KILLED BROWNIAN MOTION WITH A PRESCRIBED LIFETIME DISTRIBUTION AND MODELS OF DEFAULT

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ABSTRACT. The inverse first passage time problem asks whether, for a Brownian motion B and a nonnegative random variable ζ, there exists a time-varying barrier b such that $\mathbb{P}\{B_s > b(s), 0 \le s \le t\} = \mathbb{P}\{\zeta > t\}$. We study a "smoothed" version of this problem and ask whether there is a "barrier" b such that $\mathbb{E}[\exp(-\lambda \int_0^t \psi(B_s - b(s))\,ds)] = \mathbb{P}\{\zeta > t\}$, where λ is a killing rate parameter and $\psi : \mathbb{R} \to [0,1]$ is a non-increasing function. We prove that if ψ is suitably smooth, the function $t \mapsto \mathbb{P}\{\zeta > t\}$ is twice continuously differentiable, and the condition $0 < -\frac{d \log \mathbb{P}\{\zeta > t\}}{dt} < \lambda$ holds for the hazard rate of ζ, then there exists a unique continuously differentiable function b solving the smoothed problem. We show how this result leads to flexible models of default for which it is possible to compute expected values of contingent claims.

1. INTRODUCTION

Investors are exposed to *credit risk*, or *counterparty risk*, due to the possibility that one or more counterparties in a financial agreement will default; that is, not honor their obligations to make certain payments. Counterparty risk has to be taken into account when pricing a transaction or portfolio, and it is necessary to model the occurrence of default jointly with the behavior of asset values.

The default time is sometimes modeled as the first passage time of a credit index process below a barrier. Black and Cox [BC76] were among the first to use this approach. They define the time of default as the first time the ratio of the value of a firm and the value of its debt falls below a constant level, and they model debt as a zero-coupon bond and the value of the firm as a geometric Brownian motion. In this case, the default time has the distribution of the first-passage time of a Brownian motion (with constant drift) below a certain barrier.

Hull and White [HW01] model the default time as the first time a Brownian motion hits a given time-dependent barrier. They show that this model gives the correct market credit default swap and bond prices if the time-dependent barrier is chosen so that the first passage time of the Brownian motion has a certain distribution derived from those prices. Given a distribution for the default time, it is usually impossible to find a closed-form expression for the corresponding time-dependent barrier, and numerical methods have to be used.

We adopt a perspective similar to that of [HW01]. Namely, we model the default time as

$$(1.1) \qquad \tau := \inf \left\{ t > 0 : \lambda \int_0^t \psi(Y_s - b(s)) \, ds > U \right\}$$

where the diffusion Y is some credit index process, U is an independent mean one exponentially distributed random variable, $0 \le \psi \le 1$ is a suitably smooth, non-increasing function with $\lim_{x \to -\infty} \psi(x) = 1$ and $\lim_{x \to +\infty} \psi(x) = 0$, and $\lambda > 0$ is a rate parameter. Then,

$$(1.2) \qquad \mathbb{P}\{\tau > t\} = \mathbb{E}\left[\exp\left(-\lambda \int_0^t \psi(Y_s - b(s)) \, ds \right) \right].$$

The random time τ is a "smoothed-out" version of the stopping time of Hull and White; instead of of killing Y as soon at it crosses some sharp, time-dependent boundary, we kill Y at rate $\lambda\psi(y - b(t))$ if it is in state $y \in \mathbb{R}$ at time $t \ge 0$. That is,

$$\lim_{\Delta t \downarrow 0} \mathbb{P}\{\tau \in (t, t + \Delta t) \mid (Y_s)_{0 \le s \le t}, \, \tau > t\}/\Delta t = \lambda\psi(Y_t - b(t)).$$

When the credit index value Y_t is large, corresponding to a time t when the counterparty is in sound financial health, the killing rate $\lambda\psi(Y_t - b(t))$ is close to 0 and default in an ensuing short period of time is unlikely, whereas the killing rate is close to its maximum possible value, λ, when Y_t is low and default is more probable. Note that if we consider a family of $[0, 1]$-valued, non-increasing functions ψ that converges to the indicator function of the set $\{x \in \mathbb{R} : x < 0\}$ and λ tends to ∞, then the corresponding stopping time τ converges to the Hull and White stopping time $\inf\{t > 0 : Y_t < b(t)\}$.

The hazard rate of the random time τ is

$$
\begin{aligned}
(1.3) \qquad \frac{\mathbb{P}\{\tau \in dt \mid \tau > t\}}{dt} \quad &:= \quad \lim_{\Delta t \downarrow 0} \frac{\mathbb{P}\{\tau \in (t, t + \Delta t)\}}{\Delta t \mathbb{P}\{\tau > t\}} \\
&= \quad \lim_{\Delta t \downarrow 0} \frac{\mathbb{P}\left\{ \lambda \int_0^t \psi(Y_s - b(s)) ds \le U \le \lambda \int_0^{t+\Delta t} \psi(Y_s - b(s)) \, ds \right\}}{\Delta t \mathbb{P}\left\{ \lambda \int_0^t \psi(Y_s - b(s)) \, ds \le U \right\}} \\
&= \quad \lim_{\Delta t \downarrow 0} \frac{\mathbb{E}\left[e^{-\lambda \int_0^t \psi(Y_s - b(s)) \, ds} - e^{-\lambda \int_0^{t+\Delta t} \psi(Y_s - b(s)) \, ds} \right]}{\Delta t \mathbb{E}\left[\exp\left(-\lambda \int_0^t \psi(Y_s - b(s)) \, ds \right) \right]} \\
&= \quad \frac{\lambda \mathbb{E}\left[\psi(Y_t - b(t)) \exp\left(-\lambda \int_0^t \psi(Y_s - b(s)) \, ds \right) \right]}{\mathbb{E}\left[\exp\left(-\lambda \int_0^t \psi(Y_s - b(s)) \, ds \right) \right]}.
\end{aligned}
$$

On the other hand, suppose that ζ is a non-negative random variable with survival function $t \mapsto G(t) := \mathbb{P}\{\zeta > t\}$. Writing g for the derivative of G, the corresponding hazard rate is

$$-\frac{g(t)}{G(t)} = -\frac{d}{dt} \log G(t).$$

As a result, a necessary condition for a function b to exist such that the corresponding random time τ has the same distribution as ζ is that

$$(1.4) \qquad 0 < -g(t) < \lambda G(t), \, t \ge 0.$$

We show in Theorem 2.1 that if Y is a Brownian motion with a given suitable random initial condition, the assumption (1.4) holds, and the survival function G is twice continuously differentiable, then there is a unique differentiable function b such that the stopping time τ has the same distribution as ζ. In particular, we establish that the function b can be determined by solving a system consisting of a parabolic linear PDE with coefficients depending on b and a non-linear ODE for b with coefficients depending on the solution of the PDE. Note from (1.2) that changing the function b on a set with Lebesgue measure zero does not affect the distribution of τ, and so we have to be careful when we talk about the uniqueness of b. This minor annoyance does not appear if we restrict to continuous b.

In Theorem 4.1 we give an analogue of the existence part of the above result when ψ is the indicator of the set $\{x \in \mathbb{R} : x < 0\}$ and establish a partial uniqueness result.

Having proven the existence and uniqueness of a barrier b, we consider the pricing of certain contingent claims in Section 5. For simplicity, we take the asset price $(X_t)_{t \geq 0}$ to be a geometric Brownian motion

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t,$$

where W is a standard Brownian motion. We take the credit index $(Y_t)_{t \geq 0}$ to be given by

$$dY_t = dB_t$$

where B is another standard Brownian motion, and take the default time to be given by (1.1), where the exponential random variable U is independent of the asset price X and the credit index Y. We assume that the Brownian motions W and B are correlated; that is, that their covariation is $[B, W]_t = \rho t$ for some constant $\rho \in [-1, 1]$. We consider claims with a payoff of the form $F(X_T)1\{\tau > T\}$ for some fixed maturity T. We show how it is possible to compute conditional expected values such as

$$\mathbb{E}\left[F(X_T)1\{\tau > T\} \,\Big|\, (X_s)_{0 \leq s \leq t}, \, \tau > t \right].$$

In Section 6 we report the results of some experiments where we solved the PDE/ODE system for the barrier b numerically. Lastly, in Section 7, we follow [DP11] to demonstrate how it is possible to use market data on credit default swap prices to determine the survival function G.

1.1. The FPT and IFPT problems.
We end the this introduction with a brief discussion of the literature dealing with first passage times of diffusions across time-dependent barriers.

Consider a Brownian motion $(B_t)_{t \geq 0}$ defined on a filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$ which satisfies the usual conditions. Define the diffusion $(Y_t)_{t \geq 0}$ via the SDE

$$dY_t = \mu(Y_t, t) \, dt + \sigma(Y_t, t) \, dB_t,$$

where we assume that the coefficients $\mu : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ and $\sigma : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ are such that the SDE has a unique strong solution.

For a Borel function $b : \mathbb{R}_+ \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\pm \infty\}$, the first passage time of the diffusion process Y below the barrier b is the stopping time

$$(1.5) \qquad\qquad \tilde{\tau} = \inf\{t > 0 : Y_t < b(t)\}.$$

The following two problems related to this notion have been discussed in the literature.

The First Passage Time problem (FPT): For a given barrier $b : \mathbb{R}_+ \to \bar{\mathbb{R}}$, compute the survival function G of the first time that X goes below b; that is, find

$$(1.6) \qquad G(t) := \mathbb{P}\{\tilde{\tau} > t\}, \quad t \geq 0.$$

The Inverse First Passage Time problem (IFPT): For a given survival function G, does there exist a barrier b such that $G(t) = \mathbb{P}\{\tilde{\tau} > t\}$ for all $t \geq 0$?

