On Some Examples of Quadratic Functionals of Brownian Motion

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Abstract: During the last few years, several variants of P. Lévy's formula for the stochastic area of complex Brownian motion have been obtained, and are of interest in various domains of applied probability, particularly in relation with polymer studies.

The method used by most authors is the diagonalisation procedure of Paul Lévy.

In our paper, we derive one such variant of Lévy's formula, due to Chan, Dean, Jansons and Rogers, via a change of probability method, which reduces the computation of Laplace transforms of Brownian quadratic functionals to the computations of the means and variances of some adequate Gaussian variables.

We then show that with the help of linear algebra and invariance properties of the distribution of Brownian motion, we are able to derive simply three other variants of Lévy's formula.

Key Words: Lévy's formula, Girsanov's theorem, radius of gyration, correlated Brownian motions.

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1. Introduction:

(1.1) Let $(Z_s = X_s + i\,Y_s, s \geq 0)$ be a complex Brownian motion with $Z_0 = 0$ and let $G = \int_0^1 Z_s\,ds$ denote its centre of gravity over the time interval $[0,1]$.

In [4], we have obtained the law of (V_G, G) where $V_G = \int_0^1 |Z_s - G|^2\,ds$.

In particular, we have proved that

(1.a) $$V_G \overset{\text{(law)}}{=} \int_0^1 |\tilde{Z}_s|^2\,ds$$

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where $\tilde{Z}_s$ is a Brownian bridge over $[0,1]$; this identity in law can be obtained very simply from a Fubini theorem for double Wiener integrals:

if B is a real-valued Brownian motion, then:

(1.b) $$\int_0^1 du \left(\int_0^1 dB_s \, \phi(u,s)\right)^2 \overset{\text{(law)}}{=} \int_0^1 du \left(\int_0^1 dB_s \, \phi(s,u)\right)^2$$

for any $\phi \in L^2([0,1]^2 du\, ds)$.

(for some different derivations of (1.a), see the references in [4])

On the other hand, Chan, Dean, Jansons, and Rogers [2] have obtained the Laplace transform of the law of $(V_G, |G|^2, |Z_1|^2)$ using a diagonalization procedure for quadratic forms, close to Lévy's original computation of the Fourier transform of the stochastic area of planar Brownian motion. (In fact, this was the method used in the first preprint version of [2] the authors kindly sent us; for the method used in the final version of [2], see the discussion below in (1.2), (iii)). In this paper, we present another approach for the computation of the law of (V_G, G, Z_1) based upon Girsanov's transformation; this approach has already been used by Yor [12] to derive P. Lévy's formula for the stochastic area of complex Brownian motion (see also D. Williams [11] for a closely related computation) and by Pitman and Yor [8] to give a proof of the decomposition of Bessel bridges obtained otherwise by Pitman and Yor [9] using Ito's excursion theory.

Our interest in these questions was further aroused by the paper of Helfer-Zhongxin [7] who compute Laplace transforms of quadratic forms on the Wiener space, subject to linear conditions. One of their examples is the so-called radius of gyration tensor of a Brownian path which is a generalization in higher dimensions of the variable $\int_0^1 (B_s - G)^2 ds$ (here, B is a one-dimensional Brownian motion).

(1.2) In the previous subsection (1.1), we have presented a few examples of quadratic functionals of Brownian motion, which have been studied in recent years.

The methods used to obtain closed formulae for the corresponding characteristic functions or Laplace transforms fall essentially into the three following categories:

(i) P. Lévy's *diagonalisation procedure*, which has a strong functional analysis flavor; this method may be applied very generally and is quite powerful; however, the characteristic functions or Laplace transforms then appear as infinite products, which have to be recognized in terms of, say, hyperbolic functions... At this point, one may feel that the probabilistic interpretation gets, in some sense, out of hand, and the authors of this paper generally prefer

(ii) the *change of probability method* which, in effect, linearizes the problem, i.e: it allows to transform the study of a quadratic functional into the computation of the mean and variance of an adequate Gaussian variable,

and, finally:

(iii) the *reduction method*, which simply consists in trying to reduce the computation for a certain quadratic functional to similar computations which have already been done; the identity in law (1.b) shows why this situation may occur very naturally, and in a non-trivial manner. More simply, the relatively large number of Lévy-type formulae known nowadays makes it fairly likely that a number of new quadratic functional examples may be reduced to older ones. The paper [2] is a highly non-trivial instance of application of the "reduction method", in which important use is made of the links between Brownian quadratic functionals and the Ray-Knight theorem for diffusion local times.

(1.3) We now explain the organization of our paper.

In the second section, we present the Girsanov-type method which enables us to compute the conditional law of V_G, given G and Z_1. As a consequence, we obtain the formula given by Chan, Dean, Jansons, and Rogers in [2].

In the third section, we recover and give an extension of the formula of Helfer-Zhongxin [7]; this is obtained as a consequence of the results in section 2.

In the fourth section, we give an extension of (1.a), namely

$$\text{(1.c)} \qquad \int_0^1 (B_s - G)(B_s' - G')\, ds \overset{\text{(law)}}{=} \int_0^1 \tilde{B}_s \tilde{B}_s'\, ds$$

where B and B′ are two linearly correlated Brownian motions and $\tilde{B}$ and $\tilde{B}'$ are two Brownian bridges with the same correlation. Moreover, we compute the Laplace transform of the variables in (1.c). When B and B′ are two independent Brownian motions, this result has been obtained by Donati-Song-Yor [5], by polarizing the identity (1.b). Chou and Nualart [3] extended the previous result to correlated Brownian motions; their computation of the characteristic function of $\int_0^1 \tilde{B}_s \tilde{B}_s'\, ds$ is based on the decomposition of a Brownian motion in a series of independent Gaussian variables. We give another proof of Chou and Nualart's results using (1.a) and the rotational invariance property of the distribution of planar Brownian motion.

