On the Uniform Consistency of Bayes Estimates for Multinomial Probabilities

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Technical Report No 137
May 30, 1988

[1] Research partially supported by NSF Grant DMS 86-00235
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Abstract

A k-sided die is thrown n times, to estimate the probabilities $\theta_1,\ldots,\theta_k$ of landing on the various sides. The MLE for θ is the vector of empirical proportions $p=(p_1,\ldots,p_k)$. Consider a set of Bayesians who put uniformly positive prior mass on all reasonable subsets of the parameter space. Their posterior distributions will be uniformly concentrated near p. Sharp bounds are given, using entropy. These bounds apply to all sample sequences: there are no exceptional null sets.

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1. Introduction

This paper is about the consistency of Bayes estimates. The usual statement is that for almost all sample sequences, as the sample size goes to infinity the posterior distribution piles up near the true value of the parameter. The objective is to reformulate this idea as a finite-sample result, without exceptional null sets or "true values" of parameters.

We begin with coin tossing, and develop an explicit inequality which shows that the posterior must concentrate near the observed fraction of heads. The inequality replaces the asymptotics and eliminates the null set; the observed fraction stands in for the true parameter.

To be a little more specific, suppose there are j heads in n tosses of a coin. Consider the posterior odds ratio for a parameter interval of fixed length centered at j/n. The posterior odds are bounded below by ab^n, where $a>0$ and $b>1$ are computable constants. So the odds go to infinity at an exponential rate.

If the prior assigns measure 0 to an interval, so will the posterior. Even if the prior assigns small positive mass to the interval, it may take a long time for the data to swamp the prior. The inequality must therefore take into account the degree to which the prior covers the parameter space.

The notion of "ϕ-positivity" is introduced, to measure coverage; ϕ is a positive function on $(0,1)$. A prior μ is said to be ϕ-positive if μ assigns mass $\phi(h)$ or more to every closed interval of length h in $[0,1]$. For example, if $\phi(h)= .1h$, then μ is ϕ-positive if and only if μ is bounded below by .1 x Lebesgue measure, setwise. Priors with densities which have zeros-- like betas-- can be handled using more complicated ϕ's; so can singular priors.

The inequality on the posterior odds ratio holds uniformly in ϕ-positive priors μ, and uniformly in the fraction j/n of heads. For any parameter interval $(j/n-h, j/n+h)$, the posterior odds ratio is bounded below by

$$\psi(h,\epsilon)\, e^{n(1-\epsilon)g(h)} \tag{1.1}$$

Here, $\epsilon>0$ is a nuisance of rigor; $\psi(h,\epsilon)>0$ is computed from ϕ and does not otherwise depend on the prior; $g(h)>0$ does not depend on the data or the prior.

The rest of this paper is organized as follows: section 2 gives a careful statement of the result for coin tossing; section 3 has a heuristic proof and section 4 the rigor. The extension to the multinomial is in sections 5-6, and the last section discusses the idea of ϕ-positivity.

History. In effect, we will estimate the posterior using the method of LaPlace (1774); he showed that the posterior piles up near the MLE, but only for the uniform prior. (An easy modern proof uses Chebychev's inequality, but that was not available to LaPlace.) Some modern references on the consistency of Bayes estimates include LeCam (1953), LeCam and Schwartz (1960), Schwartz (1965), Freedman (1963), Diaconis and Freedman (1986). Edwards-Lindman-Savage (1963) must be cited too; their idea was that the data eventually swamps a non-dogmatic prior-- the principle of stable estimation (pp201-8).

A closely related development is the asymptotic normality of the posterior, which is often called the Bernstein-von Mises theorem-- although LaPlace got there first; references include Johnson (1967, 1970), Ghosh-Sinha-Joshi (1982), LeCam (1986, secs 12.3, 12.4, 17.7).

2. The theorem for coin tossing

Let ϕ be a positive function on $(0,1)$. A prior probability μ on $[0,1]$ is "ϕ-positive" if $\mu[p,p+h]\geq\phi(h)$ for all p and h with $0\leq p<p+h\leq 1$.

Let H be the relative entropy function:

$$(2.1) \qquad H(p,\theta) = -p \log \theta - (1-p) \log (1-\theta)$$

Here, $p=j/n$ is the relative frequency of heads, and θ is the parameter-- the probability of heads. (The prior is a distribution over θ.) As is well known,

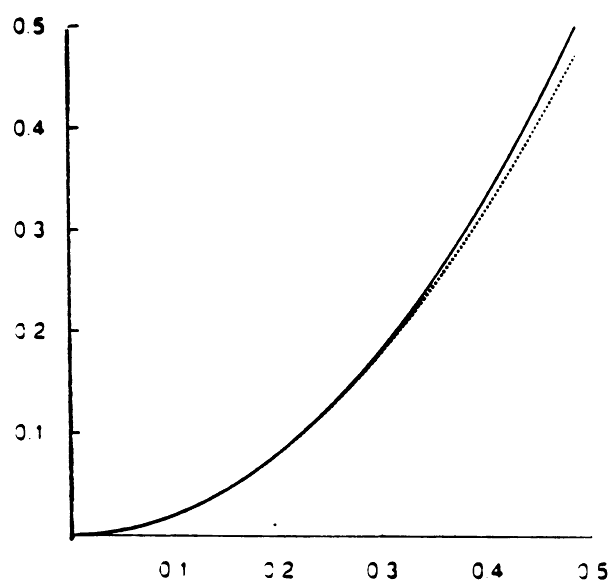
(2.2) $H(p,\bullet)$ is strictly convex, with a strict minimum at p.

For $0<h<{}^1/_2$, let

$$(2.3) \qquad g(h) = \inf_{p,\theta}\{H(p,\theta)-H(p,p): |\theta-p|\geq h\}$$

We will show later that $g(h)>0$, and the inf is attained. Clearly, g is monotone increasing. Its graph is shown in Figure 1; for details, see (5.12-19) below. Although g is defined on $(0,{}^1/_2)$, most of our results are proved only for $(0,{}^1/_4)$.

Figure 1. A graph of g(h), which is convex and increasing; $g(h) > 2h^2$, which is plotted for reference as a dashed line. The two curves are rather close.



To state the main result, suppose a coin is tossed n times, and $p=j/n$ is the fraction of heads. Let $0<h<1/2$. Let $R(n,p,h)$ be the posterior odds ratio for the inside of the parameter interval $(p-h,p+h)$ versus the outside, with respect to a ϕ-positive prior: the outside of the parameter interval is nonempty, because $h<1/2$. Let $0<\epsilon<1$. There is a $\psi(h,\epsilon)>0$, which depends on ϕ, h, and ϵ but not on n or p, such that the following inequality holds.