A large class of first passage time problems may be solved within a PDE framework. Let $u(x,t) = \frac{\partial}{\partial x}\mathbb{P}\{Y_t \leq x, \tilde{\tau} > t\}$ be the sub-probability density of the diffusion Y killed at $\tilde{\tau}$. Then, by the Kolmogorov forward equation, u satisfies

$$(1.7) \qquad \begin{cases} u_t(x,t) = \frac{1}{2}(\sigma^2 u)_{xx} - (\mu u)_x, & x > b(t),\, t > 0, \\ u(x,t) = 0, & x \leq b(t),\, t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases}$$

where f is the probability density of Y_0. For nice enough functions b this system has a unique solution and we can find the survival probability

$$G(t) = \mathbb{P}\{\tilde{\tau} > t\} = \int_{b(t)}^{\infty} u(x,t)\,dx, \quad t \geq 0.$$

This approach is used in [Ler86, Val09] to get closed form solutions for some classes of boundaries. An integral equation technique is used in [Pes02, PS06, Val09] to find the derivative $g(t) = G'(t)$ in the FPT problem for a Brownian motion. Writing $\Psi(z) := \int_z^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$, the derivative g satisfies a Volterra integral equation of the first kind of the form

$$\Psi\left(\frac{b(t)}{\sqrt{t}}\right) = -\int_0^t \Psi\left(\frac{b(t) - b(s)}{\sqrt{t-s}}\right) g(s)\,ds.$$

This and other such integral equations can be used to find g numerically.

A. Shiryaev is generally credited with introducing the IFPT problem in 1976 (we have not been able to find an explicit reference). Most authors have investigated numerical methods for finding the boundary. Details can be found in [HW00, HW01, IK02, ZS09]. It is shown in [AZ01] that for sufficiently smooth boundaries the density $u(x,t)$ and the boundary $b(t)$ are a solution of the following free boundary problem

$$(1.8) \qquad \begin{cases} u_t(x,t) = \frac{1}{2}(\sigma^2 u)_{xx} - (\mu u)_x, & x > b(t),\, t > 0, \\ u(x,t) = 0, & x \leq b(t),\, t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ G(t) = \int_{b(t)}^{\infty} u(x,t)\,dx, & t \geq 0. \end{cases}$$

where f is again the probability density of Y_0. The existence and uniqueness of a viscosity solution of (1.8) is established in [CCCS11] along with upper and lower bounds on the asymptotic behavior of b. This paper also shows that this b does in fact produce a boundary that gives the survival function G. To our knowledge it

has not be proven that a strong solution to the system (1.8) exists, nor that there is a smooth b solving the IFPT.

A variation of the IFPT is studied in [DP11]. There the barrier is fixed at zero (i.e. $b \equiv 0$) and it is the volatility parameter $\sigma(\cdot, \cdot)$ that is allowed to vary. The authors show that this problem admits an explicit solution for every differentiable survival function.

2. Global Existence and Uniqueness

Suppose for the remainder of this paper that $Y_t := Y_0 + B_t$ where $(B_t)_{t \geq 0}$ is a standard Brownian motion and Y_0 is a random variable, independent of B and with density f. In this case, (1.2) is

$$G(t) = \int_{\mathbb{R}} \mathbb{E}\left[\exp\left(-\lambda \int_0^t \psi(x + B_z - b(z))\, dz\right)\right] f(x)\, dx$$

which, by time reversal, becomes

$$G(t) = \int_{\mathbb{R}} \mathbb{E}\left[\exp\left(-\int_0^t \psi(x + B_{t-z} - b(z))dz\right) f(x + B_t)\right] dx.$$

Set

(2.1) $$u(x,t) := \mathbb{E}\left[\exp\left(-\int_0^t \psi(x + B_{t-z} - b(z))du\right) f(x + B_t)\right].$$

That is, u is the sub-probability density of Y killed at the random time τ. It is well known that the function u is the unique solution of the PDE

$$\begin{cases} u_t(x,t) = \dfrac{1}{2} u_{xx}(x,t) - \lambda \psi(x - b(t)) u(x,t), & x \in \mathbb{R},\, t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}. \end{cases}$$

Any solution to this PDE satisfies

(2.2) $$\lim_{x \to \pm\infty} u(x,t) = \lim_{x \to \pm\infty} u_x(x,t) = 0, \quad t > 0.$$

Our question as to whether we can find a "barrier" b giving us the survival function G is now equivalent to whether the system

(2.3) $$\begin{cases} u_t(x,t) = \dfrac{1}{2} u_{xx}(x,t) - \lambda \psi(x - b(t)) u(x,t),\ x \in \mathbb{R},\ t > 0, \\ u(x,0) = f(x),\ x \in \mathbb{R}, \\ \displaystyle\int_{\mathbb{R}} u(x,t)\, dx = G(t),\ t \geq 0, \end{cases}$$

has solutions (u, b). Differentiating the third equation from (2.3) with respect to t and then using the first equation together with an integration by parts, we get that

(2.4) $$-g(t) = \lambda \int_{\mathbb{R}} \psi(x - b(t)) u(x,t)\, dx,$$

where we recall that $g(t) = G'(t)$. A second differentiation in t followed by another integration by parts yields

$$\begin{aligned} g'(t) - \lambda^2 \int_{\mathbb{R}} \psi^2(x - b(t)) u(x,t) dx &= \lambda \int_{\mathbb{R}} \psi_x(x - b(t)) u(x,t) b'(t)\, dx \\ &\quad - \lambda/2 \int_{\mathbb{R}} \psi(x - b(t)) u_{xx}(x,t)\, dx \end{aligned}$$

$$= \lambda \int_{\mathbb{R}} \psi_x(x - b(t))u(x,t)b'(t)\, dx$$

(2.5)
$$+ \lambda/2 \int_{\mathbb{R}} \psi_x(x - b(t))u_x(x,t)\, dx$$

$$= \lambda \int_{\mathbb{R}} \psi_x(x - b(t))u(x,t)b'(t)\, dx$$

$$- \lambda/2 \int_{\mathbb{R}} \psi_{xx}(x - b(t))u(x,t)\, dx.$$

Note that (2.5) may be rearranged to give an ODE for b of the form $b'(t) = \Theta(b(t), t)$, where the function Θ is constructed from the function u (which, of course, depends in turn on b). Re-writing this integral equation in the form $b(t) = b(0) + \int_0^t \Theta(b(s), s)\, ds$ leads to the following theorem, our main result.

Theorem 2.1. *Suppose the following.*
- *The survival function G is twice continuously differentiable with first and second derivatives g and g' and $0 < -g(t) < \lambda G(t)$ for all $t \geq 0$ for some constant $\lambda > 0$.*
- *The initial density f satisfies $\int_{\mathbb{R}} f(x)\, dx = 1$, $f(x) > 0$ for all $x \in \mathbb{R}$, $f \in C^2(\mathbb{R})$, and the functions f, f', f'' are bounded.*
- *The function ψ is non-increasing and belongs to $C^3(\mathbb{R})$, and for some $h > 0$, $\psi(x) = 1$ for $x \leq -h$ and $\psi(x) = 0$ for $x \geq h$.*

Then, there exists a unique continuously differentiable function b such that the following three equations hold

(2.6)
$$G(t) = \int_{\mathbb{R}} \mathbb{E}\left[\exp\left(-\lambda \int_0^t \psi(x + B_u - b(u))du\right)\right] f(x)\, dx,$$

(2.7)
$$-g(t) = \lambda \int_{\mathbb{R}} \mathbb{E}\left[\exp\left(-\lambda \int_0^t \psi(x + B_u - b(u))du\right) \psi(x + B_t - b(t))\right] f(x)\, dx,$$

and

$$b(t) = b(0) \quad + \quad \int_0^t \left(\frac{g'(s) - \lambda^2 \int_{\mathbb{R}} \mathbb{E}\left[\psi^2(x + B_s - b(s))e^{-\lambda \int_0^s \psi(x+B_r - b(r))dr}\right] f(x)\, dx}{\lambda \int_{\mathbb{R}} \mathbb{E}\left[\psi_x(x + B_s - b(s))e^{-\lambda \int_0^s \psi(x+B_r - b(r))dr}\right] f(x)\, dx} \right.$$

(2.8)
$$+ \quad \left. \frac{\lambda/2 \int_{\mathbb{R}} \mathbb{E}\left[\psi_{xx}(x + B_s - b(s))e^{-\lambda \int_0^s \psi(x+B_r - b(r))dr}\right] f(x)\, dx}{\lambda \int_{\mathbb{R}} \mathbb{E}\left[\psi_x(x + B_s - b(s))e^{-\lambda \int_0^s \psi(x+B_r - b(r))dr}\right] f(x)\, dx} \right) ds$$

for all $t \geq 0$.

Proof. From now on we assume for ease of notation that $\lambda = 1$. The modifications necessary for general λ are straightforward. The proof will be via a sequence of lemmas, all of them assuming the hypotheses of Theorem 2.1 (with $\lambda = 1$). We start with the following simple observation.

Lemma 2.2. *Suppose that*

$$G(t) = \int_{\mathbb{R}} u(x,t)\, dx$$

for some continuous function $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ such that $u(x,t) > 0$ for $x \in \mathbb{R}$, $t \geq 0$.
Then, for each $t \geq 0$ there exists a unique $b(t) \in \mathbb{R}$ such that

$$-g(t) = \int \psi(x - b(t))u(x,t)\,dx.$$

Proof. Set

$$F(t,z) = \int_{\mathbb{R}} \psi(x - z)u(x,t)\,dx.$$

Then,

$$\lim_{z \to -\infty} F(t,z) = \int_{\mathbb{R}} u(x,t)\,dx = G(t),$$

$$\lim_{z \to +\infty} F(t,z) = 0,$$

and, by assumption,

$$0 < -g(t) < G(t).$$

Furthermore, F is continuous and strictly decreasing in z. So, by the intermediate value property, we can find a unique $b(t) \in \mathbb{R}$ such that $F(t, b(t)) = -g(t)$. □

Lemma 2.3 (Global Uniqueness). *Suppose there exist continuous functions b_1, b_2 such that equations (2.6), (2.7) and (2.8) are satisfied for $b = b_1$ and $b = b_2$. Then, $b_1(t) = b_2(t)$ for all $t \geq 0$.*

Proof. It follows from Lemma 2.2, and (2.7) that $b(0)$ for any solution b is uniquely specified.