In the fifth and final section, we present a further illustration of the reduction method by showing how to recover formulae obtained recently by Berthuet [1] and Foschini-Shepp [6] about the stochastic area $\int_0^1 B(s) \times dB(s)$ of a 3-dimensional Brownian motion

$(B(s), s \le 1)$, where $x \times y$ indicates the vector product of x and y in $\mathbb{R}^3$, as a consequence of Lévy's fundamental formula for the stochastic area of complex Brownian motion $(Z_t = X_t + iY_t, t \le 1)$:

$$\text{(1.d)} \quad E\left[\exp i\lambda\int_0^1 (X_s dY_s - Y_s dX_s) \,|\, Z_1 = m\right] = \left(\frac{\lambda}{\sinh\lambda}\right)\exp\frac{|m|^2}{2}(1-\lambda\coth\lambda).$$

We note, for later reference, that the left-hand side of (1.d) is also equal to:

$$\text{(1.e)} \qquad E\left[\exp -\frac{\lambda^2}{2}\int_0^1 ds\, |Z_s|^2 \,\middle|\, Z_1 = m\right].$$

2. **The joint law of** (V_G, G, Z_1).

Girsanov's transformation enables us to compute

$$\text{(2.a)} \qquad E\left[\exp -\frac{\lambda^2}{2}\int_0^1 ds\, |Z_s - \alpha G|^2 \,\middle|\, Z_1 = z, G = \xi\right]$$

for any $\alpha \in \mathbb{R}$, $\lambda \ge 0$, $z, \xi \in \mathbb{C}$. By developing $|Z_s - \alpha G|^2$, we see that (2.a) equals

$$\text{(2.b)} \qquad \exp\left(-\frac{\lambda^2}{2}(\alpha^2 - 2\alpha)\,|\xi|^2\right)\phi_\lambda(z,\xi)$$

where

$$\text{(2.c)} \qquad \phi_\lambda(z,\xi) = E\left[\exp -\frac{\lambda^2}{2}\int_0^1 ds\, |Z_s|^2 \,\middle|\, Z_1 = z, G = \xi\right].$$

Let Q be the probability defined on $\sigma\{Z_s, s \le 1\}$ by:

$$\frac{dQ}{dP} = \exp\left\{-\lambda\int_0^1 Z_s \cdot dZ_s - \frac{\lambda^2}{2}\int_0^1 |Z_s|^2 ds\right\}$$

$$= \exp\left\{-\frac{\lambda}{2}(|Z_1|^2 - 2) - \frac{\lambda^2}{2}\int_0^1 |Z_s|^2 ds\right\}.$$

Girsanov's theorem tells us that the process (Z_t) satisfies, under Q, the stochastic differential equation:

$$Z_t + \lambda\int_0^t Z_s ds = \tilde{Z}_t \quad , \text{where } \tilde{Z} \text{ is a Q–Brownian motion}.$$

Then, $(Z_t, t \le 1)$ is, under Q, an Ornstein-Uhlenbeck process with parameter $(-\lambda)$, and we have: $Z_t = \int_0^t e^{-\lambda(t-s)} d\tilde{Z}_s$. Define

$$\text{(2.d)} \qquad Z_t^{(\lambda)} = \int_0^t e^{-\lambda(t-s)} dZ_s, \quad t \le 1.$$

This is an Ornstein-Uhlenbeck process with parameter $(-\lambda)$, under the probability P. Let ϕ be a bounded measurable functional. Then,

$$E[\phi(Z_t^{(\lambda)}, t \le 1] = E_Q[\phi(Z_t, t \le 1)]$$

(2.e)
$$= E[\phi(Z_t, t \le 1)\exp\{-\frac{\lambda}{2}(|Z_1|^2 - 2) - \frac{\lambda^2}{2}\int_0^1 ds\, |Z_s|^2\}].$$

In particular, we have, for any $f: \mathbb{C} \times \mathbb{C} \to \mathbb{R}_+$,

(2.f)
$$E[f(Z_1^{(\lambda)}, G^{(\lambda)})] = E[f(Z_1, G)\exp\{-\frac{\lambda}{2}(|Z_1|^2 - 2) - \frac{\lambda^2}{2}\int_0^1 ds\, |Z_s|^2\}].$$

We deduce from (2.f) the formula

(2.g)
$$q_\lambda(z, \xi) = q_0(z, \xi)\exp(-\frac{\lambda}{2}(|z|^2 - 2))\,\phi_\lambda(z, \xi)$$

where q_λ, resp. q_0, is the density of the law of $(Z_1^{(\lambda)}, G^{(\lambda)})$, resp. (Z_1, G).

Then, we have

(2.h)
$$\phi_\lambda(z, \xi) = \exp(-\lambda + \frac{\lambda}{2}|z|^2)\,\frac{q_\lambda(z, \xi)}{q_0(z, \xi)}.$$

We now compute q_λ for $\lambda \ge 0$. $(Z_1^{(\lambda)}, G^{(\lambda)})$ is a gaussian vector.