(2.4) **Theorem.** $R(n,p,h) \geq \psi(h,\epsilon)\, e^{n(1-\epsilon)g(h)}$ for $0<h<1/4$

The first factor on the right does not depend on the data. It depends on the prior only through ϕ; it depends on h and ϵ. The second factor depends on h and ϵ too; but it depends on the data only through the sample size n. In particular, p is not involved on the right. The bound grows exponentially fast as $n\to\infty$. As it turns out, $\psi(h,\epsilon)$ is the minimal prior mass in an interval of length about ϵh^2: more rigorously, $\psi(h,\epsilon)=\phi(h^*)$, where $h^*=\min\{1/2\,\epsilon g(h),h\}$.

The unattainable ideal version of the theorem has $\psi(h,\epsilon)$ replaced by $\phi(h)$, and $\epsilon=0$ in the exponent. On the log scale, these blemishes vanish, as the corollary shows.

(2.5) Corollary. $\liminf_{n\to\infty} \inf_{p,\mu} \frac{1}{n} \log R(n,p,h) \geq g(h)$

In (2.5), the prior μ is restricted to be ϕ-positive; $0<h<1/4$;

and $g(h)$ is best possible.

As will be seen, $g(h)>2h^2$; so (2.5) implies that for suitable

$\psi(h)>0$, depending only on ϕ,

(2.6) Corollary. $R(n,p,h)\geq\psi(h)e^{2nh^2}$ for all n, all $p\in[0,1]$,

all $h\in(0,1/4)$, and all ϕ-positive priors μ.

3. Heuristics

Entropy comes into the argument when you compute the posterior odds ratio:

$$R(n,p,h) = \frac{\int_{(p-h,p+h)} \theta^{pn}(1-\theta)^{(1-p)n}\ \mu(d\theta)}{\int_{[0,p-h]\cup[p+h,1]} \theta^{pn}(1-\theta)^{(1-p)n}\ \mu(d\theta)}$$

The integrand is $e^{-nH(p,\theta)}$.

The numerator must be bounded from below, and the denominator from above. The signs may cause a little confusion: for example, the numerator is large when $H(p,\bullet)$ is small, that is, close to its minimum $H(p,p)$.

To bound the numerator, let h^* be small and positive.
For θ within h^* of p, an argument by uniform continuity will show that $H(p,\theta)$ is within $\epsilon g(h)$ of the minimal value $H(p,p)$. And the μ-measure of these θ's is at least $\psi(h,\epsilon)$. So the numerator is at least $e^{-n[H(p,p)+\epsilon g(h)]}\ \psi(h,\epsilon)$.

For the denominator: $H(p,\theta)$ is at its minimum in θ when $\theta=p$. so the worst θ in the denominator is $\theta=p+h$ or $\theta=p-h$. That suggests trying to minimize $H(p,p+h)-H(p,p)$ and $H(p,p-h)-H(p,p)$, leading to the study of $g(h)$. In the end, convexity arguments show that the denominator is at most $e^{-n[H(p,p)+g(h)]}$.

Now

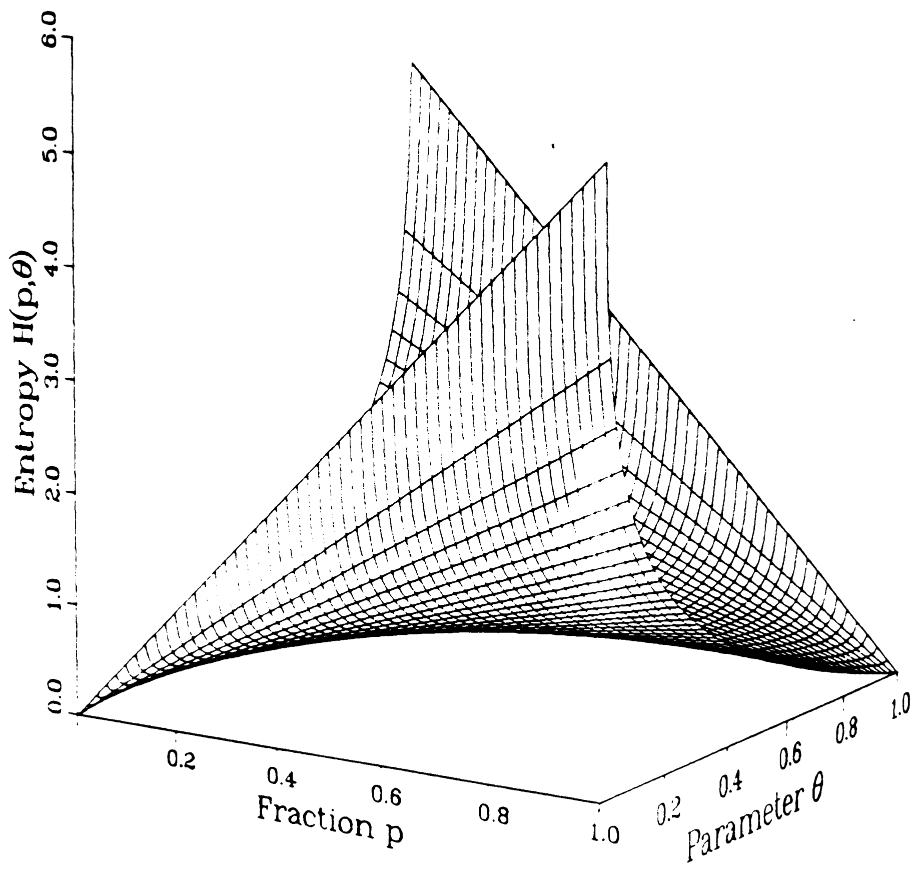
$$R(n,p,h) \geq \frac{e^{-n[H(p,p)+\epsilon g(h)]} \quad \psi(h,\epsilon)}{e^{-n[H(p,p)+g(h)]}}$$

$$= \quad \psi(h,\epsilon) \; e^{n(1-\epsilon)g(h)}$$

That is Theorem (2.4).

The entropy function $H(p,\theta)$ is naturally defined on the closed unit square, except for the corners $p=\theta=0$ and $p=\theta=1$. We set it to 0 there, but lim $H(p,\theta)$ is undefined as (p,θ) converges to either of those corners; although $H(p,p)$ tends to 0 as p tends to 0 or 1. (The discontinuity is a tedious point of difficulty in the background.) By comparison, it is safe to set $H(p,\theta)=\infty$ if $p<1$ and $\theta=1$, or $p>0$ and $\theta=0$, since ∞ is the limiting value on those edges of the unit square. A fishnet plot of $H(p,\theta)$ is shown in Figure 2.