Suppose that b_1 and b_2 are two continuous solutions. We see from equation (2.8) that any continuous solution is continuously differentiable. Set $V = \inf\{t \geq 0 : b_1(t) \neq b_2(t)\}$. Then, $b_1(t) = b_2(t)$ for $0 \leq t \leq V$ and we can suppose without loss of generality that $b_1(t) > b_2(t)$ for $V < t < V + \epsilon$ for some $\epsilon > 0$. This clearly implies that

$$\int_{\mathbb{R}} \mathbb{E}\left[\exp\left(-\int_0^t \psi(x + B_u - b_1(u))du\right)\right] f(x)\,dx$$

$$< \int_{\mathbb{R}} \mathbb{E}\left[\exp\left(-\int_0^t \psi(x + B_u - b_2(u))du\right)\right] f(x)\,dx$$

for $V < t < V + \epsilon$, and so (2.6) cannot hold for both b_1 and b_2. □

Lemma 2.4 (Global Existence). *Define S to be the supremum of the set of T such that the equations (2.6), (2.7) and (2.8) have a continuous solution on $[0, T]$. Then, $S = +\infty$.*

Proof. Suppose to the contrary that $S < +\infty$. From Lemma 2.3, the equations have a unique solution on $[0, S)$. By time-reversal, equation (2.6) is equivalent to

$$(2.9) \qquad G(t) = \int_{\mathbb{R}} \mathbb{E}\left[\exp\left(-\int_0^t \psi(x + B_{t-u} - b(u))du\right) f(x + B_t)\right]\,dx.$$

Similarly, (2.7) is equivalent to

$$-g(t)$$
$$(2.10) \qquad = \int_{\mathbb{R}} \mathbb{E}\left[\exp\left(-\int_0^t \psi(x + B_{t-u} - b(u))\,du\right) \psi(x - b(t))f(x + B_t)\right]\,dx.$$

For $0 \le t < S$ put

$$(2.11) \qquad u(x,t) := \mathbb{E}\left[\exp\left(-\int_0^t \psi(x + B_{t-u} - b(u))du\right) f(x + B_t)\right].$$

Consider $t_1 < t_2 < \ldots \uparrow S$. It follows from the continuity of the sample paths of B that as $t_n \uparrow S$

$$\exp\left(-\int_0^{t_n} \psi(x + B_{t_n-u} - b(u))\, du\right) f(x + B_{t_n})$$

$$\to \exp\left(-\int_0^S \psi(x + B_{S-u} - b(u))\, du\right) f(x + B_S)$$

almost surely for each $x \in \mathbb{R}$, and so

$$u(x, t_n) \to \mathbb{E}\left[\exp\left(-\int_0^S \psi(x + B_{S-u} - b(u))\, du\right) f(x + B_S)\right] =: u(x, S).$$

Because

$$u(x, t) \le \mathbb{E}[f(x + B_t)],$$

it follows from dominated convergence that

$$\int_{\mathbb{R}} u(x, S)\, dx = \lim_n \int_{\mathbb{R}} u(x, t_n)\, dx = \lim_n G(t_n) = G(S).$$

Also,

$$\lim_n \int_{\mathbb{R}} \psi(x - b(t_n))u(x, t_n)\, dx = -\lim_n g(t_n) = -g(S).$$

Because $0 < -g(S) < G(S)$ and

$$u(x, S) \ge e^{-S}\mathbb{E}[f(x + B_S)] > 0, \quad x \in \mathbb{R},$$

there is, by Lemma 2.2, a unique $b^* \in \mathbb{R}$ such that

$$\int_{\mathbb{R}} \psi(x - b^*)u(x, S)\, dx = -g(t).$$

We claim that $b(t_n) \to b^*$. If this was not the case, then, by passing to a subsequence we would have $b(t_n)$ converging to some other extended real c and hence, by dominated convergence,

$$\begin{aligned} -g(t) &= -\lim_n g(t_n) \\ &= \lim_n \int_{\mathbb{R}} \psi(x - b(t_n))u(x, t_n)\, dx \\ &= \int_{\mathbb{R}} \psi(x - c)u(x, S)\, dx, \end{aligned}$$

contradicting the definition of b^* (where we used the natural definitions $\psi(-\infty) := 1, \psi(+\infty) := 0$). Using dominated convergence in (2.8) we get that there exists a continuous b such that all three equations hold on $[0, S]$.

All we need to do now is show that we can extend the existence from $[0, S]$ to $[0, S + \delta]$ for some $\delta > 0$. This amounts to proving existence on $[0, \delta]$ starting at

a different initial condition – replacing the original probability density f by the density of the probability measure

$$\int_{\mathbb{R}} \mathbb{E}\left[\exp\left(-\int_0^S \psi(x + B_u - b(u))\,du\right), B_S \in \bullet\right] f(x)\,dx/G(S).$$

This will follow if we can establish the local existence, that is the existence for some $\delta > 0$, of a solution of the following PDE/ODE system

$$\begin{cases} \tilde{u}_t(x,t) = \dfrac{1}{2}\tilde{u}_{xx}(x,t) - \psi(x - \tilde{b}(t))\tilde{u}(x,t),\ x \in \mathbb{R},\ 0 < t < \delta, \\[1mm] \tilde{u}(x,0) = u(x,S)/G(S),\ x \in \mathbb{R}, \\[1mm] \tilde{b}(0) = b(S), \\[1mm] \tilde{b}'(t) = \dfrac{(g(S+t) + g'(S+t))/G(S) - \int_{\mathbb{R}}[\psi^2(x - \tilde{b}(t)) - \psi(x - \tilde{b}(t))]\tilde{u}(x,t)\,dx}{\int_{\mathbb{R}} \psi_x(x - \tilde{b}(t))\tilde{u}(x,t)\,dx} \\[3mm] \qquad\quad - \dfrac{1/2 \int_{\mathbb{R}} \psi_x(x - \tilde{b}(t))\tilde{u}_x(x,t)\,dx}{\int_{\mathbb{R}} \psi_x(x - \tilde{b}(t))\tilde{u}(x,t)\,dx},\ 0 < t < \delta. \end{cases}$$

We note that the expression for $\tilde{b}'(t)$ is not the analogue of the one for $b'(t)$ that arises immediately from differentiating (2.8), which in turn arose from rearranging (2.5) and integrating. However, adding $0 = \int_{\mathbb{R}} \psi(x - b(t))u(x,t)\,dx - g(t)$ to the right-hand side of (2.5) and then solving for $b'(t)$ leads to an expression of this form. Note that

$$u(x,S) = \mathbb{E}\left[\exp\left(-\int_0^S \psi(x + B_{S-u} - b(u))du\right) f(x + B_S)\right] > 0,$$

and, by dominated convergence, that $u(\cdot, S) \in C^2(\mathbb{R})$ with $\|u(\cdot, S)\|_{L^\infty(\mathbb{R})}$, $\|u_x(\cdot, S)\|_{L^\infty(\mathbb{R})}$, $\|u_{xx}(\cdot, S)\|_{L^\infty(\mathbb{R})}$ all finite. Therefore, we can apply Theorem 3.14 below to get that there is a time $\delta > 0$ and a unique pair $\tilde{u}, \tilde{b}$ satisfying the PDE/ODE system above with $\tilde{u}$ twice continuously differentiable in x on $\mathbb{R}$ and once continuously differentiable in t on $[0, \delta]$, i.e. $\tilde{u} \in C_x^2(\mathbb{R})C_t^1([0, \delta])$, and with $\tilde{b} \in C^1([0, \delta])$. Thus, we have proven that we have a unique continuous b satisfying equations (2.6), (2.7) and (2.8) on $[0, S + \delta]$. This contradicts the maximality of S. As a result, $S = \infty$ and we are done. $\qquad\square$

This completes the proof of Theorem 2.1. $\qquad\square$

As a corollary we get the global existence and uniqueness of the PDE/ODE system.

Corollary 2.5. *Suppose that the conditions of Theorem 2.1 hold. Then, the system*

$$(2.12) \quad \begin{cases} u_t(x,t) = \dfrac{1}{2}u_{xx}(x,t) - \psi(x - b(t))u(x,t), \\[2mm] u(x,0) = f(x), \quad x \in \mathbb{R}, \\[2mm] -g(0) = \displaystyle\int_{\mathbb{R}} \psi(x - b(0))f(x)\,dx, \\[2mm] b'(t) = \dfrac{g(t) + g'(t) - \int_{\mathbb{R}}[\psi^2(x - b(t)) - \psi(x - b(t))]u(x,t)\,dx}{\int_{\mathbb{R}} \psi_x(x - b(t))u(x,t)\,dx} \\[4mm] \qquad\quad - \dfrac{1/2\int_{\mathbb{R}} \psi_x(x - b(t))u_x(x,t)dx}{\int_{\mathbb{R}} \psi_x(x - b(t))u(x,t)\,dx}, \quad t > 0, \end{cases}$$

has a unique solution $(u,b) \in C_x^2(\mathbb{R})C_t^1(\mathbb{R}_+) \times C_t^1(\mathbb{R}_+)$.

3. Local Existence and Uniqueness

We now consider the PDE/ODE system (2.12). We have already used the standard notation F_x and F_{xx} to denote the first and second derivatives of a function F of one variable or the first and second partial derivatives with respect to the variable x of a function F of several variables. Because we repeatedly deal with the function $(x,t) \mapsto \psi(x - b(t))$, it will be convenient to recycle notation and define a function $\psi_b : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ by $\psi_b(x,t) = \psi(x - b(t))\}$. We will then set $\psi_{x,b} := \partial_x \psi_b$ and $\psi_{xx,b} := \partial_{xx} \psi_b$. We will continue to use the notation ψ_x and ψ_{xx} with its old meaning, but there should be no confusion between the different objects ψ_b and ψ_x. Similarly, we set $\phi := \psi^2 - \psi = -\psi(1 - \psi)$ and put $\phi_b(x,t) = \phi(x - b(t))$. Lastly, for two functions f,g define $\langle f,g \rangle = \int_{\mathbb{R}} f(x,t)g(x,t)\,dx$.