Let

(2.i)
$$G^{(\lambda)} = \alpha_\lambda Z_1^{(\lambda)} + H^{(\lambda)}$$

be the orthogonal decomposition of $G^{(\lambda)}$ with respect to $Z_1^{(\lambda)}$.

We denote by σ_λ^2, resp. h_λ^2, the variance of the real part of $Z_1^{(\lambda)}$, resp. of $H^{(\lambda)}$.

Then, we have:

(2.j)
$$q_\lambda(z, \xi) = \frac{1}{(2\pi)^2\sigma_\lambda^2 h_\lambda^2}\exp(-\frac{1}{2}[\frac{|z|^2}{\sigma_\lambda^2} + \frac{|\xi - \alpha_\lambda z|^2}{h_\lambda^2}]).$$

It now remains to compute σ_λ^2, h_λ^2, α_λ. From (2.d) and (2.g), we deduce

$$\sigma_\lambda^2 = \int_0^1 e^{-2\lambda(1-s)}ds = \frac{e^{-\lambda}}{\lambda}\sinh\lambda; \quad \sigma_0^2 = 1,$$

and $\alpha_\lambda \sigma_\lambda^2 = E[\mathrm{Re}(G^{(\lambda)})\mathrm{Re}(Z_1^{(\lambda)})] = \int_0^1 ds\, E[X_s^{(\lambda)} X_1^{(\lambda)}]$.

From (2.d), we get: $E[X_s^{(\lambda)} X_1^{(\lambda)}] = \int_0^s e^{-\lambda(s-u)} e^{-\lambda(1-u)}du = \frac{e^{-\lambda}}{\lambda}\sinh(\lambda s)$, so that $\alpha_\lambda \sigma_\lambda^2 = \frac{e^{-\lambda}}{\lambda^2}(\cosh\lambda - 1)$ and $\alpha_\lambda = \frac{1}{\lambda}\frac{\cosh\lambda - 1}{\sinh\lambda}$, $\alpha_0 = \frac{1}{2}$. To compute h_λ^2, we note that, from (2.i),

$$g_\lambda^2 = \alpha_\lambda^2 \cdot \sigma_\lambda^2 + h_\lambda^2 \quad \text{, where } g_\lambda^2 = E[(\int_0^1 ds\, X_s^{(\lambda)})^2].$$

We have:

$$g_\lambda^2 = 2\int_0^1 ds \int_s^1 dt\, E[X_s^{(\lambda)} X_t^{(\lambda)}] = \frac{1}{\lambda}\int_0^1 ds \int_s^1 dt\,(e^{-\lambda(t-s)} - e^{-\lambda(t+s)})$$

$$= \frac{1}{\lambda^2}[1 - \frac{3}{2\lambda} + 2\frac{e^{-\lambda}}{\lambda} - \frac{e^{-2\lambda}}{2\lambda}].$$

On the other hand, $\sigma_\lambda^2 \alpha_\lambda^2 = \frac{e^{-\lambda}}{\lambda^3}\frac{(\cosh\lambda - 1)^2}{\sinh\lambda}$, so that

$$h_\lambda^2 = \frac{1}{\lambda^2} + \frac{1}{\lambda^3}[-\frac{3}{2} + 2e^{-\lambda} - \frac{e^{-2\lambda}}{2} - e^{-\lambda}\frac{(\cosh\lambda - 1)^2}{\sinh\lambda}]$$

$$= \frac{1}{\lambda^2} + \frac{1}{\lambda^3 \sinh\lambda}[1 - \frac{3}{4}e^{\lambda} + \frac{1}{2}e^{-\lambda} - e^{-2\lambda} + \frac{e^{-3\lambda}}{4}$$

$$-e^{-\lambda}\{\frac{e^{2\lambda}}{4} + \frac{e^{-2\lambda}}{4} + \frac{1}{2} - e^{\lambda} - e^{-\lambda} + 1\}]$$

$$= \frac{1}{\lambda^2} + \frac{1}{\lambda^3 \sinh\lambda}[2 - e^{\lambda} - e^{-\lambda}]$$

and, finally: $h_\lambda^2 = \frac{1}{\lambda^2} + \frac{2(1-\cosh\lambda)}{\lambda^3 \sinh\lambda}$; $h_0^2 = \frac{1}{12}$.

From (2.h), we deduce

$$\text{(2.k)} \quad \begin{aligned} \phi_\lambda(z,\xi) &= \frac{e^{-\lambda}\sigma_0^2 h_0^2}{\sigma_\lambda^2 h_\lambda^2}\exp\{-\frac{1}{2}[|z|^2(\frac{1}{\sigma_\lambda^2} + \frac{\alpha_\lambda^2}{h_\lambda^2} - \frac{1}{\sigma_0^2} - \frac{\alpha_0^2}{h_0^2} - \lambda) \\ &\quad + |\xi|^2(\frac{1}{h_\lambda^2} - \frac{1}{h_0^2}) - 2z\cdot\xi(\frac{\alpha_\lambda}{h_\lambda^2} - \frac{\alpha_0}{h_0^2})]\}. \end{aligned}$$

