subset p(D_1) \cup p(D_2)$, the intersection of $p^{-1}(N)$ with any horizontal line $t_2 = x$ is either the empty set if $x \notin \phi(N)$, or a countable union of horizontal segments if $x \in \phi(N)$. In particular,

$$\mu_2(N) = \int_0^{L(0,\bar{u})} dt_2\, l(t_2),$$

where $l(t_2) > 0 \Longleftrightarrow t_2 \in \phi(N)$. This clearly implies (a).

(b) Observe that $\{u : \text{there is } k \in \mathbb{N} \text{ s.t. } \psi_k(u) \in N\} \subset \phi(N)$, so the conclusion follows from (a). □

It will be convenient to number elements of a given level set from right to left instead of from left to right. On $\{\tau < +\infty\}$, set

$$\tilde{\psi}_i(u) = \psi_{\tau(u)-i}(u), \quad 0 \le i \le \tau(u).$$

Provided $\tilde{\psi}_k(u)$ is a point of increase or decrease, for $0 \le k \le \tau(u)$, we have $\tilde{\psi}_{2i}(u) \in p(D_1)$ if $2i \le \tau(u)$ and $\tilde{\psi}_{2i+1} \in p(D_2)$ if $2i+1 \le \tau(u)$.

If $f : [0,\bar{u}] \to \mathbb{R}$ is Borel, we set $f(+\infty) = 0$, and define two Borel functions $\alpha_1(f)$, $\alpha_2(f) : [0,\bar{u}] \to \mathbb{R}$ by

$$\alpha_1(f)(u) = f(u),$$

$$\alpha_2(f)(u) = \Big[\sum_{0 \le 2i+1 \le \tau(u)} f(\tilde{\psi}_{2i+1}(u)) - \sum_{0 \le 2i \le \tau(u)} f(\tilde{\psi}_{2i}(u))\Big]\, I_{\{\tau<+\infty\}}(u),$$

and set $K(f) = J(\alpha_1(f), \alpha_2(f))$, $L(f) = I(\alpha_1(f), \alpha_2(f))$ if $K(f) \in L^2(T, dt)$.

Fix $a,b \in \mathbb{R}$, and suppose $f_1, f_2 : [0,\bar{u}] \to \mathbb{R}$ are Borel.

5.11. **Lemma.** (a) We have $K(af_1 + bf_2) = aK(f_1) + bK(f_2)$ and $K(f_1 + a) = K(f_1)$.

(b) Assume in addition that $(\alpha_1(f_i), \alpha_2(f_i)) \in L_1 \times L_2$, $i = 1,2$. Then $L(f_1) = L(f_2)$ if and only if there is $k \in \mathbb{R}$ such that $f_1 = f_2 + k$ μ_1-a.s. and μ_2-a.s.

Proof. The first statement in (a) holds because α_1 and α_2 depend linearly on f and the definition of J implies

$$J(a\alpha_1(f_1) + b\alpha_1(f_2),\, a\alpha_2(f_1) + b\alpha_2(f_2)) = aJ(\alpha_1(f_1), \alpha_2(f_1)) + bJ(\alpha_1(f_2), \alpha_2(f_2)).$$

To prove the second statement in (a), note that

$$\tau(u) = 0 \Rightarrow \alpha_2(f_1 + a)(u) = \alpha_2(f_1)(u) - a,$$

$$\tau(u) = 1 \Rightarrow \alpha_2(f_1 + a)(u) = \alpha_2(f_1)(u),$$

and by induction, we get for $u \in \{\tau < +\infty\}$ that

$$u \in p(D_1) \Rightarrow \alpha_2(f_1 + a)(u) = \alpha_2(f_1)(u) - a$$

$$u \in p(D_2) \Rightarrow \alpha_2(f_1 + a)(u) = \alpha_2(f_1)(u).$$

But since $\alpha_1(f_1 + a) = \alpha_1(f_1) + a$, Theorem 5.5 (b) implies that $K(f_1 + a) = K(f_1)$.