In the notation we have introduced, we wish to consider the system

$$(3.1) \quad \begin{cases} u_t(x,t) = \dfrac{1}{2}u_{xx}(x,t) - \psi(x - b(t))u(x,t), \quad x \in \mathbb{R}, t > 0, \\[2mm] u(x,0) = f(x), \quad x \in \mathbb{R}, \\[2mm] b(0) = b_0, \\[2mm] b'(t) = \dfrac{g(t) + g'(t) - \langle \phi_b, u \rangle - 1/2\langle \psi_{x,b}, u_x \rangle}{\langle \psi_{x,b}, u \rangle}, \quad t > 0, \end{cases}$$

for some $b_0 \in \mathbb{R}$. (In the proof of Theorem 2.1 we choose b_0 to satisfy $-g(0) = \int_{\mathbb{R}} \psi(x - b_0)f(x)\,dx$, but we may take an arbitrary value for b_0 and still obtain a local existence and uniqueness result.)

We have assumed in the statement of Theorem 2.1 that $f \in C^2(\mathbb{R})$ and $\psi \in C^3(\mathbb{R})$ with $\|\psi\|_{L^\infty} = 1$, $\|\psi\|_{L^\infty} =: B$, $\|\psi_{xx}\|_{L^\infty} =: C$, and $\|\psi_{xxx}\|_{L^\infty} =: F$ for finite constants B, C, F. Furthermore, we have assumed for some $h > 0$ that $\psi(x) = 1$ for $x \leq -h$, that $\psi(x) = 0$ for $x \geq h$, and that $\psi \geq 0$ and $\psi_x \leq 0$ for all $x \in \mathbb{R}$. Set $\int_{\mathbb{R}} |\psi_x(x)|\,dx =: D$ and note that $0 < D < \infty$. It is immediate that $\|\phi\|_{L^\infty} \leq 1$ and $\|\phi_x\|_{L^\infty} = \|\psi_x(1 - 2\psi)\|_{L^\infty} \leq \|\psi_x\|_{L^\infty} = B$. Moreover, the functions ϕ and ϕ_x are supported on $[-h, h]$ and $0 < \int_{\mathbb{R}} |\phi(x)|\,dx =: E < \infty$.

Definition 3.1. For $T > 0$, let $(\mathcal{L}^T, \|\cdot\|_T)$ be the Banach space consisting of pairs of functions (u, b) such that $u \in C_x^2(\mathbb{R})C_t([0,T])$, $b \in C([0,T])$ and

$$
\begin{aligned}
\|(u,b)\|_T := \ & \|u\|_{L_x^\infty(\mathbb{R})L_t^\infty([0,T])} \\
& + \|u_x\|_{L_x^\infty(\mathbb{R})L_t^\infty([0,T])} + \|u_{xx}\|_{L_x^\infty(\mathbb{R})L_t^\infty([0,T])} \\
& + \|b\|_{L^\infty([0,T])} \\
& < \infty.
\end{aligned}
$$

(3.2)

Definition 3.2. Given constants $M, N, P, A, L > 0$, $b_0 \in \mathbb{R}$ and $T > 0$, define the closed subset $\Gamma_{MNPALb_0}^T \subset \mathcal{L}^T$ by

$$
\begin{aligned}
\Gamma_{MNPALb_0}^T := \Big\{ (u,b) \in \mathcal{L}^T : \ & \\
& \|u\|_{L_x^\infty L_t^\infty([0,T])} \le M, \\
& \|u_x\|_{L_t^\infty([0,T])L_x^\infty} \le N, \\
& \|u_{xx}\|_{L_t^\infty([0,T])L_x^\infty} \le P, \\
& b(0) = b_0, \\
& \|b\|_{L^\infty([0,T])} \le A/2, \\
& \inf_{x \in [-A,A], t \in [0,T]} u(x,t) \ge L \Big\}.
\end{aligned}
$$

(3.3)

The following is the main result of this section.

Theorem 3.3. *Suppose that the assumptions of Theorem 2.1 hold. Suppose also that the constants $M, N, P, A, L > 0$ and $b_0 \in \mathbb{R}$ are such that*

- $|b_0| \le A/4$,
- $f(x) \ge 4L > 0$ *for $x \in [-A, A]$,*
- $\|f\|_{L^\infty(\mathbb{R})} \le M/2$,
- $\|f_x\|_{L^\infty(\mathbb{R})} \le N/2$,
- $\|f_{xx}\|_{L^\infty(\mathbb{R})} \le P/2$.

Then, for $T > 0$ sufficiently small there is a contractive map $\Phi : \Gamma_{MNPALb_0}^T \to \Gamma_{MNPALb_0}^T$ defined by $\Phi(v, b) = (u, c)$, where

$$
(3.4) \quad
\begin{cases}
u_t(x,t) = \dfrac{1}{2}u_{xx}(x,t) - \psi(x - b(t))v(x,t), & x \in \mathbb{R}, t > 0, \\[2mm]
u(x,0) = f(x), & x \in \mathbb{R}, \\[2mm]
c'(t) = \dfrac{g(t) + g'(t) - \langle \phi_b, v \rangle - 1/2\langle \psi_{x,b}, v_x \rangle}{\langle \psi_{x,b}, v \rangle}, & 0 < t \le T, \\[2mm]
c(0) = b_0.
\end{cases}
$$

We will prove Theorem 3.3 in a series of lemmas. Each lemma will assume the hypotheses of Theorem 3.3 and the bounds established in the previous lemmas.

Remark 3.4. Since f is continuous and positive, for any $A > 0$ there exists $L > 0$ such that $f(x) \ge 4L$ for $x \in [-A, A]$. Therefore, we are not restricting the possible values of $b(0)$ by the above assumptions. We will also assume without loss of generality that $h \le A/4$.

Lemma 3.5 (Boundedness of u). *Suppose that $(u, c) = \Phi((v, b))$, with $(v, b) \in \Gamma^T_{MNPALb_0}$. Then, there exists a time $T > 0$ such that*

$$\|u\|_{L^\infty_x L^\infty_t ([0,T])} \leq M.$$

Proof. Using Duhamel's formula (see (8.2)),

$$
\begin{aligned}
|u(x,t)| &= \left| \int_{\mathbb{R}} G(y,t) f(x-y)\, dy - \int_0^t \int_{\mathbb{R}} G(x-y, t-s) \psi_{c(s)}(y) v(y,s)\, dy\, ds \right| \\
&\leq \int_{\mathbb{R}} G(y,t) f(x-y)\, dy + \int_0^t \int_{\mathbb{R}} G(x-y, t-s) |\psi_{c(s)}(y)| |v(y,s)|\, dy\, ds \\
&\leq M/2 \int_{\mathbb{R}} G(y,t)\, dy + M \int_0^t \int_{\mathbb{R}} G(x-y, t-s)\, dy\, ds \\
&\leq M/2 + Mt \\
&\leq M
\end{aligned}
$$

when $t \leq \frac{1}{2}$, where

$$G(x,t) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}, \quad x \in \mathbb{R},\ t > 0.$$

$\square$

Lemma 3.6 (Boundedness of u_x). *Suppose that $(u, c) = \Phi((v, b))$ with $(v, b) \in \Gamma^T_{MNPALb_0}$. Then, there exists a time $T > 0$ such that*

$$\|u_x\|_{L^\infty_t ([0,T]) L^\infty_x} \leq N.$$

Proof. Since u_x solves

$$
\begin{cases}
\left(\partial_t - \dfrac{\partial_{xx}}{2} \right) u_x = -\psi_{x,c} v - \psi_c v_x, \quad x \in \mathbb{R},\ t > 0, \\
u_x(x, 0) = f_x(x),
\end{cases}
$$

we have via Duhamel's formula that

$$
\begin{aligned}
|u_x(x,t)| &= \left| \int_{\mathbb{R}} G(y,t) f_x(x-y)\, dy \right. \\
&\quad \left. + \int_0^t \int_{\mathbb{R}} G(x-y, t-s)(-\psi_{x,c} v - \psi_c v_x)(y,s)\, dy\, ds \right| \\
&\leq \int_{\mathbb{R}} G(y,t) |f_x(x-y)|\, dy + \int_0^t \int_{\mathbb{R}} G(x-y, t-s) |\psi_{x,c}| |v(y,s)|\, dy\, ds \\
&\quad + \int_0^t \int_{\mathbb{R}} G(x-y, t-s) |\psi(y-c(s))| |v_x(y,s))|\, dy\, ds \\
&\leq \frac{N}{2} + MB \int_0^t \int_{\mathbb{R}} G(x-y, t-s)\, dy\, ds + N \int_0^t \int_{\mathbb{R}} G(x-y, t-s)\, dy\, ds \\
&\leq \frac{N}{2} + MBt + Nt.
\end{aligned}
$$

Thus,

$$\|u_x\|_{L^\infty_t([0,T]) L^\infty_x} \leq \frac{N}{2} + (MB + N) T \leq N$$

whenever $T \leq T^*$, where

$$T^* = \frac{N}{2(MB + N)}.$$

$\square$

Lemma 3.7 (Boundedness of u_{xx}). *Suppose that $(u, c) = \Phi((v, b))$ with $(v, b) \in \Gamma^T_{MNPALb_0}$. Then, there exists a time $T > 0$ such that*

$$\|u_{xx}\|_{L^\infty_t([0,T])L^\infty_x} \leq P.$$

Proof. Noting that u_{xx} solves

$$\begin{cases} \left(\partial_t - \frac{\partial_{xx}}{2}\right) u_{xx} = -\psi_{xx,c}v - 2\psi_{x,c}v_x - \psi_c v_{xx}, & x \in \mathbb{R},\ t > 0, \\ u_{xx}(x, 0) = f_{xx}(x), \end{cases}$$

analogous manipulations to those from Lemma 3.6 yield the result. $\square$

Lemma 3.8 (Lower bound for u and boundedness of c' and c). *Suppose that $(u, c) = \Phi((v, b))$ with $(v, b) \in \Gamma^T_{MNPALb_0}$. Then, there exists a time $T > 0$ such that*

$$(3.5) \qquad\qquad u \geq L \text{ on } x \in [-A, A],\ t \in [0, T],$$

and $c(t) \in [-A/2, A/2]$ for $t \in [0, T]$.