We denote

$$A_\lambda = \frac{e^{-\lambda}\sigma_0^2 h_0^2}{\sigma_\lambda^2 h_\lambda^2};\; B_\lambda = \frac{1}{\sigma_\lambda^2} + \frac{\alpha_\lambda^2}{h_\lambda^2} - \frac{1}{\sigma_0^2} - \frac{\alpha_0^2}{h_0^2} - \lambda;$$

$$C_\lambda = \frac{\alpha_\lambda}{h_\lambda^2} - \frac{\alpha_0}{h_0^2};\; D_\lambda = \frac{1}{h_\lambda^2} - \frac{1}{h_0^2}.$$

From the expressions of α_λ, σ_λ^2 and h_λ^2, we obtain:

$$\text{(2.l)} \quad \begin{aligned} A_\lambda &= \frac{\lambda^4}{12(\lambda\sinh\lambda - 2(\cosh\lambda - 1))};\; B_\lambda = \frac{\lambda(\lambda\cosh\lambda - \lambda)}{\lambda\sinh\lambda - 2(\cosh\lambda - 1)} - 4; \\ C_\lambda &= \frac{\lambda^2(\cosh\lambda - 1)}{\lambda\sinh\lambda - 2(\cosh\lambda - 1)} - 6;\; D_\lambda = \frac{\lambda^3\sinh\lambda}{\lambda\sinh\lambda - 2(\cosh\lambda - 1)} - 12. \end{aligned}$$

We may now derive the joint law of (V_G, G, Z_1). In the next proposition, we give the Laplace transform of $(\int_0^1 |Z_s|^2 ds, |G|^2, |Z_1|^2)$ which was computed by Chan-Dean-Jansons-Rogers [2].

Proposition 1: *1) For* $\lambda > 0$, $\rho, \sigma \geq 0$, *with* ρ *and* σ *small enough, we have:*

$$E[\exp - \frac{\lambda^2}{2}\{\int_0^1 |Z_s|^2 ds - \rho\, |G|^2 - \sigma\, |Z_1|^2\}]$$

(2.m)

$$= ((1-\rho)\cosh\lambda + \rho\frac{\sinh\lambda}{\lambda} + \sigma[(\rho-1)\lambda\sinh\lambda - 2\rho(\cosh\lambda - 1)])^{-1}.$$

2) *In particular, the conditional law of* V_G *given* Z_1 *is characterized by:*

$$E[\exp - \frac{\mu^2}{2} V_G \big| Z_1 = z] = \frac{\mu^2}{2(\cosh\mu - 1)} \exp\frac{-|z|^2}{2}\left(\frac{\mu\sinh\mu}{2(\cosh\mu - 1)} - 1\right)$$

(2.n)

$$= \frac{(\frac{\mu}{2})^2}{(\sinh\frac{\mu}{2})^2} \exp\frac{-|z|^2}{2}\left(\frac{\mu}{2}\coth\frac{\mu}{2} - 1\right).$$

Proof:

1) Let

$$\phi_\lambda(z,\xi) = E[\exp - \frac{\lambda^2}{2}\int_0^1 |Z_t|^2 dt \big| Z_1 = z, G = \xi]$$

$$= A_\lambda \exp - \frac{1}{2}\{B_\lambda\, |z|^2 - 2C_\lambda\, z\cdot\xi + D_\lambda\, |\xi|^2\}$$

where A_λ, B_λ, C_λ, D_λ are given by (2.1).

We now consider

$$F \stackrel{\text{def}}{=} E[\exp - \frac{\lambda^2}{2}(\int_0^1 |Z_t|^2 dt - \rho|G|^2 - \sigma|Z_1|^2)]$$

$$= \int \phi_\lambda(z,\xi) \exp - \frac{\lambda^2}{2}(-\rho\, |\xi|^2 - \sigma\, |z|^2) P_{(Z_1,G)}(dz, d\xi).$$

From (2.j), we have:

$$P_{(Z_1,G)}(dz, d\xi) = \frac{1}{(2\pi)^2\sigma_0^2 h_0^2}\exp\left(-\frac{1}{2}\left[\frac{|z|^2}{\sigma_0^2} + \frac{|\xi - \alpha_0 z|^2}{h_0^2}\right]\right) dz\, d\xi$$

$$= \frac{1}{(2\pi)^2\frac{1}{12}}\exp\left(-\frac{1}{2}\left[|z|^2 + \frac{|\xi - \frac{z}{2}|^2}{\frac{1}{12}}\right]\right) dz\, d\xi.$$

Hence,

$$F = \int_{\mathbb{C}^2} \frac{12A_\lambda}{(2\pi)^2} \exp - \frac{1}{2} ({}^t Z \Gamma^{-1} Z)\, dz\, d\xi$$

with $Z = \begin{pmatrix} z_1 \\ \xi_1 \\ z_2 \\ \xi_2 \end{pmatrix}$, $\Gamma^{-1} = \begin{pmatrix} a_\lambda & b_\lambda & 0 & 0 \\ b_\lambda & c_\lambda & 0 & 0 \\ 0 & 0 & a_\lambda & b_\lambda \\ 0 & 0 & b_\lambda & c_\lambda \end{pmatrix}$, and: $\begin{aligned} a_\lambda &= B_\lambda + 4 - \sigma\lambda^2; \\ b_\lambda &= \frac{C_\lambda}{2} - 6; \\ c_\lambda &= D_\lambda + 12 - \rho\lambda_2. \end{aligned}$

So, $F = 12A_\lambda \sqrt{\det \Gamma} = 12A_\lambda \dfrac{1}{a_\lambda c_\lambda - b_\lambda^2}$. By a straightforward computation, we then obtain (2.m).