(b) Suppose $f_1 = f_2 + k$ μ_1- and μ_2-a.s. Then $\alpha_1(f_1) = \alpha_1(f_2 + k)$ μ_1-a.s. and $\alpha_2(f_1) = \alpha_2(f_2 + k)$ μ_2-a.s., so by (a),

$$L(f_1) = L(f_2 + k) = L(f_2).$$

To show the converse, assume $L(f_1) = L(f_2)$. We then use Theorem 5.5 (b) to get $k \in \mathbf{R}$ such that

$$\alpha_1(f_1) = \alpha_1(f_2) + k \quad \mu_1 \text{ a.s.},$$

$$(5.15) \quad \alpha_2(f_1) I_{p(D_1)} = (\alpha_2(f_2) - k) I_{p(D_1)} \quad \mu_2-\text{a.s.},$$

$$(5.16) \quad \alpha_2(f_1) I_{p(D_2)} = \alpha_2(f_2) I_{p(D_2)} \quad \mu_2-\text{a.s.}$$

Let N be a μ_2-null set outside of which (5.15) and (5.16) hold. By Lemma 5.10 (b), we may assume that

$$\tau(u) < +\infty \Rightarrow \psi_k(u) \notin N \cap (p(D_1) \cup p(D_2)), \ 0 \le k \le \tau(u).$$

But then, when $u \notin N$ and $\tau(u) < +\infty$, $\tilde{\psi}_0(u) \in p(D_1)$, so by (5.15),

$$-f_1(\tilde{\psi}_0(u)) I_{p(D_1)}(u) = (\alpha_2(f_1)(\tilde{\psi}_0(u)) - k) I_{p(D_1)}(u) = -f_2(\tilde{\psi}_0(u)) - k.$$

We then use (5.16) and the fact that $\tilde{\psi}_1(u) \in p(D_2)$ to get $f_1(\tilde{\psi}_1(u)) = f_2(\tilde{\psi}_1(u)) + k$ (if $\tau(u) \ge 1$). By induction, we get for μ_2-almost all u

$$f_1(\tilde{\psi}_l(u)) \doteq f_2(\tilde{\psi}_l(u)) + k, \ 0 \le l \le \tau(u).$$

Setting $l = \tau(u)$ gives the desired result. □

Set $\tilde{H}(\Gamma) = \{Y \in G(\Gamma)$: there is f Borel such that $Y = L(f)\}$.

The following lemma gives, under Assumption 5.7, a characterization of the elements of Y of $G(\Gamma)$ which belong to $\tilde{H}(\Gamma)$. It is a generalization of [DR; Lemma 3.11].

5.12. **Proposition** Let $(f_1, f_2) \in \underline{\underline{L}}$. Then $Y = I(f_1, f_2) \in \tilde{H}(\Gamma)$ if and only if there is a

μ_i-null set N_i, $i = 1,2$, such that if $u \in [0,\bar{u}]\setminus(N_1 \cup N_2)$, then

$$f_1(u) I_{p(D_1)}(u) = (-f_2(u) + f_2(\psi_1(u))) I_{p(D_1)}(u), \tag{5.17}$$

and

$$f_1(u) I_{p(D_2)}(u) = (f_2(u) - f_2(\psi_1(u))) I_{p(D_2)}(u). \tag{5.18}$$

Proof. If $Y = I(f_1, f_2) = L(f)$ for some Borel $f : [0,\bar{u}] \to \mathbb{R}$, then (5.17) and (5.18) follows from Theorem 5.5 (b) and the definitions of $\alpha_1(f)$ and $\alpha_2(f)$. Now suppose (5.17) and (5.18) hold. We define $f : [0,\bar{u}] \to \mathbb{R}$ successively on the sets $\{\tau = 0\}$, $\{\tau = 1\}$,... as follows. First, we define f on the set $\{\tau = 0\}$ by setting

$$f I_{\{\tau=0\}} = (f_1 I_{N_1^c} - f_2 I_{N_1}) I_{\{\tau=0\}},$$

and then, assuming by induction that f has been defined on $\{\tau = k-1\}$, we set

$$f I_{\{\tau=k\}} = (f_1 I_{N_1^c} + (f_2 - f_2 \circ \psi_1) I_{N_1}) I_{\{\tau=k\}} \tag{5.19}$$

if k is odd, and

$$f I_{\{\tau=k\}} = (f_1 I_{N_1^c} + (-f_2 + f_2 \circ \psi_1) I_{N_1}) I_{\{\tau=k\}} \tag{5.20}$$

if k is even. This defines f on $\{\tau < +\infty\}$, and we set

$$f I_{\{\tau=+\infty\}} = f_1 I_{\{\tau=+\infty\}}.$$

Clearly, $f_1(u) = \alpha_1(f)(u)$, $u \notin N_1$.