Figure 2. Fishnet plot of $H(p,\theta)$. The singularities at (0,0) and (1,1) are not visible. $H(\bullet,\theta)$ is linear; $H(p,\bullet)$ is convex; $p \to H(p,p)$ is concave; $p \to H(p,p+h)$ is neither convex nor concave.



4. Proofs for the coin.

Let $0<h<{}^1/_4$. Recall $g(h)$ from (2.3). Confirm that

$$(4.1) \qquad g(h) = \min_p \{H(p,p+h)-H(p,p):\ 0\le p<1-h\} = \min_p \{H(p,p-h)-H(p,p):\ h<p\le 1\} > 0$$

For example, $p\to H(p,p+h)-H(p,p)$ is continuous on $(0,1-h)$; positive by (2.2); tends to $\log {}^1/_{1-h}>0$ as $p\to 0^+$; tends to ∞ as $p\to(1-h)^-$. And $H(1-p,1-\theta)=H(p,\theta)$. For details, see the next section.

The ψ-function and the proof of Theorem (2.4)

Assume without real loss of generality that $0\le p\le {}^1/_2$. Keep $h\in(0,{}^1/_4)$. Clearly,

$$(4.2) \qquad \frac{\partial}{\partial\theta} H(p,\theta) = -\frac{p}{\theta} + \frac{1-p}{1-\theta}.$$

If $p<\theta<p+h$, the first term on the right in (4.2) is trapped in $(-1,0)$. The second term is positive, and at most $(1-p)/(1-p-h)$; this bound increases with p, to a maximum of $({}^1/_2)/({}^1/_2-h)$: the latter is at most 2. This is where we use the condition that $h<{}^1/_4$. Thus

$$(4.3) \qquad \left|\frac{\partial}{\partial\theta} H(p,\theta)\right| < 2, \quad \text{provided } p<\theta<p+h,\ 0\le p\le {}^1/_2,\ 0<h<{}^1/_4$$

Fix $\epsilon>0$. Let $h^*=\min\{{}^1/_2\epsilon g(h),h\}$. Let $\psi(h,\epsilon)=\phi(h^*)$, a positive lower bound on the prior μ-mass in $(p,p+h^*)$. By (4.3), $p<\theta<p+h^*$ entails $H(p,\theta)<H(p,p)+\epsilon g(h)$. The odds ratio for the inside of $(p-h,p+h)$ vs the outside can now be estimated, as follows.

$$R(n,p,h) = \frac{\int_{(p-h,p+h)} e^{-nH(p,\theta)}\mu(d\theta)}{\int_{[0,p-h]\cup[p+h,1]} e^{-nH(p,\theta)}\mu(d\theta)}$$

Since $h^* \leq h$, the numerator is bounded below by

$$\int_{(p,p+h^*)} e^{-nH(p,\theta)} \mu(d\theta) \geq e^{-n[H(p,p)+\epsilon g(h)]} \mu(p,p+h^*)$$

$$\geq e^{-n[H(p,p)+\epsilon g(h)]} \psi(h,\epsilon)$$

The denominator is bounded above by $e^{-n[H(p,p)+g(h)]}$. For example, suppose $p+h \leq \theta \leq 1$. Then $H(p,\theta) \geq H(p,p+h)$ by (2.2); and $H(p,p+h) \geq H(p,p)+g(h)$ because $g(h)$ is the worst-case entropy differential. (Since $p<1$, $H(p,1)=\infty$; and the inequalities are trivial if $p+h<1$ but $\theta=1$; or if $p+h=\theta=1$.) In consequence, and this completes the proof of the theorem,

$$R(n,p,h) \geq \psi(h,\epsilon)\, e^{-n(1-\epsilon)g(h)} \qquad \square$$

Proof of (2.5). The inequality is immediate from (2.4). To see that $g(h)$ is best possible, fix h. For now, fix j and n too. Abbreviate $p=j/n$. We must bound $R(n,p,h)$ from above. As (2.2) shows, the numerator is bounded above by $e^{-nH(p,p)}$.

Let $\delta>0$. The denominator is bounded below by the integral over $[p+h,p+h+\delta]$. For θ in that interval, $H(p,\theta)$ is at most $H(p+h+\delta)$, by (2.2). So the denominator is at least

$$\mu(p+h,p+h+\delta)\bullet e^{-n[H(p,p+h+\delta)]} .$$

If $p+h+\delta<1$, then,

$$\frac{1}{n}\log R(n,p,h) \leq O(\frac{1}{n}) + H(p,p+h+\delta) - H(p,p)$$

To complete the argument, let $n\to\infty$; let $p=j/n$ tend to a point where $H(p,p+h)-H(p,p)$ takes its minimum value $g(h)$; and let $\delta\to 0$. (Eventually $p+h<1$, because $H(p,1-)=\infty$; for details on the minimization, see the next section.) □

5. The function g(h).

The outline of this section is as follows. The entropy differential $D_{+h}(p)=H(p,p+h)-H(p,p)$ is introduced, and its derivative computed in terms of the auxiliary "left hand" and "right hand" functions L_h and R_h. The point p_h where D_{+h} achieves its minimum is computed by solving $L_h(p)=R_h(p)$. The g(h) of (2.3) is the worst-case entropy differential, $D_{+h}(p_h)$; some properties of g(h) are developed.

Entropy differentials

Let $h\in(0,{}^1/_2)$, so $h<1-h$. For $0\leq p<1-h$, let

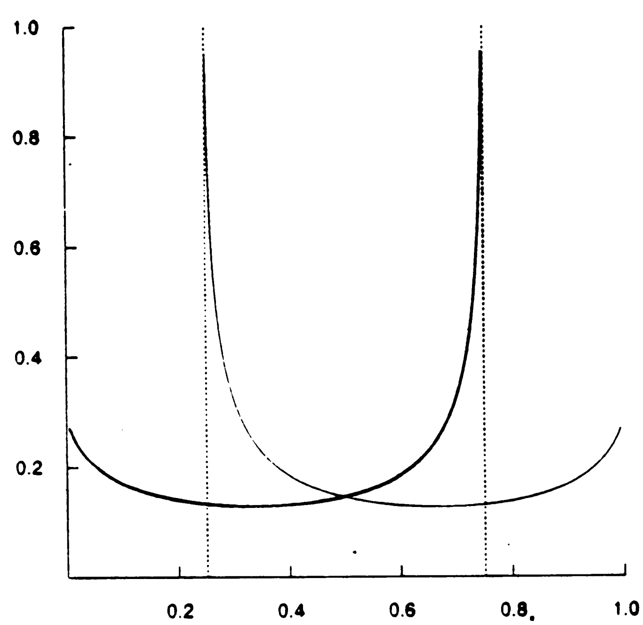
$$D_{+h}(p)=H(p,p+h)-H(p,p)$$

For $h<p\leq 1$, let

$$D_{-h}(p)=H(p,p-h)-H(p,p)$$

These are the "entropy differentials," which are to be bounded below. They are plotted in Figure 3.