2) To prove (2.n), we may either use (2.m) for $\rho = 1$ and then condition with respect to $\frac{1}{2}|Z_1|^2$, which is exponentially distributed, or we may integrate $E[\exp - \frac{\lambda^2}{2}\int_0^1 ds\, |Z_s - G|^2 \,|\, Z_1 = z,\ G = \xi]$, which is given by (2.b) and (2.k), with respect to the conditional law of G given Z_1; this latter conditional law is, by (2.i) and (2.j), equal to:

$$P_{G/Z_1}(d\xi \,|\, z) = \frac{1}{2\pi h_0^2}\, e^{-\frac{1}{2h_0^2}|\xi - \alpha_0 z|^2}\, d\xi .$$

The details of the computation are left to the reader.

3. **An application to the Brownian radius of gyration.**

We now want to derive a result of Helfer-Zhongxin [7] from formula (2.n). In [7], the authors define the radius of gyration of a Brownian path $B = (B_1, \ldots, B_n)$ in $\mathbb{R}^n$ by

$$T_{i,j} = \int_0^1 B_i(t) B_j(t)\, dt - \left(\int_0^1 B_i(t)\, dt\right)\left(\int_0^1 B_j(t)\, dt\right) \equiv \int_0^1 (B_i(t) - G_i)(B_j(t) - G_j)\, dt$$

where G denotes the centre of gravity of B over the time interval $[0,1]$, and $1 \le i,j \le n$.

We want to characterize the law of the symmetric, positive matrix $(T_{i,j})_{1 \le i,j \le n}$, and the conditional law for closed paths, i.e.: its law for the Brownian bridge. Formula (2.n) gives the answer in the case $n = 1$:

$$E[\exp - \frac{\lambda^2}{2} T \,|\, B(1) = 0] = \frac{\frac{\lambda}{2}}{(\sinh \frac{\lambda}{2})}$$

(Note that we are now dealing with the real valued Brownian motion rather than with complex Brownian motion as in section 2).

We also have the formula $E[\exp -\frac{\lambda^2}{2}T] = \sqrt{\frac{\lambda}{\sinh\lambda}}$. For the general case, let $\sigma = (\sigma_{i,j})$ be a symmetric positive matrix; we want to compute:

$$E[\exp -\frac{1}{2}\sum_{i,j}\sigma_{i,j}T_{i,j} \mid B(1) = l] \qquad (l \in \mathbb{R}^n).$$

Let α be the symmetric positive square root of σ. α is diagonalizable, so: $\alpha = p^{-1}\delta p$ with p orthogonal and δ diagonal. Then, $\sigma = p^{-1}\delta^2 p$. We define $M(s) = \alpha B(s)$; then, if we denote by $x \cdot y$ the euclidean scalar product of x and y, two generic vectors of $\mathbb{R}^n$, we have:

$$\sum_{i,j}\sigma_{i,j}T_{i,j} = \int_0^1 (B(s) - G) \cdot \sigma(B(s) - G)\,ds$$

$$= \int_0^1 |\alpha(B(s) - G)|^2 ds = \int_0^1 |M(s) - \int_0^1 M(u)\,du|^2 ds.$$

Using $M(s) = p^{-1}\delta p B(s)$ and denoting by $\bar{B}$ the Brownian motion pB, we obtain:

$$\sum\sigma_{i,j}T_{i,j} = \int_0^1 |\delta(\bar{B}(s) - \int_0^1 \bar{B}(u)\,du)|^2 ds = \sum_{i=1}^n \lambda_i^2 \int_0^1 (\bar{B}_i(s) - \bar{G}_i)^2 ds$$

where λ_i^2 are the eigenvalues of σ. Now, we have:

$$E[\exp -\frac{1}{2}\sum_{i,j}\sigma_{i,j}T_{i,j} \mid B_1 = l] = E[\exp -\frac{1}{2}\sum_i \lambda_i^2 \int_0^1 (\bar{B}_i(s) - \bar{G}_i)^2 ds \mid \bar{B}(1) = pl].$$

Using the independence of the components of $\bar{B}$, we may now apply formula (2.n), although in its one dimensional version, so that the last quantity appears to be equal to

$$\text{(3.a)} \qquad \prod_{i=1}^n \frac{(\frac{\lambda_i}{2})}{\sinh(\frac{\lambda_i}{2})} \exp\left(-\frac{1}{2}\sum_{i=1}^n ((Pl)_i)^2 \left(\frac{\lambda_i}{2}\coth\frac{\lambda_i}{2} - 1\right)\right).$$

The formula (3.a) is given in [7] only for $l = 0$. We also obtain, in the same way:

$$\text{(3.b)} \qquad E[\exp -\frac{1}{2}\sum_{i,j}\sigma_{i,j}T_{i,j}] = \prod_{i=1}^n \sqrt{\frac{\lambda_i}{\sinh\lambda_i}}.$$

4. The covariance of correlated Brownian motions.

Let $(B_t, t \geq 0)$ and $(B'_t, t \geq 0)$ be two independent real valued Brownian motions starting from 0. In [5], the authors obtained the following identities: for small λ, (4.a) $$E\left[\exp \lambda^2 \int_0^1 ds\, B_s B_s'\right] = [(\cos\lambda)(\cosh\lambda)]^{-1/2}$$