By Lemma 5.10 (b), let N_2' be a μ_2-null set such that $u \notin N_2'$ implies $\psi_k(u) \notin N_2 \cap (p(D_1) \cup p(D_2))$, $0 \le k \le \tau(u)$. Clearly $N_2' \supset N_2$, and if $u \notin N_2'$, we have by (5.17)

$$\alpha_2(f)(u) I_{\{\tau=0\}} = -f(u) I_{\{\tau=0\}}(u) = -(f_1(u) I_{N_1^c}(u) - f_2(u) I_{N_1}(u)) I_{\{\tau=0\}}$$
$$= f_2(u) I_{\{\tau=0\}}.$$

Similarly, equations (5.19) and (5.20) become respectively

$$f I_{\{\tau=k\}\setminus N_2'} = (f_2 - f_2 \circ \psi_1) I_{\{\tau=k\}\setminus N_2'}, \quad k \text{ odd},$$

$$f I_{\{\tau=k\}\setminus N_2'} = (-f_2 + f_2 \circ \psi_1) I_{\{\tau=k\}\setminus N_2'}, \quad k \text{ even}.$$

Proceeding by induction on k, one checks that

$$\alpha_2(f)(u) = f_2(u), \quad u \in \{\tau < +\infty\} \setminus N_2'.$$

Since $\mu_2(\{\tau < +\infty\} \setminus N_2') = \mu_2([0,\bar{u}])$, we have shown that $\alpha_i(f) = f_i$ μ_i-a.s., $i = 1,2$. Thus

$$Y = I(f_1, f_2) = L(f),$$

concluding the proof. □

5.13. **Lemma.** (a) Fix $0 \le u_0 \le \bar{u}$. If $f = I_{[0,u_0]}$, then $L(f) = W_{u_0,\phi(u_0)}$.

(b) $\tilde{H}(\Gamma)$ is a closed linear subspace of $G(\Gamma)$.

Proof. (a) To begin with, we have in this case $\alpha_1(f) = I_{[0,u_0]}$, and clearly $\alpha_2(f)(u) = 0$ if $u > u_0$. In addition, we will check by induction that μ_2-a.s., if $\phi(u) > \phi(u_0)$ then

$$u \in p(D_1) \Rightarrow \alpha_2(f)(u) = -1,$$

$$u \in p(D_2) \Rightarrow \alpha_2(f)(u) = 0,$$

whereas if $\phi(u) < \phi(u_0)$, then

$$u \in p(D_1) \Rightarrow \alpha_2(f)(u) = 0,$$

$$u \in p(D_2) \Rightarrow \alpha_2(f)(u) = 1.$$

Indeed, if $\tau(u) < +\infty$, let $u = v_0 < v_1 < \cdots < v_k < u_0$ be such that

$$\{v_0, \ldots, v_k\} = \{v \in [u, u_0] : \phi(v) = \phi(u)\}.$$

Now if $u \in p(D_1)$, u is a point of decrease of ϕ, and so when $\phi(u) > \phi(u_0)$, k must be even. On the other hand, if $\phi(u) < \phi(u_0)$, then k will be odd. Together with a similar argument for $p(D_2)$, all four implications above follow from the definition of $\alpha_2(f)$.

It is then easy to see that

$$K(f) = J(\alpha_1(f), \alpha_2(f)) = I_{R_{u_0,\phi(u_0)}} \text{ dt-a.s.},$$

and thus $I(f) = W_{u_0,\phi(u_0)}$.

(b) Let $(Y^n, n \in \mathbb{N})$ be a sequence of elements of $\tilde{H}(\Gamma)$ converging to Y in $L^2(\Omega, \underline{\underline{F}}, P)$. Since $G(\Gamma)$ is closed, $Y \in G(\Gamma)$. By Theorem 5.5, there is $(f_1^n, f_2^n) \in \underline{\underline{L}}$ and $(f_1, f_2) \in \underline{\underline{L}}$ such that $Y^n = I(f_1^n, f_2^n)$, $n \in \mathbb{N}$, and $Y = I(f_1, f_2)$. We may of course assume that f_1^n and f_1 satisfy (5.3). But then Lemmas 5.1(b) and 5.2(d) imply that for each $\varepsilon > 0$,

$$f_1 I_{[0,\tau_\varepsilon]} = \lim_{n\to\infty} f_1^n I_{[0,\tau_\varepsilon]} \text{ in } L^2([0,\bar{u}], d\mu_1),$$

$$f_2 I_{[0,\tau_\varepsilon]} = \lim_{n\to\infty} f_2^n I_{[0,\tau_\varepsilon]} \text{ in } L^2([0,\bar{u}], d\mu_2).$$