Figure 3. The entropy differentials are convex functions of p, and $D_{+h}<D_{-h}$ on $(h,{}^1/_2)$. The figure shows $h={}^1/_4$. The heavier curve is D_{+h}; the ligher one, D_{-h}. Vertical lines are shown at h and 1-h.



The main object of this section is to give some description of g and the entropy differentials, using convexity arguments; and the next lemma is basic to the development.

(5.1) **Lemma.** Let $p>0$ and let δ be real, with $p+\delta>0$ and $\delta\neq 0$. Let $\Phi(p)=p\log[p/(p+\delta)]$. Then Φ is strictly convex.

Proof. First, suppose $\delta<0$. Then $p>|\delta|$ and

$$\Phi(p) = -p\log\left(1-\frac{|\delta|}{p}\right) = \sum_{j=1}^{\infty} \frac{1}{j}|\delta|^{j}\frac{1}{p^{j-1}}$$

Each term is convex.

Next, suppose $\delta>0$. Let $u=p+\delta>\delta$. Now

$$\Phi(p) = (u-\delta)\log\left(1-\frac{\delta}{u}\right)$$

$$= -(u-\delta)\sum_{j=1}^{\infty} \frac{1}{j}\delta^{j}\frac{1}{u^{j}}$$

$$= -\delta + \sum_{j=1}^{\infty}\left\{\frac{1}{j}-\frac{1}{j+1}\right\}\delta^{j+1}\frac{1}{u^{j}}$$

Again, each term is convex. □

(5.2) **Lemma.**

a) $D_{-h}(p)=D_{+h}(1-p)$.

b) $D_{-h}(^1/_2)=D_{+h}(^1/_2)$.

c) $D_{+h}(0+)<\infty$, but $D_{+h}(1-h-)=\infty$.

d) $D_{-h}(1-)<\infty$, but $D_{-h}(h+)=\infty$.

e) The entropy differentials are strictly convex functions of p.

f) $D_{-h}(p)-D_{+h}(p)$ is strictly decreasing as p increases from h to 1-h.

g) $D_{-h}(p)>D_{+h}(p)$ for $h<p<^1/_2$.

Proof. Claims a)-d) are clear.

Claim e) follows from (5.1).

Claim f). Clearly,

$$\begin{aligned} D_{-h}(p)-D_{+h}(p) &= H(p,p-h)-H(p,p+h) \\ &= p\log\frac{p+h}{p-h} - (1-p)\log\frac{1-p+h}{1-p-h} \\ &= p\log\left[\frac{1+\frac{h}{p}}{1-\frac{h}{p}}\right] - (1-p)\log\left[\frac{1+\frac{h}{1-p}}{1-\frac{h}{1-p}}\right] \\ &= 2\sum_{j=0}^{\infty} h^{2j+1}\left\{\frac{1}{p^{2j}} - \frac{1}{(1-p)^{2j}}\right\} \end{aligned}$$

This is legitimate because $0<h/p<1$ and $0<h/(1-p)<1$. Finally,

$$p \to \left\{\frac{1}{p^{2j}} - \frac{1}{(1-p)^{2j}}\right\}$$

is monotone decreasing as p increases, at least for $j>0$.

Claim g) is immediate from b) and f). □

Next, we have to differentiate the entropy differentials, and it is enough to consider $D_{+h}(\bullet)$. For $0<p<1-h$ let

(5.3) $$L_h(p) = \log(p+h) - \log p + \log(1-p) - \log(1-p-h)$$

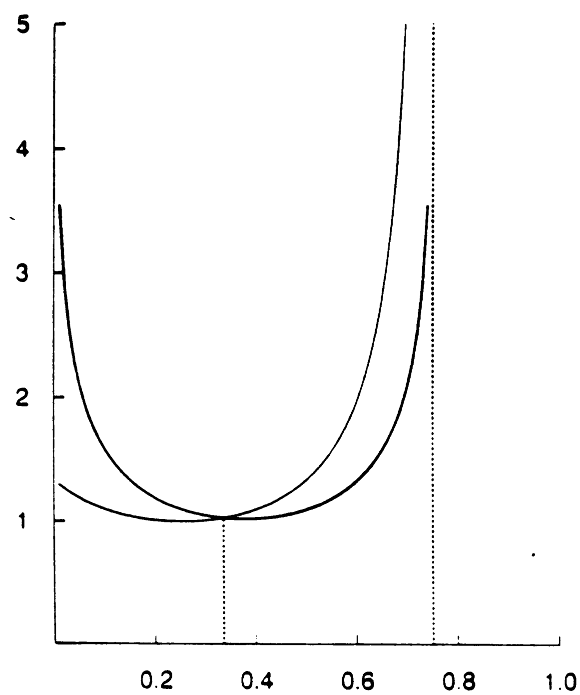
(5.4) $$R_h(p) = \frac{1-p}{1-p-h} - \frac{p}{p+h} = \frac{h}{(p+h)(1-p-h)}$$

As is easily seen,

(5.5) $$\frac{\partial}{\partial p} D_{+h}(p) = -L_h(p) + R_h(p)$$

To minimize D_{+h}, we have to solve $L_h(p) = R_h(p)$; hence the "L" and "R", for "left" and "right" hand sides of the equation. Some facts about D, L and R are developed as lemmas: see Figure 4.

Figure 4. The functions $L_h(\bullet)$ and $R_h(\bullet)$ are convex. Their graphs cross at p_h. The figure shows $h=1/4$ and $p_h=.3345$. The heavier curve is L_h; the lighter one, R_h. A vertical line is shown at $1-h$.