Proof. Recall that $b(0) \in [-A/4, A/4]$. Then, it is immediate that

$$(3.6) \qquad \left| \int_{\mathbb{R}} \psi_x(x - b(t))v(x, t)\, dx \right| = \left| \int_{\mathbb{R}} \psi_x(y)v(y + b(t))\, dy \right| \geq DL, \quad t \in [0, T],$$

because on the support $[-h, h]$ of ψ_x we have $y \in [-h, h] \subseteq [-A/4, A/4]$ which together with the bound on $b(t)$ implies $y + b(t) \in [-A, A]$. Therefore, $v(y + b(t)) \geq L$ for $t \in [0, T]$ which, since $\psi_x \leq 0$, yields

$$\int_{\mathbb{R}} \psi_x(y)v(y + b(t))\, dy \leq L \int_{\mathbb{R}} \psi_x(y)\, dy = -LD < 0, \quad t \in [0, T].$$

We see from these bounds that

$$|c'(t)| \leq \frac{\sup_{[0,t]}(|g + g'|) + ME + ND/2}{LD}$$

and, by integrating,

$$|c(t)| \leq |c(0)| + \frac{\sup_{[0,t]}(|g + g'|) + ME + ND/2}{LD}t.$$

Thus, there is $T > 0$ such that for $t \in [0, T]$,

$$|c(t)| \in [-A/2, A/2].$$

Using the assumptions, equation (8.2) gives

$$\begin{aligned} u(x, t) &= \int_{\mathbb{R}} G(y, t)f(x - y)\, dy - \int_0^t \int_{\mathbb{R}} G(x - y, t - s)\psi_{c(s)}(y)v(y, s)\, dy\, ds \\ &\geq 4L \int_{x-A}^{x+A} G(y, t)\, dy - M \int_0^t \int_{\mathbb{R}} G(x - y, t - s)\, dy\, ds \\ &\geq 4L \int_{x-A}^{x+A} G(y, t)\, dy - Mt. \end{aligned}$$

If $0 \leq x \leq A$ then $x - A \leq 0$ and $x + A \geq A > 0$ so for small enough t we have

$$\int_{x-A}^{x+A} G(y,t)\,dy \geq \int_0^A G(y,t)\,dy \geq \frac{1}{3}.$$

If $-A \leq x < 0$ then $x + A \geq 0$ and $x - A \leq -A < 0$. So, for small enough t,

$$\int_{x-A}^{x+A} G(y,t)\,dy \geq \int_{-A}^0 G(y,t)\,dy \geq \frac{1}{3}.$$

Therefore, there exists a time $T > 0$ such that whenever $t \in [0,T]$ and $x \in [-A, A]$,

$$\begin{aligned} u(x,t) &\geq & \frac{4}{3}L - Mt \\ &\geq & L. \end{aligned}$$

$\square$

Lemma 3.9. *For a sufficiently small time $T > 0$, the set $\Gamma_{MNPALb_0}^T$ is mapped into itself by Φ.*

Proof. The above lemmas provided the necessary bounds. Now, note that if we start with $(v,b) \in \Gamma_{MNPALb_0}^T$, then we first get the function c from the last two equations in (3.4) by simply integrating. The integration is well-defined because the denominator is bounded in absolute value below by $DL > 0$ and the numerator is bounded above. Thus, $c \in C^1([0,t])$. Next, having c in hand we get the function u from the first two equations of (3.4). We note that, by Duhamel's formula, the function u has actually more than the desired smoothness, namely, $u \in C_x^2(\mathbb{R})C_t^1([0,T])$. $\square$

Lemma 3.10. *Suppose that $(v_1, b_1), (v_2, b_2) \in \Gamma_{MNPALb_0}^T$. Set $(u_1, c_1) = \Phi((v_1, b_1))$ and $(u_2, c_2) = \Phi((v_2, b_2))$. For any $\epsilon > 0$ there exists $T > 0$ such that*

$$(3.7) \qquad \|c_2 - c_1\|_{L_t^\infty([0,T])} \leq \epsilon \|(v_2, b_2) - (v_1, b_1)\|_T.$$

Proof. Note that the functions c_1, c_2 satisfy

$$(3.8) \qquad \begin{cases} c_1'(t) = \dfrac{g(t) + g'(t) - \langle \phi_{b_1}, v_1 \rangle - 1/2\langle \psi_{x,b_1}, \partial_x v_1 \rangle}{\int_{\mathbb{R}} \langle \psi_{x,b_1}, v_1 \rangle}, & t > 0, \\[3mm] c_2'(t) = \dfrac{g(t) + g'(t) - \langle \phi_{b_2}, v_2 \rangle - 1/2\langle \psi_{x,b_2}, \partial_x v_2 \rangle}{\int_{\mathbb{R}} \langle \psi_{x,b_2}, v_2 \rangle}, & t > 0. \end{cases}$$

Subtracting the two equations gives

$$\begin{aligned} c_2'(t) - c_1'(t) &= [g(t) + g'(t)]\left(\frac{\langle \psi_{x,b_1}, v_1 \rangle - \langle \psi_{x,b_1}, v_2 \rangle}{\langle \psi_{x,b_1}, v_1 \rangle \langle \psi_{x,b_2}, v_2 \rangle} + \frac{\langle \psi_{x,b_1}, v_2 \rangle - \langle \psi_{x,b_2}, v_2 \rangle}{\langle \psi_{x,b_1}, v_1 \rangle \langle \psi_{x,b_2}, v_2 \rangle}\right) \\ &\quad + \frac{(\langle \phi_{b_1}, v_1 \rangle - \langle \phi_{b_2}, v_1 \rangle)\langle \psi_{x,b_2}, v_2 \rangle}{\langle \psi_{x,b_1}, v_1 \rangle \langle \psi_{x,b_2}, v_2 \rangle} + \frac{(\langle \phi_{b_2}, v_1 \rangle - \langle \phi_{b_2}, v_2 \rangle)\langle \psi_{x,b_2}, v_2 \rangle}{\langle \psi_{x,b_1}, v_1 \rangle \langle \psi_{x,b_2}, v_2 \rangle} \\ &\quad + \frac{(\langle \phi_{b_2}, v_2 \rangle - \langle \phi_{b_1}, v_2 \rangle)\langle \phi_{b_2}, v_2 \rangle}{\langle \psi_{x,b_1}, v_1 \rangle \langle \psi_{x,b_2}, v_2 \rangle} + \frac{(\langle \phi_{b_1}, v_2 \rangle - \langle \phi_{b_1}, v_1 \rangle)\langle \phi_{b_2}, v_2 \rangle}{\langle \psi_{x,b_1}, v_1 \rangle \langle \psi_{x,b_2}, v_2 \rangle} \\ &\quad + \frac{(\langle \psi_{x,b_1}, \partial_x v_1 \rangle - \langle \psi_{x,b_2}, \partial_x v_1 \rangle)\langle \psi_{x,b_2}, v_2 \rangle}{2\langle \psi_{x,b_1}, v_1 \rangle \langle \psi_{x,b_2}, v_2 \rangle} \\ &\quad + \frac{(\langle \psi_{x,b_2}, \partial_x v_1 \rangle - \langle \psi_{x,b_2}, \partial_x v_2 \rangle)\langle \psi_{x,b_2}, v_2 \rangle}{2\langle \psi_{x,b_1}, v_1 \rangle \langle \psi_{x,b_2}, v_2 \rangle} \end{aligned}$$

$$+ \frac{(\langle \psi_{x,b_2}, v_2 \rangle - \langle \psi_{x,b_1}, v_2 \rangle) \langle \psi_{x,b_2}, \partial_x v_2 \rangle}{2 \langle \psi_{x,b_1}, v_1 \rangle \langle \psi_{x,b_2}, v_2 \rangle}$$

$$+ \frac{(\langle \psi_{x,b_1}, v_2 \rangle - \langle \psi_{x,b_1}, v_1 \rangle) \langle \psi_{x,b_2}, \partial_x v_2 \rangle}{2 \langle \psi_{x,b_1}, v_1 \rangle \langle \psi_{x,b_2}, v_2 \rangle}.$$

Using the fact that the functions ψ, ψ_x and ϕ are Lipschitz, that v_1 and v_2 are bounded, and that their first derivatives are bounded, we find that

$$
\begin{aligned}
\|c_2' - c_1'\|_{L_t^\infty([0,T])} \quad \leq \quad & \frac{\sup_{[0,T]} |g + g'| \|v_1 - v_2\|_{L_x^\infty L_t^\infty([0,T])}}{L^2 D^2} \\
& + \frac{\sup_{[0,T]} |g + g'| MC(A + 2h) \|b_2 - b_1\|_{L_t^\infty([0,T])}}{L^2 D^2} \\
& + \frac{DM^2 B(A + 2h) \|b_2 - b_1\|_{L_t^\infty([0,T])}}{L^2 D^2} \\
& + \frac{DME \|v_2 - v_1\|_{L_x^\infty L_t^\infty([0,T])}}{L^2 D^2} \\
& + \frac{EM^2 B(A + 2h) \|b_2 - b_1\|_{L_t^\infty([0,T])}}{L^2 D^2} \\
& + \frac{ME^2 \|v_2 - v_1\|_{L_x^\infty L_t^\infty([0,T])}}{L^2 D^2} \\
& + \frac{NMDC(A + 2h) \|b_2 - b_1\|_{L_t^\infty([0,T])}}{2L^2 D^2} \\
& + \frac{MD^2 \|\partial_x v_2 - \partial_x v_1\|_{L_x^\infty L_t^\infty([0,T])}}{2L^2 D^2} \\
& + \frac{NMDC(A + 2h) \|b_2 - b_1\|_{L_t^\infty([0,T])}}{2L^2 D^2} \\
& + \frac{ND^2 \|v_2 - v_1\|_{L_x^\infty L_t^\infty([0,T])}}{2L^2 D^2}.
\end{aligned}
$$