(4.b) $$E\left[\exp \lambda^2 \int_0^1 ds\, (B_s - G)(B_s' - G')\right] = E\left[\exp \lambda^2 \int_0^1 ds\, \tilde{B}_s \tilde{B}_s'\right] = \frac{\lambda}{[(\sin\lambda)(\sinh\lambda)]^{1/2}}$$

where $G = \int_0^1 B_s ds$, $G' = \int_0^1 B_s' ds$ and $\tilde{B}$ and $\tilde{B}'$ are two independent standard Brownian bridges. Chou and Nualart [3] extended these two results when B and B′ are linearly correlated; we now present this extension, as taken from [3]:

Proposition 2: *Let* B *and* B′ *be two Brownian motions, starting from 0 with correlation* ρ *(therefore,* $|\rho| \le 1$*), i.e.:*

$$B_s' = \rho B_s + \sqrt{1-\rho^2}\, B_s'', \quad 0 \le s \le 1,$$

where B *and* B″ *are independent Brownian motions. In the same way, let* $\tilde{B}$ *and* $\tilde{B}'$ *be two correlated standard Brownian bridges with* $\tilde{B}_s' = \rho \tilde{B}_s + \sqrt{1-\rho^2}\, \tilde{B}_s''$, *where* $\tilde{B}$ *and* $\tilde{B}''$ *are two independent Brownian bridges. Then,*

(4.c) $$E\left[\exp \lambda^2 \int_0^1 B_s B_s' ds\right] = [(\cos \lambda\sqrt{1+\rho})(\cosh\lambda\sqrt{1-\rho})]^{-1/2}$$

and

$$E\left[\exp \lambda^2 \int_0^1 (B_s - G)(B_s' - G')\, ds\right] \overset{(i)}{=} E\left[\exp \lambda^2 \int_0^1 \tilde{B}_s \tilde{B}_s' ds\right]$$

(4.d) $$\overset{(ii)}{=} \frac{(\lambda^2\sqrt{1-\rho^2})^{1/2}}{[(\sin\lambda\sqrt{1+\rho})(\sinh\lambda\sqrt{1-\rho})]^{1/2}}.$$

Chou and Nualart [3] prove this result using a diagonalization procedure.

We now show that we can recover (4.c) and (4.d) in a very simple way from the Laplace transforms of $\int_0^1 B_s^2 ds$ and $\int_0^1 \tilde{B}_s^2 ds$.

First, we notice that the Laplace transform obtained in (4.c) is the product of two Laplace transforms, namely:

$$(\cos\lambda\sqrt{1+\rho})^{-1/2} = E\left[\exp \frac{\lambda^2}{2}(1+\rho)\int_0^1 ds\, B_s^2\right]$$

and

$$(\cosh\lambda\sqrt{1-\rho})^{-1/2} = E\left[\exp - \frac{\lambda^2}{2}(1-\rho)\int_0^1 ds\, B_s^2\right].$$

Thus, (4.c) is equivalent to the identity in law:

$$\int_0^1 B_s B_s' ds \overset{(law)}{=} \frac{1}{2}\{(1+\rho)\int_0^1 \beta_s^2 ds - (1-\rho)\int_0^1 \gamma_s^2 ds\} \tag{4.e}$$

where β and γ are two independent Brownian motions. However, (4.e) may be proved directly using an orthogonal transformation of the planar Brownian motion (B, B''). Indeed, if we define the matrix $R_\mu = \begin{pmatrix} \mu & \sqrt{1-\mu^2} \\ \sqrt{1-\mu^2} & -\mu \end{pmatrix}$, with

$$\mu = \sqrt{\frac{1+\rho}{2}} \in [0,1], \tag{4.f}$$

then R_μ is orthogonal, and thus $\begin{pmatrix} X \\ Y \end{pmatrix} = R_\mu \begin{pmatrix} B \\ B'' \end{pmatrix}$ is a planar Brownian motion. Therefore, we have,

$$\int_0^1 B_s B_s' ds = \rho \int_0^1 B_s^2 ds + \sqrt{1-\rho^2}\int_0^1 ds\, B_s B_s'' \overset{(law)}{=} \rho \int_0^1 X_s^2 ds + \sqrt{1-\rho^2}\int_0^1 ds\, X_s Y_s$$

so that the left-hand side of (4.e) is equal, in law, to:

$$\rho \int_0^1 (\mu B_s + \sqrt{1-\mu^2} B_s'')^2 ds + \sqrt{1-\rho^2}\int_0^1 ds\, (\mu B_s + \sqrt{1-\mu^2} B_s'')(\sqrt{1-\mu^2} B_s - \mu B_s'')$$

$$= \int_0^1 (\rho\mu^2 + \sqrt{1-\rho^2}\,\mu\sqrt{1-\mu^2}) B_s^2 ds + \int_0^1 (\rho(1-\mu^2) - \sqrt{1-\rho^2}\,\mu\sqrt{1-\mu^2})(B_s'')^2 ds$$

$$+ \int_0^1 (2\rho\mu\sqrt{1-\mu^2} + \sqrt{1-\rho^2}(1-2\mu^2)) B_s B_s'' ds$$

$$= \frac{1}{2}(\rho+1)\int_0^1 B_s^2 ds - \frac{1}{2}(1-\rho)\int_0^1 (B_s'')^2 ds, \text{ using (4.f)}.$$

Thus, (4.e) holds and, as a consequence, we obtain (4.c).