(5.6) **Lemma.** Fix α with $0 \leq \alpha \leq 1$. Abbreviate $\beta = 1 - \alpha$.

a) $$L_h(1/2 - \alpha h) = 4h + 2\sum_{j=1}^{\infty} \frac{\alpha^{2j+1} + \beta^{2j+1}}{2j+1} (2h)^{2j+1}$$

b) $$R_h(1/2 - \alpha h) = 4h + 2\sum_{j=1}^{\infty} \beta^{2j} (2h)^{2j+1}$$

Proof. As is easily verified,

$$L_h(1/2 - \alpha h) = \log(1+2\beta h) - \log(1-2\beta h) + \log(1+2\alpha h) - \log(1-2\alpha h)$$

$$R_h(1/2 - \alpha h) = \frac{4h}{1 - 4\beta^2 h^2}$$ □

(5.7) **Lemma.**

a) $R_h(0+) < \infty = L_h(0+)$.

b) $R_h(1/2) > L_h(1/2)$.

c) $$\left. \frac{\partial}{\partial p} D_{+h}(p) \right|_{p=1/2} > 0$$

d) $$\left. \frac{\partial}{\partial p} D_{-h}(p) \right|_{p=1/2} < 0$$

e) $D_{+h}(\bullet)$ is decreasing near 0.

f) $D_{-h}(\bullet)$ is increasing near 1.

Proof. Claim a) is easy.

Claim b) follows from (5.6), with $\alpha = 0$.

Claim c) is immediate from b), and d) is symmetric.

Claim e) follows from a), and f) is symmetric. □

Let p_h be the unique point where $D_{+h}(\bullet)$ attains its minimum.

(5.8) **Lemma.**

a) $L_h(\bullet)$ and $R_h(\bullet)$ are strictly convex.

b) $R_h(1-h-\delta)/L_h(1-h-\delta)\to\infty$ as $\delta\to 0^+$.

c) $L_h(p)-R_h(p)$ is strictly decreasing as p increases from 0 to 1-h.

d) p_h is the unique solution to $L_h(p)=R_h(p)$.

e) L_h is minimum at $^1/_2-^1/_2h$; and R_h, at $^1/_2-h$.

f) $^1/_2-h < p_h < ^1/_2-^1/_2h$.

Proof. Claim a). As is easily seen,

$$R_h(p) = h\left\{\frac{1}{p+h} + \frac{1}{1-p-h}\right\}$$

Each term is convex. Another easy calculation shows

$$\frac{\partial^2}{\partial p^2}L_h(p) = \frac{1}{p^2} - \frac{1}{(p+h)^2} + \frac{1}{(1-p-h)^2} - \frac{1}{(1-p)^2} > 0$$

Claim b) is easy.

Claim c). $R_h(p)-L_h(p)=\frac{\partial}{\partial p}D_{+h}(p)$, which increases because D_{+h} is convex.

Claim d) follows.

Claim e) is easy.

Claim f). This is equivalent to the pair of inequalities

$$L_h(^1/_2-h) > R_h(^1/_2-h)$$

$$L_h(^1/_2-^1/_2h) < R_h(^1/_2-^1/_2h)$$

which follow from (5.6). □

Recall that D_{+h} takes its minimum at p_h. We now make some remarks about the function $h \to p_h$. Abbreviate

$$p' = \frac{\partial}{\partial h} p_h .$$

(5.9) **Remark.** $p'<0$ for $0<h<{}^1/_2$

Proof. The implicit function theorem and (5.8d) imply that p_h is a smooth function of h. Take $p=p_h$ in (5.3-4) and differentiate with respect to h; equate the results to see

$$\frac{p'+1}{p+h} - \frac{p'}{p} - \frac{p'}{1-p} + \frac{p'+1}{1-p-h} = \frac{(p+h)(1-p-h) - h(p'+1)(1-2p-2h)}{[(p+h)(1-p-h)]^2}$$

This simplifies to

$$p'\left\{ \frac{1}{p+h} - \frac{1}{p} - \frac{1}{1-p} + \frac{1}{1-p-h} \right\} = \frac{h(p'+1)(2p-1+2h)}{[(p+h)(1-p-h)]^2}$$

and then

$$(5.10) \qquad p'\frac{2p-1+h}{p(1-p)} = (p'+1)\frac{2p-1+2h}{(p+h)(1-p-h)}$$

Now $2p-1+h<0$ because $p_h<{}^1/_2-{}^1/_2 h$ while $2p-1+2h>0$ because $p_h>{}^1/_2-h$; see (5.8f). If $p'+1>0$ then $p'<0$ by (5.10); on the other hand, if $p'+1\le 0$ then $p'<0$. $\square$

A graph of p_h is shown in Figure 5. Numerical calculations suggest that p_h is a convex function of h, but practically linear on $(0,{}^1/_2)$:

$$ {}^1/_2 - {}^2/_3 h < p_h < {}^1/_2 - .64h $$

Some of this can be proved; also see (5.8f).

(5.11) **Remarks.**

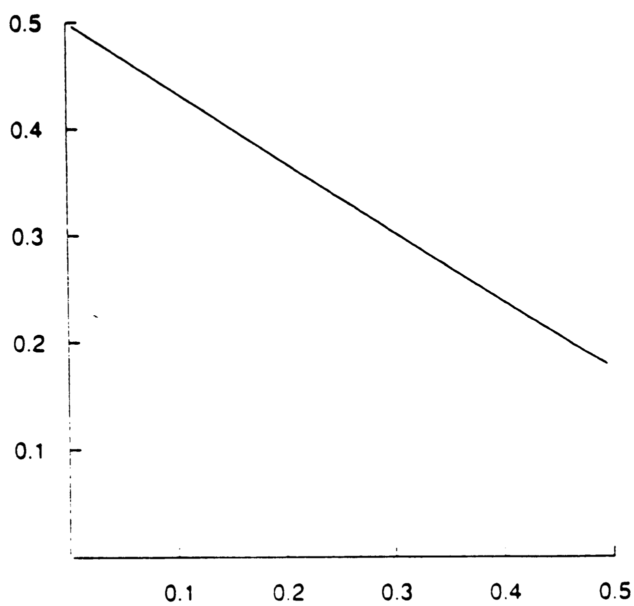
a) $p_h > {}^1/_2 - {}^2/_3 h$ for $0<h<{}^1/_2$, by (5.6).

b) $p_h < {}^1/_2 - ({}^2/_3-\epsilon)h$ for $0<h<\delta$, again by (5.6).

c) The derivative of p_h at 0 is ${}^2/_3$.

d) $p' \to {}^2/_3$ as $h \to 0^+$, by (5.10) and c).

Figure 5. A graph of $h \to p_h$; this function is virtually a straight line on $(0,{}^1/_2)$.