Integrating and recalling that $c_1(0) = c_2(0) = b_0$ leads to

$$
\begin{aligned}
\left| \int_0^t (c_2'(s) - c_1'(s)) \, ds \right| \quad &= \quad |c_2(t) - c_1(t) - (c_2(0) - c_1(0))| \\
&\leq \quad \int_0^t |c_2'(s) - c_1'(s)| \, ds \\
&\leq \quad t \|c_2' - c_1'\|_{L_t^\infty([0,t])}.
\end{aligned}
$$

Hence,

$$\|c_2 - c_1\|_{L_t^\infty([0,T])} \leq T \|c_2' - c_1'\|_{L_t^\infty([0,T])},$$

and by the above bound on $\|c_2' - c_1'\|_{L_t^\infty([0,T])}$ for any $\epsilon > 0$ we can choose T small enough that

$$\|c_2 - c_1\|_{L_t^\infty([0,T])} \leq \epsilon \|(v_2, b_2) - (v_1, b_1)\|_T.$$

$\square$

Lemma 3.11. *Suppose that $(v_1, b_1), (v_2, b_2) \in \Gamma_{MNPALb_0}^T$. Set $(u_1, c_1) = \Phi((v_1, b_1))$ and $(u_2, c_2) = \Phi((v_2, b_2))$. For any $\epsilon > 0$ there exists $T > 0$ such that*

$$(3.9) \qquad \|u_2 - u_1\|_{L_x^\infty L_t^\infty([0,T])} \leq \epsilon \|(v_2, b_2) - (v_1, b_1)\|_T.$$

Proof. The following equations hold

$$
\begin{cases}
\left(\partial_t - \dfrac{\partial_{xx}}{2}\right) u_1 = -\psi(x - c_1(t))v_1, & x \in \mathbb{R},\ t > 0, \\[2mm]
\left(\partial_t - \dfrac{\partial_{xx}}{2}\right) u_2 = -\psi(x - c_2(t))v_2, & x \in \mathbb{R},\ t > 0, \\[2mm]
u_1(x,0) = f(x), & x \in \mathbb{R}, \\[1mm]
u_2(x,0) = f(x), & x \in \mathbb{R}.
\end{cases}
$$
(3.10)

By Duhamel's formula we have

$$
u_1 = G * (f\delta_{t=0}) + G * (-\psi_{c_1} v_1)
$$
(3.11)

and

$$
u_2 = G * (f\delta_{t=0}) + G * (-\psi_{c_2} v_2),
$$
(3.12)

where we recall that $*$ denotes convolution on $\mathbb{R}_+ \times \mathbb{R}$. Subtracting the two equations gives

$$
u_1 - u_2 = G * ((\psi_{c_2} - \psi_{c_1})v_1 + \psi_{c_2}(v_2 - v_1)).
$$

Bounding in terms of the sup norm and using the fact that

$$
|\psi(x - c_1(t)) - \psi(x - c_2(t))| \le \|\psi_x\|\,|c_1(t) - c_2(t)|,
$$

we have

$$
\begin{aligned}
|u_1(x,t) - u_2(x,t)| \ &\le\ \int_0^t \int_{\mathbb{R}} G(x - y, t - s)|\psi_{c_1}(y,s) - \psi_{c_2}(y,s)||v_1(y,s)|\, dy\, ds \\
&\quad + \int_0^t \int_{\mathbb{R}} G(x - y, t - s)|\psi_{c_2}(y,s)||v_2(y,s) - v_1(y,s)|\, dy\, ds \\
&\le\ \|\psi_x\|_{L_x^\infty}\|v_1\|_{L^\infty L_t^\infty([0,T])}\|c_1 - c_2\|_{L_x^\infty} t \\
&\quad + \|\psi\|_{L_x^\infty}\|v_1 - v_2\|_{L_x^\infty L_t^\infty([0,t])} t \\
&=\ BM\|c_1 - c_2\|_{L_x^\infty} t + \|v_1 - v_2\|_{L_x^\infty L_t^\infty([0,t])} t.
\end{aligned}
$$

Thus,

$$
\|u_1 - u_2\|_{L_x^\infty L_t^\infty([0,T])} \le B\|c_1 - c_2\|_{L_x^\infty} T + \|v_1 - v_2\|_{L_x^\infty L_t^\infty([0,T])} T.
$$

so for small enough T we see that (3.9) holds. $\square$

Lemma 3.12. *Suppose that* $(v_1, b_1), (v_2, b_2) \in \Gamma_{MNPALb_0}^T$. *Set* $(u_1, c_1) = \Phi((v_1, b_1))$ *and* $(u_2, c_2) = \Phi((v_2, b_2))$. *For any* $\epsilon > 0$ *there exists* $T > 0$ *such that*

$$
\|\partial_x u_1 - \partial_x u_2\|_{L_x^\infty L_t^\infty([0,T])} \le \epsilon \|(v_2, b_2) - (v_1, b_1)\|_T.
$$
(3.13)

Proof. Differentiating (3.10) with respect to x

$$(3.14) \quad \begin{cases} \left(\partial_t - \dfrac{\partial_{xx}}{2} \right) \partial_x u_1(x,t) = -\psi_{x,c_1}(x,t) v_1(x,t) - \psi_{c_1}(x,t) \partial_x v_1(x,t), \\ \qquad\qquad\qquad x \in \mathbb{R}, t > 0, \\ \left(\partial_t - \dfrac{\partial_{xx}}{2} \right) \partial_x u_2(x,t) = -\psi_{x,c_2}(x,t) v_2(x,t) - \psi_{c_2}(x,t) \partial_x v_2(x,t), \\ \qquad\qquad\qquad x \in \mathbb{R}, t > 0, \\ \partial_x u_1(x,0) = f_x(x), \quad x \in \mathbb{R}, \\ \partial_x u_2(x,0) = f_x(x), \quad x \in \mathbb{R}, \end{cases}$$

Via Duhamel's formula,

$$(3.15) \qquad \partial_x u_1 = G * (f_x \delta_{t=0}) + G * (-\psi_x(\cdot - c_1(\cdot)) v_1 - \psi(\cdot - c_2(\cdot)) \partial_x v_1)$$

and

$$(3.16) \qquad \partial_x u_1 = G * (f_x \delta_{t=0}) + G * (-\psi_x(\cdot - c_2(\cdot)) v_2 - \psi(\cdot - c_2(\cdot)) \partial_x v_2).$$

Subtracting and rearranging,

$$\begin{aligned} (\partial_x u_1 - \partial_x u_2)(x,t) &= \int_0^t \int_{\mathbb{R}} G(x-y, t-s)[\psi_{x,c_2} v_2(y,s) - \psi_{x,c_1} v_1(y,s)] \, dy \, ds \\ &\quad + \int_0^t \int_{\mathbb{R}} G(x-y, t-s)[\psi_{c_2} \partial_x v_2(y,s) - \psi_{c_1} \partial_x v_1(y,s)] \, dy \, ds \\ &= \int_0^t \int_{\mathbb{R}} G(x-y, t-s)[\psi_{x,c_2} v_2(y,s) - \psi_{x,c_2} v_1(y,s)] \, dy \, ds \\ &\quad + \int_0^t \int_{\mathbb{R}} G(x-y, t-s)[\psi_{x,c_2} v_1(y,s) - \psi_{x,c_1} v_1(y,s)] \, dy \, ds \\ &\quad + \int_0^t \int_{\mathbb{R}} G(x-y, t-s)[\psi_{c_2} \partial_x v_2(y,s) - \psi_{c_2} \partial_x v_1(y,s)] \, dy \, ds \\ &\quad + \int_0^t \int_{\mathbb{R}} G(x-y, t-s)[\psi_{c_2} \partial_x v_1(y,s) - \psi_{c_1} \partial_x v_1(y,s)] \, dy \, ds. \end{aligned}$$

Using estimates similar to those in the proof of Lemma 3.11

$$\begin{aligned} \|\partial_x u_1 - \partial_x u_2\|_{L_x^\infty L_t^\infty([0,T])} &\leq BM \|v_2 - v_1\|_{L_x^\infty L_t^\infty([0,T])} T \\ &\quad + CM \|c_2 - c_1\|_{L_t^\infty([0,T])} T \\ &\quad + \|\partial_x v_2 - \partial_x v_1\|_{L_x^\infty L_t^\infty([0,T])} T \\ &\quad + BN \|c_2 - c_1\|_{L_t^\infty([0,T])} T \\ &= BMT \|v_2 - v_1\|_{L_x^\infty L_t^\infty([0,T])} \\ &\quad + (CM + BN)T \|c_2 - c_1\|_{L_t^\infty([0,T])} \\ &\quad + T \|\partial_x v_2 - \partial_x v_1\|_{L_x^\infty L_t^\infty([0,T])}. \end{aligned}$$

so for T small we recover (3.13). $\square$

Lemma 3.13. *Suppose that* $(v_1, b_1), (v_2, b_2) \in \Gamma^T_{MNPALb_0}$. *Set* $(u_1, c_1) = \Phi((v_1, b_1))$ *and* $(u_2, c_2) = \Phi((v_2, b_2))$. *For any* $\epsilon > 0$ *there exists* $T > 0$ *such that*

$$(3.17) \qquad \|\partial_{xx} u_1 - \partial_{xx} u_2\|_{L_x^\infty L_t^\infty([0,T])} \leq \epsilon \|(v_2, b_2) - (v_1, b_1)\|_T.$$