Using a diagonal subsequence argument, we get (Borel) sets M_1 and M_2 with

$\mu_i(M_i) = 0$, $i = 1,2$, and subsequences $(f_1^{n_k}, k \in \mathbb{N})$ and $(f_2^{n_k}, k \in \mathbb{N})$ such that

$$(5.21) \qquad f_i(u) = \lim_{n\to\infty} f_i^{n_k}(u), \quad u \notin M_i,\ i = 1,2.$$

Since $Y^{n_k} \in \tilde{H}(\Gamma)$, Proposition 5.12 affirms the existence of μ_i-null sets N_i^k, $i = 1,2$, such that if $u \in [0,\bar{u}] \setminus (N_1^k \cup N_2^k)$,

$$(5.22) \qquad \begin{aligned} f_1^{n_k}(u)\, I_{p(D_1)}(u) &= (-f_2^{n_k}(u) + f_2^{n_k}(\psi_1(u)))\, I_{p(D_1)}(u), \\ f_2^{n_k}(u)\, I_{p(D_2)}(u) &= (f_2^{n_k}(u) - f_2^{n_k}(\psi_1(u)))\, I_{p(D_2)}(u). \end{aligned}$$

Now set

$$N_1' = (\bigcup_{k\in\mathbb{N}} N_1^k) \cup M_1, \quad N_2' = (\bigcup_{k\in\mathbb{N}} N_2^k) \cup M_2.$$

By Lemma 5.10 (b), there is a μ_2-null set $N_2'' \supset N_2'$ such that $u \notin N_2'' \Rightarrow \psi_1(u) \notin N_2' \cap (p(D_1) \cup p(D_2))$. Finally, we set

$$N_2 = N_2'' \cup (p(D_1) \cup p(D_2))^c \cup \{\tau = +\infty\},$$

which is a μ_2-null set, such that

$$u \notin N_2 \Rightarrow (u \notin N_2^k \cup M_2 \text{ and } \psi_1(u) \notin N_2^k \cup M_2).$$

We can now pass to the limit in (5.22) using (5.21), to see that if $u \in [0,\bar{u}] \setminus (N_1 \cup N_2)$, then

$$\begin{aligned} f_1(u)\, I_{p(D_1)}(u) &= (-f_2(u) + f_2(\psi_1(u)))\, I_{p(D_1)}(u), \\ f_2(u)\, I_{p(D_2)}(u) &= (f_2(u) + f_2(\psi_1(u)))\, I_{p(D_2)}(u). \end{aligned}$$

By Proposition 5.12 this means that $Y \in \tilde{H}(\Gamma)$, concluding the proof. □

The following theorem is a generalization of [DR; Theorem 3.12], which contains the case "ϕ monotone non-increasing".

5.14. **Theorem.** Suppose ϕ satisfies Assumption 5.7. Then the following conditions are equivalent:

(a) $\underline{\underline{H}}(\Gamma) = \underline{\underline{G}}(\Gamma)$;

(b) $W(D_1)$ is $\underline{\underline{H}}(\Gamma)$-measurable:

(c) μ_1 and μ_2 are mutually singular.

Proof. *(a) ⇒ (b).* This follows from Proposition 3.1 and the fact that $W(D_1) = W_{\bar{u}}^1$.

(b) ⇒ (c). Note that $W(D_1) = I(f_1, f_2)$ where $f_1 \equiv 1$ and $f_2 \equiv 0$. Now if $W(D_1)$ belongs to $\underline{\underline{H}}(\Gamma)$, it will in fact belong to $H(\Gamma)$. In particular, by Lemma 5.13, $W(D_1)$ would belong to $\tilde{H}(\Gamma)$. But then Proposition 5.12 would yield μ_i-null sets N_i, $i = 1,2$ such that

$$u \in [0,\bar{u}] \setminus (N_1 \cup N_2) \Rightarrow 1\mathrm{I}_{p(D_1) \cup p(D_2)} = 0.$$

This means that $p(D_1) \cup p(D_2) \subset N_1 \cup N_2$. Since

$$\mu_2([0,\bar{u}] \setminus (p(D_1) \cup p(D_2))) = 0,$$

we can assume that

$$[0,\bar{u}] \setminus (p(D_1) \cup p(D_2)) \subset N_2.$$

But then $[0,\bar{u}] = N_1 \cup N_2$, with $\mu_i(N_i) = 0$, $i = 1,2$, so μ_1 and μ_2 are mutually singular.