The construction of g(h) and its properties

Let g(h) be the worst-case entropy differential:

$$g(h) = D_{+h}(p_h)$$
$$= H(p_h, p_h+h) - H(p_h, p_h)$$
$$= H(1-p_h, 1-p_h-h) - H(1-p_h, 1-p_h)$$

Numerical calculations suggest that g(h) is convex, that $g(h)/(2h^2)$ is convex, and the latter is practically constant, increasing from 1 at h=0 to 1.07 at $h={}^1/_2$. Some of this can be proved, and will be discussed now.

(5.12) **Remarks.**

a) $g(h) \approx 2h^2$ as $h \to 0^+$. Indeed,

$$H(p,p+h) - H(p,p) = \frac{1}{2}\,\frac{h^2}{p(1-p)} + O(h^3)$$

as $h \to 0^+$, uniformly in p bounded away from 0 and 1;

recall from (5.11) that $p_h \to {}^1/_2$ as $h \to 0^+$.

b) g(h) is monotone. Indeed,

$$\text{(5.13)} \qquad \frac{\partial}{\partial h} D_{+h}(p) = \frac{h}{(p+h)(1-p-h)} > 0$$

Since the whole curve $D_{+h}(\bullet)$ shifts up when h increases, so does its minimum value, which is g(h).

(5.14) Proposition. Let $0<p<1$ and $0<h<1-p$. Then $D_{+h}(p)>2h^2$. Indeed, there is equality at $h=0$; the difference is strictly increasing (but not convex).

Proof. Equation (5.13) shows

$$(5.15) \qquad \frac{\partial}{\partial h} D_{+h}(p) = \frac{h}{(p+h)(1-p-h)} \geq 4h$$

because $(p+h)(1-p-h)\leq{}^1/_4$, with equality only at $p+h={}^1/_2$. This proves strict monotonicity. $\square$

(5.16) Corollary. Let $0<h<{}^1/_2$. Then $g(h)>2h^2$.

Proof. $g(h) = \inf\,\{D_{+h}(p)\colon 0<p<{}^1/_2\}$. If $0<h<{}^1/_2$ and $0<p<{}^1/_2$ then $0<h<1-p$, and (5.14) applies. $\square$

(5.17) Corollary. Let $0<h<{}^1/_2$. Then $g(h)-2h^2$ is strictly increasing in h.

Proof. Use (5.15). $\square$

(5.18) Remark. $h\to D_{+h}(p)/2h^2$ is not increasing in h. For example, with $p=.3$, the min occurs near $h=.27$; the function decreases from 0 to .27, then increases.

(5.19) Proposition. Let $0<h<{}^1/_2$. Then $g(h)<3h^2$.

Proof. Choose $p={}^1/_2-h$. Then $g(h)<D_{+h}(p)$, so

$$\begin{aligned} g(h) &< ({}^1/_2-h)\log(1-2h) + ({}^1/_2+h)\log(1+2h) \\ &= \sum_{j=1}^{\infty}\left[\frac{1}{2j-1} - \frac{1}{2j}\right](2h)^{2j} \\ &= 2h^2 + 4h^2\left\{\sum_{j=2}^{\infty}\left[\frac{1}{2j-1} - \frac{1}{2j}\right](2h)^{2(j-1)}\right\} \end{aligned}$$

But $2h<1$, so $g(h)/2h^2$ is bounded above by

$$1 + 2\left\{\sum_{j=2}^{\infty}\left[\frac{1}{2j-1} - \frac{1}{2j}\right]\right\} < 1 + 2\left\{\sum_{j=2}^{4}\left[\frac{1}{2j-1} - \frac{1}{2j}\right] + \frac{1}{9}\right\} < 1.5 \quad \square$$

If $0<h<{}^1/_4$, a similar argument starting from $p={}^1/_2$ gives a much tighter upper bound, $g(h)<2.1h^2$.

6. The theorem for the multinomial.

Let S_k be the simplex of all k-vectors θ with non-negative coordinates θ_i adding to 1. Consider a die with k sides, labelled 1,...,k. In n tosses, the relative frequencies with which these sides land form a vector $p=(p_1,...,p_k)$ in S_k.

(6.1) For $0<h<1/k$, let $N_k(h,p)$ be the polyhedral neighborhood of p consisting of the $\theta \in S_k$ with $|\theta_i - p_i|<h$ for all i.

Plainly, $N_k(h,p)$ is the sphere around p of radius h-- in the sup norm.

(6.2) Let $H_k(p,\theta)$ be the relative entropy:

$$H_k(p,\theta) = -\sum_{i=1}^{k} p_i \log \theta_i$$

This can be defined everywhere by the convention $0\times\infty=0$, but the limit of $H_k(p,\theta)$ is not well defined if eg p_1 and θ_1 both tend to zero.

The first main result in this section, to be proved later, shows that the minimum entropy differentials do not depend on the dimension k. Recall g from (2.3).

(6.3) **Proposition.**

$$\inf_{p\in S_k,\ \theta\notin N_k(h,p)} [H_k(p,\theta) - H_k(p,p)] = g(h)$$

A prior μ on the simplex S_k is "ϕ-positive" if it is uniformly positive on a certain class of sub-simplexes. More specifically, let v be a vector of length k, whose entries are either +1 or -1.

(6.4) Definition. Let $T_k(v)$ be the set of k-vectors $x=(x_1,...,x_k)$ satisfying:

$$\sum_{i=1}^{k} x_i = 0 \quad \text{and} \quad 0 \le v_i x_i \le 1 \text{ for } i=1,...,k$$

For $p \in S_k$ and $0<h<{}^1/_k$ let

$$T_k(p,h,v) = p+hT_k(v) = \{p+hx: x \in T_k(v)\}$$

Say μ is ϕ-positive iff $\mu\{T_k(p,h,v)\}>\phi(h)$ whenever $T_k(p,h,v) \subset S_k$.

To state the next result, suppose the k-sided die is tossed n times, and p is the vector of empirical frequencies. Let $0<h<{}^1/_k$. Let $R(n,p,h)$ be the posterior odds ratio for the inside of $N_k(h,p)$ versus the outside, with respect to a ϕ-positive prior: the outside is nonempty, because $h<{}^1/_k$. Let $\epsilon>0$. There is a $\psi(h,\epsilon)>0$, which depends on ϕ, h, and ϵ but not on n or p, such that the following inequality holds.

(6.5) **Theorem.** $R(n,p,h) \ge \psi(h,\epsilon)\, e^{n(1-\epsilon)g(h)}$ for $0<h<{}^1/_{2k}$.