Using the same transformation R_μ, we have the corresponding identity for the Brownian bridges:

$$\int_0^1 \tilde{B}_s \tilde{B}_s' ds \overset{(law)}{=} \frac{1}{2}\{(\rho+1)\int_0^1 \tilde{\beta}_s^2 ds - (1-\rho)\int_0^1 \tilde{\gamma}_s^2 ds\}. \tag{4.g}$$

To prove the identity (i) in (4.d), we still use the transformation R_μ and the identity in law $\int_0^1 (B_s - G)^2 ds \overset{(law)}{=} \int_0^1 \tilde{B}_s^2 ds$. (4.d) is then proved using (4.g) and the well-known formula:

$$E[\exp \frac{\lambda^2}{2}\int_0^1 ds\, \tilde{B}_s^2] = \left[\frac{\lambda}{\sin\lambda}\right]^{1/2}.$$

Moreover, the above method enables us to compute easily the conditional law of $\int_0^1 B_s B_s' ds$ given B_1 and B_1', which extends the celebrated Paul Lévy formulae (1.d)-(1.e).

Proposition 3: *Let* B *and* B′ *be defined as in the preceding proposition. We suppose* $|\rho| < 1$. *Then, we have, for small enough* λ*:*

$$E[\exp \lambda^2 \int_0^1 B_s B_s' ds \,|\, B_1 = x, B_1' = y]$$

(4.h)
$$= \left[\frac{\lambda^2 \sqrt{1-\rho^2}}{(\sin \lambda \sqrt{1+\rho})(\sinh \lambda \sqrt{1-\rho})} \right]^{1/2} \exp(-C),$$

where:

$$C = \frac{(x+y)^2}{4(1+\rho)} (\lambda \sqrt{1+\rho} \operatorname{cotg}(\lambda \sqrt{1+\rho}) - 1) + \frac{(x-y)^2}{2} (\lambda \sqrt{1-\rho} \coth(\lambda \sqrt{1-\rho}) - 1).$$

Proof:

$$E[\exp \lambda^2 \int_0^1 B_s B_s' ds \,|\, B_1 = x, B_1' = y]$$

$$= E[\exp \lambda^2 (\rho \int_0^1 B_s^2 ds + \sqrt{1-\rho^2} \int_0^1 B_s B_s'' ds) \,|\, B_1 = x, B_1'' = \frac{y - \rho x}{\sqrt{1-\rho^2}}]$$

$$= E\left[\exp \frac{\lambda^2}{2} \{(\rho+1) \int_0^1 B_s^2 ds - (1-\rho) \int_0^1 B_s''^2 ds\} \,\middle|\, \begin{array}{l} \mu B_1 + \sqrt{1-\mu^2} B_1'' = x, \\ \sqrt{1-\mu^2} B_1 - \mu B_1'' = \dfrac{y - \rho x}{\sqrt{1-\rho^2}} \end{array} \right]$$

(using the same transformation R_μ as before)

$$\text{(4.i)} = E\left[\exp \frac{\lambda^2}{2} \{(\rho+1) \int_0^1 B_s^2 ds - (1-\rho) \int_0^1 B_s''^2 ds \,\middle|\, \begin{array}{l} B_1 = \mu x + \sqrt{1-\mu^2} \dfrac{y - \rho x}{\sqrt{1-\rho^2}} \\ B_1'' = \sqrt{1-\mu^2} x - \mu \dfrac{y - \rho x}{\sqrt{1-\rho^2}} \end{array} \right]$$

Since $\mu = \sqrt{1 + \frac{\rho}{2}}$ (see: (4.f)), then the values of B_1 and B_1'' in (4.i) are:

(4.j)
$$\begin{cases} B_1 = \left[\sqrt{\dfrac{1+\rho}{2}} - \rho \sqrt{\dfrac{1-\rho}{2}} \dfrac{1}{\sqrt{1-\rho^2}} \right] x + \sqrt{\dfrac{1-\rho}{2(1-\rho^2)}} y = \dfrac{1}{\sqrt{2(1+\rho)}} (x+y) \\ B_1'' = \dfrac{1}{\sqrt{2(1-\rho)}} (x-y). \end{cases}$$

Now, (4.h) follows from (4.i)-(4.j) and Lévy's formula:

$$E[\exp - \frac{\lambda^2}{2} \int_0^1 B_s^2 ds \,|\, B_1 = x] = \sqrt{\frac{\lambda}{\sinh \lambda}} \exp[- \frac{x^2}{2} (\lambda \coth \lambda - 1)],$$

which is the one-dimensional version of (1.e).

5. The stochastic area of a three-dimensional Brownian motion.

The aim of this section is to bring together some results of Berthuet [1] on one hand, and Foschini-Shepp [6] on the other hand, which concern the study of the distribution of the 6-dimensional random variable:

$$(B(1), S(1)),$$

where $(B(t), t \geq 0)$ is a 3-dimensional Brownian motion starting from 0, and $S(t) = \int_0^t B(s) \times dB(s)$, and to show how these results may be deduced from Lévy's formula (1.d).

Berthuet [1] is more particularly interested in the quantity:

$$V = \iiint_{0<t_1<t_2<t_3<1} \det(dB(t_3), dB(t_2), dB(t_1)),$$

and he notes that:

(5.a) $$V = B(1) \cdot S(1).$$

The main result in Foschini and Shepp [6] is now presented as formula (5.c).