(c) ⇒ (a). By Remark 4.11, it suffices to show that

$$u \mapsto L(u_0, u) \text{ and } u \mapsto l(u_0, u)$$

are singular functions, for all $u_0 \in [0,\bar{u}]$. But this is a consequence of Proposition 6.1 (a) below. □

5.15. **Corollary.** Assume ϕ has bounded variation. Then $\underline{\underline{H}}(\Gamma) = \underline{\underline{G}}(\Gamma)$ if and only if ϕ is singular with respect to Lebesgue measure.

Proof. This is an immediate consequence of Theorem 5.14 and Proposition 6.7 (b) below. □

6. Conditions under which μ_2 is singular with respect to Lebesgue-measure.

In view of Theorem 5.14, it becomes of interest to determine for which functions ϕ the measures μ_1 and μ_2 are mutually singular. The following proposition gives a complete answer when ϕ has bounded variation, and, in the general case, relates the property to singularity of $l(u_0, \cdot)$ and $L(u_0, \cdot)$, $u_0 \in [0,\bar{u}]$.

6.1. **Proposition.**

(a) For an arbitrary continuous function ϕ, the following two conditions are equivalent.

(6.1) There is a countable dense set $Q \subset [0, u_0]$ such that $l(u_0, \cdot)$ and $L(u_0, \cdot)$ are singular (with respect to Lebesgue measure), $\forall u_0 \in Q$.

(6.2) μ_2 is singular with respect to Lebesgue measure.

(b) If ϕ has bounded variation, then (6.2) holds if and only if ϕ is a singular function.

Proof. *(6.1) ⇒ (6.2).* Observe that there is a sequence of rectangles ($R_k = [a^k, b^k] \times [c^k, d^k]$, $k \in \mathbb{N}$), where each R_k is contained either in D_1 or in D_2, such that

$$E_k \cap E_l \;=\; \emptyset \text{ a.s.}, \; k \neq l, \quad \text{and} \quad \bigcup_k E_k \;=\; D_1 \cup D_2,$$

where $E_k = (D_1 \cup D_2) \cap p^{-1}(p(R_k))$. It is thus sufficient to show that for a fixed rectangle $R = [a, b] \times [c, d]$ ($a < b, c < d$), the restriction of μ_2 to the Borel set $p(R)$ is singular with respect to μ_1. We only examine the case $R \subset D_1$, since the case $R \subset D_2$ is similar.

Let $u_0 \in Q \cap [a, b]$, and set $f(u) = l(u_0, u)$, $u \geq u_0$. By (6.1), there is a Borel set $N \subset p(R)$ such that $\mu_1(N) = 0$ and $p(R) \setminus N$ is a df-null set. By the above considerations, the proof will be complete provided we show that $\mu_2(p(R) \setminus N) = 0$.

For $c \leq t_2 \leq d$, set $l(t_2) = p(u_0, t_2) - q(u_0, t_2)$, where

$$q(u_0, t_2) \;=\; \begin{cases} \sup\{u \leq u_0 : \phi(u) = t_2\} & \text{if } \{\quad\} \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases}$$

We then have (recall that $\hat{f}^{-1}$ is defined prior to Lemma 4.1)

$$\begin{aligned} \mu_2(p(R) \setminus N) \;&=\; \int_{\mathbb{R}_+^2} I_{p^{-1}(p(R)\setminus N)}(t)\, dt \\ &=\; \int_c^d dt_2\, l(t_2)\, I_{\{\hat{f}^{-1} \notin N\}}(t_2) \\ &\leq\; \int_c^d dt_2\, \hat{f}^{-1}(t_2)\, I_{N^c}(\hat{f}^{-1}(t_2)) \\ &=\; \int_{\hat{f}^{-1}(c)}^{\hat{f}^{-1}(d)} u\, I_{N^c}(u)\, df(u) \\ &=\; 0 \end{aligned}$$

(the last equality uses the change of variables formula of [DM; Chap. VI. 2, (55.1)]). This completes the proof of (6.1) ⇒ (6.2).

(6.2) ⟹ (6.1). We shall only prove that $f(u) = l(u_0, u)$ is singular with respect to μ_1, for all $u_0 \in [0, \bar{u}]$, since the other case is similar.