Proof. Differentiating (3.10) twice with respect to x

(3.18)
$$
\begin{cases}
\left(\partial_t - \dfrac{\partial_{xx}}{2}\right)\partial_{xx}u_1 = -\psi_{xx,c_1}v_1 - 2\psi_{x,c_1}\partial_x v_1 - \psi_{c_1}\partial_{xx}v_1, \\
\qquad\qquad\qquad x \in \mathbb{R},\ t > 0, \\
\left(\partial_t - \dfrac{\partial_{xx}}{2}\right)\partial_{xx}u_2 = -\psi_{xx,c_2}v_2 - 2\psi_{x,c_2}\partial_x v_2 - \psi_{c_2}\partial_{xx}v_2, \\
\qquad\qquad\qquad x \in \mathbb{R},\ t > 0, \\
\partial_{xx}u_1(x,0) = f_{xx}(x), \quad x \in \mathbb{R}, \\
\partial_{xx}u_2(x,0) = f_{xx}(x), \quad x \in \mathbb{R}.
\end{cases}
$$

Duhamel's formula and similar manipulations to Lemmas 3.11 and 3.12 give

$$
\begin{aligned}
\|\partial_{xx}u_1 - \partial_{xx}u_2\|_{L_x^\infty L_t^\infty([0,T])} &\leq\ CM\|v_2 - v_1\|_{L_t^\infty([0,T])L_x^\infty}T \\
&\quad + FM\|c_2 - c_1\|_{L_t^\infty([0,T])}T \\
&\quad + 2B\|\partial_x v_2 - \partial_x v_1\|_{L_x^\infty L_t^\infty([0,T])}T \\
&\quad + 2CN\|c_2 - c_1\|_{L_t^\infty([0,T])}T \\
&\quad + \|\partial_{xx}v_2 - \partial_{xx}v_1\|_{L_x^\infty L_t^\infty([0,T])}T \\
&\quad + BP\|c_2 - c_1\|_{L_t^\infty([0,T])}T \\
&=\ CMT\|v_2 - v_1\|_{L_x^\infty L_t^\infty([0,T])} \\
&\quad + 2BT\|\partial_x v_2 - \partial_x v_1\|_{L_x^\infty L_t^\infty([0,T])} \\
&\quad + T\|\partial_{xx}v_2 - \partial_{xx}v_1\|_{L_x^\infty L_t^\infty([0,T])} \\
&\quad + (FM + 2CN + BP)T\|c_2 - c_1\|_{L_t^\infty([0,T])}.
\end{aligned}
$$

so when $T > 0$ is small (3.17) holds. $\qquad\square$

Theorem 3.14. *[Local existence and uniqueness] Suppose that the conditions of Theorem 2.1 hold. Then, there exists a time $T > 0$ such that the system*

$$
\begin{cases}
u_t(x,t) = \dfrac{1}{2}u_{xx}(x,t) - \psi(x - b(t))u(x,t), \quad x \in \mathbb{R},\ t > 0, \\
u(x,0) = f(x), \quad x \in \mathbb{R}, \\
b'(t) = \dfrac{g(t) + g'(t) - \langle\phi_b, u\rangle - 1/2\langle\psi_{x,b}, u_x\rangle}{\langle\psi_{x,b}, u\rangle}, \quad t > 0, \\
b(0) = b_0,
\end{cases}
$$

has a unique solution $(u,b) \in C_x^2(\mathbb{R})C_t^1([0,T]) \times C^1([0,T])$.

Proof. Note there exist strictly positive constants A, M, N and P such that $b_0 \in \left[-\frac{A}{4}, \frac{A}{4}\right]$, $f(x) \geq L > 0$, when $x \in [-A, A]$, $\|f\|_{L^\infty(\mathbb{R})} \leq M$, $\|f_x\|_{L^\infty(\mathbb{R})} \leq N/2$, and $\|f_x\|_{L^\infty(\mathbb{R})} \leq P/2$. Putting all the estimates from the above lemmas together we have that, if $0 < \epsilon < 1$ is fixed, then for $T > 0$ small enough

$$
\|(u_2, c_2) - (u_1, c_1)\| \leq \epsilon\|(v_2, b_2) - (v_1, b_1)\|.
$$

Thus, there exists a $T > 0$ such that the map $\Phi : \Gamma_{MNPALb_0}^T \to \Gamma_{MNPALb_0}^T$ is a contraction. Since $\Gamma_{MNPALb_0}^T$ is a closed subset of the Banach space $\mathcal{L}^T$, the Contraction Mapping Theorem gives that there exists a unique fixed point, that is,

a pair $(u, b) \in C_x^2(\mathbb{R})C_t([0, T]) \times C([0, T])$ with $b(0) = b_0$ such that

(3.19)
$$\begin{cases} u_t(x, t) = \dfrac{1}{2}u_{xx}(x, t) - \psi(x - b(t))u(x, t) \\[2mm] u(x, 0) = f(x) \\[2mm] b'(t) = \dfrac{g(t) + g'(t) - \langle \phi_b, u \rangle - 1/2\langle \psi_{x,b}, u_x \rangle}{\langle \psi_{x,b}, u \rangle} \\[2mm] b(0) = b_0. \end{cases}$$

We can now argue that our fixed point (u, b) has more smoothness than it seems *a priori*. The third equation in (3.19) implies that b must be continuously differentiable with a bounded derivative. This, together with the first equation from (3.19) then tells us that u has a continuous derivative in time. Therefore, we must have $(u, b) \in C_x^2(\mathbb{R})C_t^1([0, T]) \times C^1([0, T])$. $\qquad\qquad\square$

Corollary 3.15. *Assume the hypotheses of Theorem 3.14 and the extra conditions*

(3.20)
$$\begin{cases} G(0) = \displaystyle\int_{\mathbb{R}} f(x)\, dx, \\[2mm] -g(0) = \displaystyle\int_{\mathbb{R}} \psi(x - b(0))f(x)\, dx, \\[2mm] 0 < -g(t) < G(t), \quad t \in [0, T]. \end{cases}$$

Then, there exists a time $T > 0$ such that the system

$$\begin{cases} u_t(x, t) = \dfrac{1}{2}u_{xx}(x, t) - \psi(x - b(t))u(x, t), \quad x \in \mathbb{R},\, 0 < t < T, \\[2mm] u(x, 0) = f(x), \quad x \in \mathbb{R}, \\[2mm] G(t) = \displaystyle\int_{\mathbb{R}} u(x, t)\, dx, \quad t \in [0, T], \end{cases}$$

has a unique solution $(u, b) : \mathbb{R} \times [0, T] \to \mathbb{R}$. Furthermore, $u \in C_x^2(\mathbb{R})C_t^1([0, T])$ and $b \in C^1([0, T])$.

Proof. First note that by Lemma 2.2 we have that $b(0)$ is uniquely determined. From Theorem 3.14 we have that there exist unique u, b with $u \in C_x^2(\mathbb{R})C_t^1([0, T])$ and $b \in C^1([0, T])$ satisfying the PDE and having everywhere in $[0, T]$

$$b'(t) = \frac{g(t) + g'(t) - \langle \phi_b, u \rangle - 1/2\langle \psi_{x,b}, u_x \rangle}{\langle \psi_{x,b}, u \rangle}.$$

Set $F(t) := G(t) - \int_{\mathbb{R}} u(x, t)\, dx$ and note that the first two conditions from (3.20) yield, together with the PDE, $F_t(0) = F(0) = 0$. The function F belongs to $C^1([0, T])$ and F_t belongs to $C([0, T])$. The above equation for b' is equivalent, after using the PDE, to

$$F_{tt}(t) - F_t(t) = 0, \quad t \in [0, T].$$

Integrating and using the fundamental theorem of calculus, we get

$$F_t(t) - F(t) = F_t(0) - F(0) = 0, \quad t \in [0, T].$$

The unique solution to this differential equation is $F(t) = Ce^t$ for some constant $C \in \mathbb{R}$. This together with $F(0) = 0$ yields $F(t) = 0$ for $t \in [0, T]$. Thus,

$$G(t) = \int_{\mathbb{R}} u(x, t)\, dx, \quad t \in [0, T].$$

Then, taking a derivative and using the PDE,

$$-g(t) = \int_{\mathbb{R}} \psi(x - b(t))u(x,t)\,dx, \quad t \in [0,T].$$

Because $|\psi(x)| \leq 1$ for $x \in \mathbb{R}$, $\psi = 0$ for $x \geq h$ and $u(x,t) > 0$ we see that

$$0 < \int_{\mathbb{R}} \psi(x - b(t))u(x,t)\,dx = -g(t) < \int_{\mathbb{R}} u(x,t)\,dx = G(t).$$

$\square$

4. Discontinuous killing

Next, we consider the existence of a barrier when killing is done non-smoothly. That is, we ask whether there exists a function b such that, for a given G

$$(4.1) \qquad G(t) = \int_{\mathbb{R}} \mathbb{E}\left[\exp\left(-\int_0^t \mathbf{1}_{(-\infty,0]}(x + B_u - b(u))\,du\right)f(x)\right]\,dx$$

Note that $\int_0^t \mathbf{1}_{(-\infty,0]}(x + B_u - b(u))\,du$ is the time during the interval $[0,t]$ spent by a Brownian motion started at x below the barrier b.