(6.6) Corollary. $\liminf_{n\to\infty} \inf_{p,\mu} \frac{1}{n} \log R(n,p,h) \ge g(h)$

In (6.6), the prior μ is restricted to be ϕ-positive, $0<h<{}^1/_{2k}$, and $g(h)$ is best possible.

7. Proofs for the multinomial.

The proof of (6.3)

Suppose $k \geq 3$. Since the entropy function (6.2) is convex in θ with its minimum at p, the infimum outside the convex polyhedron $N_k(h,p)$ is attained on the boundary. Consider eg the intersection of the boundary with

$$F = \{\theta:\ \theta \in S_k \text{ and } \theta_k = p_k + h\}$$

Assume for the sake of argument that this face is nonempty, so $p_k + h \leq 1$. Consider

$$\inf_{\theta \in F} H_k(p,\theta) - H_k(p,p) \tag{7.1}$$

Now

$$\begin{aligned} H_k(p,\theta) &= -\sum_{i=1}^{k} p_i \log \theta_i \\ &= -\sum_{i=1}^{k-1} p_i \log \theta_i - p_k \log(p_k+h) \\ &= -\sum_{i=1}^{k-1} p_i \log(\theta_i/1-p_k-h) \\ &\qquad - p_k \log(p_k+h) - (1-p_k)\log(1-p_k-h) \end{aligned}$$

The last sum in the display can be written as $(1-p_k)H_{k-1}(\tilde{p},\tilde{\theta})$ where $\tilde{p}_i = p_i/1-p_k$ and $\tilde{\theta}_i = \theta_i/1-\theta_k$ for $i=1,\ldots,k-1$. So $\tilde{p},\ \tilde{\theta} \in S_{k-1}$. Now $(1-p_k)H_{k-1}(\tilde{p},\tilde{\theta})$ is minimized in $\tilde{\theta}$ at $\tilde{\theta}=\tilde{p}$, and the value of the minimum is

$$-\sum_{i=1}^{k-1} p_i \log(p_i/1-p_k) = -\sum_{i=1}^{k-1} p_i \log p_i + (1-p_k)\log(1-p_k)$$

We pause to confirm that our minimum is on the boundary of $N_k(h,p)$: by construction, it is only on the hyperplane $\theta_k = p_k + h$. Switching back to the original coordinate system, the minimum is at θ with

$$\theta_i = \frac{(1-p_k-h)}{1-p_k} p_i \quad \text{for } i=1,\ldots,k-1 \quad \text{and} \quad \theta_k = p_k + h$$

Clearly, the θ_i are non-negative and sum to 1. For $i<k$, θ_i falls below p_i by the amount

$$\frac{p_i}{1-p_k} h < h.$$

Coming back to the main line of argument, the infimum in (7.1) is obtained by subtracting $H_k(p,p)$, and equals

$$-p_k \log(p_k+h) - (1-p_k)\log(1-p_k-h) + p_k \log p_k + (1-p_k)\log(1-p_k)$$

which can be recognized as $D_{+h}(p_k)$. The latter is minimized when $p_k = p_h$, and the minimum value is $g(h)$. This completes the proof of (6.3). □

(7.2) Remark. If h is near 0, then p_h is near $^1/_2$. So the $p \in S_k$ with the worst entropy differentials have one or two coordinates near $^1/_2$. By renumbering, suppose $p_1 \leq \ldots \leq p_k$. There are two possibilities for the worst-case p's:

i) $p_k = 1-p_h > {}^1/_2$ and $p_1, \ldots, p_{k-1}$ are free;

ii) $p_k = p_h < {}^1/_2$ and $p_1, \ldots, p_{k-1}$ are free.

Case i) includes eg the possibility that $p_k = 1-p_h$ and $p_{k-1} = p_h$ and $p_1 = \ldots = p_{k-2} = 0$.

Recall the definition (6.4) of the simplex $T_k(v)$. The proof of the next lemma is omitted as standard.

(7.3) Lemma. The extreme points of $T_k(v)$ consist of all k-vectors x which sum to zero, with $x_i = v_i$ or $x_i = 0$ for all i.

The proof of (6.5)

Suppose by renumbering the sides that $p_1 \leq \ldots \leq p_k$. Let $v_1 = \ldots = v_{k-1} = 1$ and $v_k = -1$. We work on the simplex $T_k(p,h,v)$, which is wholly in the interior of S_k, because $p_k \geq {}^1/_k > h$. Indeed, the simplex has k extreme points by (7.3):

$$\begin{array}{llllll}
p_1 & p_2 & p_3 & \ldots\ldots & p_{k-1} & p_k \\
p_1+h & p_2 & p_3 & \ldots\ldots & p_{k-1} & p_k-h \\
p_1 & p_2+h & p_3 & \ldots\ldots & p_{k-1} & p_k-h \\
p_1 & p_2 & p_3+h & \ldots & p_{k-1} & p_k-h \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & & \\
p_1 & p_2 & p_3 & \ldots\ldots & p_{k-1}+h & p_k-h
\end{array}$$

And each extreme point is in S_k.

The rest of the argument is as for the coin. Indeed, as a function of $\theta \varepsilon T_k(p,h,v)$, $H_k(p,\theta)$ still has Lipschitz constant 2. For the proof, set $p_k = 1 - p_1 - \ldots - p_{k-1}$ and $\theta_k = 1 - \theta_1 - \ldots - \theta_{k-1}$; then differentiate with respect to θ_i for $i<k$:

$$\frac{\partial}{\partial \theta_i} H(p,\theta) = -\frac{p_i}{\theta_i} + \frac{p_k}{\theta_k}$$

The first term is trapped in $[-1,0]$, because $\theta_i \geq p_i$ for $\theta \varepsilon T_k(p,h,v)$. The second term is at most 2; indeed, $\theta_k \geq p_k - h$, so the second term is bounded above by $p_k/(p_k-h)$: but $p_k \geq {}^1/_k$ and $h < {}^1/_{2k}$. This completes the Lipschitz estimate.

To estimate the odds ratio, bound the numerator below by integrating over $T_k(p,h,v)$, using the Lipschitz estimate. Bound the denominator above using (6.3).

$\square$

8. Some facts about ϕ-positivity

This section has some remarks and examples on the idea of ϕ-positivity; we hope to explore the theory more systematically in the future. Recall that ϕ is a positive function on $(0,1)$; and the prior μ is ϕ-positive iff it assigns mass $\phi(h)$ or more to every closed interval of length h in $[0,1]$.

(8.1) Remark. If $\phi(h) > ah$ for all h, and μ is ϕ-positive, then μ is bounded setwise below by a times Lebesgue measure.