Since ϕ is continuous at u_0, for any fixed $\varepsilon > 0$, there exist a, b and d such that

$$a < u_0 < b,\ \phi(u_0) - \varepsilon < d < \phi(u_0)$$

and $R = [a, b] \times [0, d] \subset D_1$. By (6.2), there is a Borel set N with Lebesgue measure 1 such that $\mu_2(N) = 0$. Since

$$l(t_2) \setminus \hat{f}^{-1}(t_2) \ \geq\ (b - a) \setminus \bar{u},\ 0 \leq t_2 \leq d,$$

we get

$$\begin{aligned} 0 &= \mu_2(p(R) \cap N) \\ &= \int_{D_1} I_{p^{-1}(p(R) \cap N)}(t)\, dt \\ &= \int_0^d l(t_2)\, I_{p^{-1}(p(R) \cap N)}(u_0, t_2)\, dt_2 \\ &\geq \frac{b-a}{\bar{u}} \int_0^d \hat{f}^{-1}(t_2)\, I_N(\hat{f}^{-1}(t_2))\, dt_2 \\ &= \frac{b-a}{\bar{u}} \int_{\bar{u}}^{\hat{f}^{-1}(d)} u\, I_N(u)\, df(u). \end{aligned}$$

Thus $N \cap [f^{-1}(d), \bar{u}]$ is a df-null set. Since ε was arbitrary, $N \cap [u_0, \bar{u}]$ is a df-null set. But this implies that f is singular with respect to μ_1, concluding the proof of (6.2) ⟹ (6.1).

We now turn to the proof of (b). To begin with, assume ϕ is singular. By (a), it is sufficient to show that $l(u_0, \cdot)$ and $L(u_0, \cdot)$ are singular, for all $u_0 \in [0, \bar{u}]$. In Example 4.10 (a), this was done for $L(u_0, \cdot)$, and the other case is similar.

We now show that if (6.2) holds, then ϕ is singular. Since ϕ has bounded variation, its derivative ϕ' exists almost everywhere [S; Chap. IV. Th. (9.1)] (ϕ' may take on the values $+\infty$ and $-\infty$). In order to check that ϕ is singular, it suffices to show that $\phi' = 0$ μ_1-a.s. [S; Chap. IV. Th. (7.8)].

Set $N = \{u \in [0, \bar{u}]:\ \phi'(u)$ exists and $\phi'(u) \neq 0\}$. We assume $\mu_1(N) > 0$ and show that this leads to a contradiction. Now $N = N_1 \cup N_2$, where

$$N_1 = \{u \in [0, \bar{u}] : \phi'(u) \text{ exists and } -\infty \leq \phi'(u) < 0\}$$

and $N_2 = N \setminus N_1$. So we may as well assume $\mu_1(N_1) > 0$. Now observe that $N_1 \subset p(D_1)$. To derive a contradiction, we are going to show that for some

$u_0 \in [0, \bar{u}]$, $f(\cdot) = l(u_0, \cdot)$ satisfies $f' = \phi'$ on a subset of N_1 with positive Lebesgue measure.

Since $p(D_1)$ is a countable union of sets of the form $p(R_k)$, where R_k is a rectangle contained in D_1, there is a rectangle $R \subset D_1$ such that $\mu_1(N_1 \cap p(R)) > 0$. Let $a < b$, $c < d$ be such that $R = [a, b] \times [c, d]$, and set $u_0 = b$, $f(u) = l(u_0, u)$, $u \geq u_0$. Fix $u > u_0$, $u \in N_1 \cap p(R)$. We show that $f'(u) = \phi'(u)$. Now since $\phi'(u) < 0$ and $u \in p(R)$, we must have $f(u) = \phi(u)$. But then, for sufficiently small $h > 0$, we have

$$D(f, h) = \frac{f(u+h) - f(u)}{h} \leq \frac{\phi(u+h) - \phi(u)}{h},$$

so

$$\limsup_{h \downarrow 0} D(f, h) \leq \phi'(u). \tag{6.3}$$

Now fix $\varepsilon > 0$. Then for sufficiently small $h > 0$,

$$\phi(u+h) \geq \phi(u) + h(\phi'(u) - \varepsilon).$$

The right-hand side of this inequality is a decreasing function of h, so

$$f(u+h) \geq f(u) + h(\phi'(u) - \varepsilon).$$

But then

$$\liminf_{h \downarrow 0} D(f, h) \geq \phi'(u) - \varepsilon.$$

Since ε is arbitrary, we combine this inequality with (6.3) to get

$$\lim_{h \downarrow 0} D(f, h) = \phi'(u).$$

A similar argument for $h \uparrow 0$ gives $f'(u) = \phi'(u) < 0$.

Thus $f' < 0$ on a set of positive Lebesgue measure, so f cannot be singular with respect to μ_1. By part (a), this contradicts the assumption that (6.2) holds, and the proof is complete. □

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