Proposition 4: *1) For every* $m \in \mathbb{R}^3$, $\xi \in \mathbb{R}^3$, *with* $|\xi| = 1$, *and* $\lambda \in \mathbb{R}$, $\lambda \neq 0$, *one has*

(5.b) $$E[\exp(i\lambda\xi \cdot S(1)) \,|\, B(1) = m] = \left(\frac{\lambda}{\sinh\lambda}\right) \exp \frac{|m|^2 - (\xi \cdot m)^2}{2}(1 - \lambda\coth\lambda)$$

2) For every $z \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$, $\xi \in \mathbb{R}^3$, *with* $|\xi| = 1$, *one has:*

(5.c) $$E[\exp i(z \cdot B(1) + \lambda\xi \cdot S(1))] = \left(\frac{1}{\cosh\lambda}\right) \exp - \frac{1}{2}\left(|z|^2 \frac{\tanh\lambda}{\lambda} + (z \cdot \xi)^2 (1 - \frac{\tanh\lambda}{\lambda})\right).$$

Proof: 1) We assume $m \neq 0$, and we introduce $m' = \frac{m}{|m|}$. Let $\xi = pm' + qm''$ be the orthogonal decomposition of ξ with respect to m, with $|m''| = 1$, and let $\eta = m' \times m''$; then, the triple $(\xi, \eta, \xi \times \eta)$ is an orthonormal basis of $\mathbb{R}^3$; elementary vector calculus shows that:

$$\xi \cdot \int_0^1 B(s) \times dB(s) = \int_0^1 (-B(s) \cdot (\xi \times \eta)) d_s(\eta \cdot B(s)) + (B(s) \cdot \eta) d_s((\xi \times \eta) \cdot B(s))$$

and formula (5.b) now follows from Lévy's formula (1.d).

2) As a consequence of formula (5.b), and writing m in a *fixed* orthonormal basis $(\xi, \eta_0, \xi \times \eta_0)$, we obtain, with the notation:

$$\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) = (z\cdot\xi,\ z\cdot\eta_0,\ z\cdot(\xi \times \eta_0)):$$

$E[\exp i(z\cdot B(1) + \lambda\xi\cdot S(1))]$

$$= \frac{\lambda}{(\sinh\lambda)} \int_{\mathbb{R}^3} \frac{dn_1 dn_2 dn_3}{(2\pi)^{3/2}} \exp\{i\tilde{z}\cdot n + \frac{n_2^2 + n_3^2}{2}(1 - \lambda\coth\lambda) - \frac{|n|^2}{2}\}$$

$$= \frac{\lambda}{(\sinh\lambda)}(\exp - \frac{1}{2}\{\tilde{z}_1^2 + (\tilde{z}_2^2 + \tilde{z}_3^2)(\frac{\tanh\lambda}{\lambda})\})\frac{\tanh\lambda}{\lambda}$$

$$= \frac{1}{(\cosh\lambda)} \exp - \frac{1}{2}\{|z|^2(\frac{\tanh\lambda}{h}) + (z\cdot\xi)^2(1 - \frac{\tanh\lambda}{\lambda})\};$$

formula (5.c) is proven.

□

From formula (5.b), it is easy, by making the same sort of manipulation as in the proof of Proposition 4, to obtain a complete, and relatively simple, description of the law of the random vector:

(5.d) $$(B(1); \theta(1)\cdot S(1), \theta(1) \times S(1), \gamma(1))$$

where: $\theta(1) = \frac{B(1)}{|B(1)|}$ is the angular part of $B(1)$, and $\gamma(1) = S(1) - (\theta(1)\cdot S(1))\theta(1)$ is the component of $S(1)$ which is orthogonal to $\theta(1)$. In the following proposition, the proof of which is very similar to that of Proposition 4, we present a (slightly incomplete) description of the law of the vector (5.d).

Proposition 5: *1) The variables* $B(1)$ *and* $\theta(1)\cdot S(1)$ *are independent, and moreover, for every* $\lambda \in \mathbb{R}$, $\lambda \neq 0$, *one has:*

(5.e) $$E[\exp i\lambda(\theta(1)\cdot S(1))] = \frac{\lambda}{\sinh\lambda}$$

2) Conditionally on $B(1) = m$, *the variables* $\theta(1) \times S(1)$ *and* $\gamma(1)$ *have the same distribution (but are not independent). Let* Y *denote either of these variables, and* $\xi \in \mathbb{R}^3$ *satisfy:* $|\xi| = 1$, *and* $\xi \cdot m = 0$. *Then, we have for every* $\nu \in \mathbb{R}$, $\nu \neq 0$:

(5.f) $$E[\exp i\nu(\xi \cdot Y) | B(1) = m] = \left(\frac{\nu}{\sinh\nu}\right) \exp \frac{|m|^2}{2}(1 - \nu\coth\nu).$$

We leave it to the interested reader to deduce the complete characteristic function of

$$(\theta(1) \cdot S(1), \theta(1) \times S(1), \gamma(1)),\ \text{given } B(1) = m,$$

from formula (5.b). The assertions of Proposition 5 may then be read off from this complete characteristic function.

To conclude on this topic, we remark that Berthuet's result about V (see [1]) may be deduced from formula (5.a) and the first assertion of Proposition 5, which, put together, imply:

$$E[\exp(i\lambda V)] = E\left[\frac{\lambda|B(1)|}{\sinh(\lambda|B(1)|)}\right].$$

In a different direction, Price-Rogers-Williams [10] obtain a skew-product representation of the stochastic area process $(\int_0^t B(s) \times dB(s), t \geq 0)$.

□

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