It is natural to conjecture that a ϕ-positive class of measures is bounded below (setwise) by a positive measure, but that turns out to be wrong; ϕ-positivity is a more general idea.

(8.2) Example. There is a ϕ-positive class of probability measures $M = \{\mu\}$ on $[0,1]$ such that if α is a measure and $\alpha \leq \mu$ setwise for all $\mu \in M$, then $\alpha = 0$.

Construction. The class M will be countable. Let λ be Lebesgue measure on $[0,1]$. Let λ_n assign mass $1/n+1$ to each of $0/n$, $1/n$, $2/n$, ..., n/n. Let

$$\mu_n = \frac{n+1}{n+2}\lambda_n + \frac{1}{n+2}\lambda$$

Let $R=\{r\}$ be the rationals in $[0,1]$, and Q the irrationals. If $\alpha\le\mu_n$, then $\alpha\{r\}\le 1/n+2$ and $\alpha(Q)\le 1/n+2$, so in the end $\alpha\{r\}=0$, $\alpha(R)=0$, and $\alpha(Q)=0$.

We claim that $\{\mu_n\}$ is ϕ-positive, with $\phi(h)=h^2/4$. To verify this, consider the interval $[x,x+h]$. Suppose $\frac{a-1}{n} < x \le \frac{a}{n}$ and $\frac{b}{n} \le x+h < \frac{b+1}{n}$. Clearly, $\frac{b-a}{n} \ge h - \frac{2}{n}$; so $b-a \ge nh-2$. So, there are at least $b-a+1$ rationals of order n in $[x,x+h]$, and

$$\lambda_n[x,x+h] \ge \frac{nh-1}{n+1}$$

Now

$$\begin{aligned}\mu_n[x,x+h] &\ge \frac{nh-1}{n+2}\\ &\ge \frac{1}{2}h - \frac{1}{n+2}\\ &\ge \frac{1}{4}h \quad \text{if } n+2 \ge \frac{4}{h}\\ &\ge \frac{1}{4}h^2\end{aligned}$$

If $n+2 < \frac{4}{h}$, a lower bound on $\mu_n[x,x+h]$ is still $\frac{1}{4}h^2$, from the λ-term only. In fact, $\phi(h)$ is of order h^2, as one sees by taking n of order $1/h$. $\square$

There is a connection with monotone rearrangements (Hardy-Littlewood-Polya, 1934).

(8.3) Remark. Let ϕ be convex, with derivative f, and $\phi(1)=1$. So f is monotone nondecreasing, and its integral is 1. All rearrangements of f are ϕ-positive. Some rearrangements have bigger (and nonconvex) ϕ's; for such a ϕ, all rearrangements of its density will no longer be ϕ-positive. If $\phi(h)=ah^2$, the rearrangements can be bounded below only be a trivial measure.

We endow the space of probabilities on [0,1] with the weak-star topology, which is compact and metrizable.

(8.4) Remark Let M be a ϕ-positive class. Then the closed convex hull of M is ϕ-positive too.

If M consists of one prior, or finitely many priors, then M is ϕ-positive; the next result is a small generalization.

(8.5) Remark Let M be a closed, convex class of probabilities on [0,1]. Suppose that each element of M assigns positive mass to every open interval. Then M is ϕ-positive.

Proof. Fix h with $0<h<1$. Let $0\leq x\leq 1-h$. Let the continuous function f_x on [0,1] vanish to the left of x and to the right of x+h; let $f_x=1$ at $x+\frac{1}{2}h$; complete f_x by linear interpolation. Now $\mu(f_x)$ is a continuous positive function of $\mu \in M$ and x; so it has a positive minimum: $\phi(h)$ can be defined as this minimum, over μ and x. □

Let M_ϕ be the class of ϕ-positive μ. When is M_ϕ nonempty? When is ϕ the exact inf, that is, $\phi(h) = \inf\{\mu[x,x+h]: \mu \in M_\phi$ and $0\leq x<x+h\leq 1\}$? What are the extreme points of M_ϕ? At this point, we only have some scattered remarks as partial answers.

(8.6) Example. Let $\phi(h)=h/10$, for $0<h<1$. One compact convex class M of ϕ-positive μ is the set of μ of the form

$$.1*\text{Lebesgue} + .9*\nu,$$

where ν is any probability. The extreme points have $\nu=\delta_x$. This class is maximal, by a standard extension argument off intervals. There seem to be two other compact convex ϕ-positive classes M, which are minimal: take $\nu=\delta_0$ or δ_1. To get intermediate classes, mix over any compact set of δ_x's containing x = 0 or 1.

(8.7) Example. Let $\phi(h)={}^{1}/_{2}h$ for $h<{}^{2}/_{3}$ and $\phi(h)=2h$ for ${}^{2}/_{3}<h<1$. The extreme points of the class of ϕ-positive μ seem to be as follows:

$$ {}^{1}/_{2}\text{ Lebesgue} + {}^{1}/_{2}\ \delta_a \quad \text{with } {}^{1}/_{3}\le a\le {}^{2}/_{3} $$

$$ {}^{1}/_{2}\text{ Lebesgue} + {}^{1}/_{2}\{3a\ \delta_a + \text{density 3 on } ({}^{2}/_{3}+a,1)\} \quad \text{for } a<{}^{1}/_{3} $$

(8.8) Remark. Let $M=\{\mu\}$ be ϕ-positive. Then $\phi(1/n)\le 1/n$, otherwise μ has mass greater than 1. Likewise, if ϕ is the exact inf of M, then $\phi(h/n)\le\phi(h)/n$.

(8.9) Example. $a_n = 2^n\phi(1/2^n)$ can decrease arbitrarily rapidly.

Construction. Let $a_1 < 1/2$, and $a_{n+1} < a_n$. Let μ_n have density equal to a_n on $[0,1/2^n]$ and equal to b_n on $(1/2^n,1]$. So b_n can be computed from a_n, and $b_n > 1$. Let $M = \{\mu_n\}$. We claim that M is ϕ-positive for suitable ϕ; and if ϕ is the exact inf, $\phi(1/2^n) = a_n/2^n$. Indeed, if $m \le n$, then

$$\mu_m[0,1/2^n] = a_m/2^n$$

On the other hand, if $m>n$,

$$\mu_m[0,1/2^n] > a_n/2^n$$

Indeed,

$$\begin{aligned}\mu_m[0,1/2^n] &> b_m(1/2^n - 1/2^m)\\ &> b_m/2^{n+1}\\ &> 1/2^{n+1}\\ &> a_n/2^n\end{aligned}$$

□

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