Theorem 4.1. *There exists a function b such that, for a given, twice continuously differentiable G satisfying $0 < -g(t)/G(t) < 1$, $t \geq 0$ equation (4.1) holds for all $t \geq 0$. Furthermore, there exists at most one locally Lipschitz such b.*

Proof. Let ϕ be a smooth decreasing function supported on $[0,1]$ with $\int_{\mathbb{R}} \phi(x)\,dx = 1$. Put

$$\underline{\psi}_\epsilon(x) = \int_x^\infty \phi((y - \epsilon)/\epsilon))(1/\epsilon)\,dy$$

and

$$\overline{\psi}_\epsilon(x) = \int_x^\infty \phi(y/\epsilon))(1/\epsilon)dy,$$

so that

$$(4.2) \qquad \underline{\psi}_\epsilon(x) \leq 1\{x \leq 0\} \leq \overline{\psi}_\epsilon.$$

Note also that

$$(4.3) \qquad \underline{\psi}_\epsilon(x) \text{ increases with } \epsilon \text{ for all } x$$

and

$$(4.4) \qquad \overline{\psi}_\epsilon(x) \text{ decreases with } \epsilon \text{ for all } x.$$

Let $\underline{b}_\epsilon$ and $\overline{b}_\epsilon$ be the two barriers corresponding to $\underline{\psi}_\epsilon(x)$ and $\overline{\psi}_\epsilon$. The existence and uniqueness of these two barriers follows by Theorem 2.1. From (4.2) we have that

$$\overline{b}_\epsilon(t) \leq \underline{b}_\epsilon(t)$$

for all t and from (4.3), (4.4) that

$$\overline{b}_\epsilon(t) \text{ is increasing in } \epsilon \text{ for each } t$$

and

$$\underline{b}_\epsilon(t) \text{ is decreasing in } \epsilon \text{ for each } t.$$

Put

$$\overline{b}_*(t) = \lim_{\epsilon \downarrow 0} \overline{b}_\epsilon(t)$$

and

$$\underline{b}_*(t) = \lim_{\epsilon \downarrow 0} \underline{b}_\epsilon(t).$$

Then,

(4.5) $$\overline{b}_*(t) \leq \underline{b}_*(t),$$

and both of these functions give a stopping time with the correct distribution for the case where ψ is the indicator of $(-\infty, 0]$. Because of (4.5), it must be the case that $\overline{b}_*(t) = \underline{b}_*(t)$ for Lebesgue almost all t. If b is locally Lipschitz it follows by an argument similar to the one in Lemma 2.3 that this b is unique. $\qquad\square$

5. Pricing Claims

Suppose that the asset price $(X_t)_{t \geq 0}$ is a geometric Brownian motion given by

(5.1) $$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t,$$

where $(W_t)_{t \geq 0}$ is a standard Brownian motion. We model default using a diffusion $(Y_t)_{t \geq 0}$ where

(5.2) $$dY_t = dB_t,$$

with $(B_t)_{t \geq 0}$ another standard Brownian motion. We assume that the Brownian motions W and B are correlated with correlation $-1 \leq \rho \leq 1$; that is, the cross-variation of the two processes satisfies

$$[B, W]_t = \rho t, \quad t \geq 0.$$

We can assume without loss of generality that for two independent Brownian motions B', B'' we have

$$\begin{cases} W_t = B'_t, \\ B_t = \rho B'_t + \sqrt{1 - \rho^2} B''_t. \end{cases}$$

In the following we will look at pricing contingent claims with a fixed maturity $T > 0$ and payoff of the form

$$F(X_T) 1\{\tau > T\}$$

for the random time

$$\tau := \inf \left\{ t > 0 : \lambda \int_0^t \psi(Y_s - b(s)) \, ds > U \right\},$$

where U is an independent exponentially distributed random variable with mean one.

Note that

$$\mathbb{E}^x[F(X_T) 1\{\tau > T\}] = \mathbb{E}^x \left[F(X_T) \exp \left(-\lambda \int_0^T \psi(Y_s - b(s)) \, ds \right) \right].$$

More generally, we will be interested in expressions of the form

$$\mathbb{E}^x \left[F(X_T) 1\{\tau > T\} \,\Big|\, (X_s)_{0 \leq s \leq t}, \, \tau > t \right]$$
$$= \mathbb{E}^x \left[F(X_T) \exp \left(-\lambda \int_t^T \psi(Y_s - b(s)) \, ds \right) \,\Big|\, (X_s)_{0 \leq s \leq t}, \, \tau > t \right],$$

which we interpret as the price of the payoff at time $0 \leq t \leq T$ given that default has not yet occurred.

Consider the Markov process $Z = (X, Y, V)$ where X, Y are as above, and V is a process that, when started at v is at $v + t$ after t units of time, that is, $V_t = V_0 + t$. The generator of Z is

$$A = (1/2)\sigma^2 x^2 D_x^2 + \mu x D_x + (1/2)D_y^2 + \rho\sigma x D_x D_y + D_v.$$

We want to compute

$$\mathbb{E}^{(x,y)}\left[F(X_T)e^{-\int_0^T \lambda\psi(Y_s - b(s))\,ds}\right] = \mathbb{E}^{(x,y,0)}\left[F(X_T)e^{-\int_0^T \lambda\psi(Y_s - b(V_s))\,ds}\right].$$

The Feynman-Kac formula says that the solution to the PDE

(5.3)
$$\begin{cases} D_t u(x,y,v,t) = Au(x,y,v,t) - \lambda\psi(y - b(v))u(x,y,v,t), \\ u(x,y,v,0) = F(x), \end{cases}$$

satisfies

$$\mathbb{E}^{(x,y)}\left[F(X_T)\exp\left(-\int_0^T \lambda\psi(Y_s - b(s))ds\right)\right] = u(x,y,0,T).$$

Thus, if we assume the Brownian motion Y has an random starting point Y_0 with density f that is independent of $(Y_t - Y_0)_{t \geq 0}$, then

$$\mathbb{E}^x\left[F(X_T)\exp\left(-\int_0^T \lambda\psi(Y_s - b(s))ds\right)\right] = \int_{\mathbb{R}} u(x,y,0,T)f(y)\,dy.$$

Using this and the Markov property, one can find the function $K(x,y,t)$ satisfying

$$K(X_t, Y_t, t) = \mathbb{E}^x\left[F(X_T)\exp\left(-\lambda\int_t^T \psi(Y_s - b(s))\,ds\right)\,\Big|\,(X_s)_{0 \leq s \leq t},\,(Y_s)_{0 \leq s \leq t},\,\tau > t\right].$$

The price at time t, given that we know the history of the price process X_t and that default has not happened up to time t, is

$$\mathbb{E}\left[F(X_T)1\{\tau > T\}\,\Big|\,(X_s)_{0 \leq s \leq t},\,\tau > t\right] = \mathbb{E}\left[K(X_t, Y_t, t)\,\Big|\,(X_s)_{0 \leq s \leq t},\,\tau > t\right]$$
$$= \frac{\mathbb{E}[K(X_t, Y_t, t)1\{\tau > t\}\,|\,(X_s)_{0 \leq s \leq t}]}{\mathbb{E}[1\{\tau > t\}\,|\,(X_s)_{0 \leq s \leq t}]}.$$

It follows from the SDE for X that

$$B_t' = W_t = \frac{1}{\sigma}\left[\log X_t - \log X_0 + \left(\frac{\sigma^2}{2} - \mu\right)t\right],$$

so if we observe the asset price X, then we can reconstruct the standard Brownian motion B'. On the other hand,

$$X_t = X_0\exp\left(\sigma B_t' - \left(\frac{\sigma^2}{2} - \mu\right)t\right).$$

Now,

$$\mathbb{E}[K(X_t, Y_t, t)1\{\tau > t\} \mid (X_s)_{0 \le s \le t}]$$

$$= \mathbb{E}\left[K\left(X_0 \exp\left(\sigma B_t' - \left(\frac{\sigma^2}{2} - \mu\right)t\right), Y_0 + \rho B_t' + \sqrt{1 - \rho^2}B_t'', t\right)\right.$$

$$\times \mathbf{1}\left\{\int_0^t \psi\left(Y_0 + \rho B_s' + \sqrt{1 - \rho^2}B_s'' - b(s)\right)\, ds \le U\right\}$$

$$\left.\Bigg| X_0, (B_s')_{0 \le s \le t}\right].$$

We therefore want to be able to compute for a function $c : \mathbb{R}_+ \to \mathbb{R}$ the conditional expected value

$$\mathbb{E}\left[K\left(X_0 \exp\left(\sigma c(t) - \left(\frac{\sigma^2}{2} - \mu\right)t\right), Y_0 + \rho c(t) + \sqrt{1 - \rho^2}B_t'', t\right)\right.$$

$$\left.\times \mathbf{1}\left\{\int_0^t \psi\left(Y_0 + \rho c(s) + \sqrt{1 - \rho^2}B_s'' - b(s)\right)\, ds \le U\right\} \Bigg| X_0\right]$$

$$= \mathbb{E}\left[K\left(X_0 \exp\left(\sigma c(t) - \left(\frac{\sigma^2}{2} - \mu\right)t\right), Y_0 + \rho c(t) + \sqrt{1 - \rho^2}B_t'', t\right)\right.$$

$$\left.\times \exp\left(-\int_0^t \psi\left(Y_0 + \rho c(s) + \sqrt{1 - \rho^2}B_s'' - b(s)\right)\, ds\right) \Bigg| X_0\right],$$

with $(B_t'')_{t \ge 0}$ a standard Brownian motion independent of X_0. We can do this using Feynman-Kac.

The denominator in the formula for the price at time t is a special case of the numerator we have just calculated with $K \equiv 1$, and it can be dealt with in the same way.

We have thus observed that computing the price of a contingent claim reduces to solving certain PDEs with coefficients depending on the path of the asset price.

6. Numerical Results

In this section we present the results of some numerical experiments. We solved the PDE/ODE system (2.12) using the pseudo-spectral Implicit-Explicit Fourth Order Runge-Kutta scheme ARK4(3)6L[2]SA-ERK, taking 8192 nodes and period 16, developed in [KC03]. For the function ψ we used the Fejér kernel of order 512 applied to the indicator of the set $\{x \in \mathbb{R} : x < 0\}$; in other words ψ is the Cesàro sum of the truncated Fourier series of order 512 of the indicator of the set $\{x \in \mathbb{R} : x < 0\}$. The time horizon was taken to be $T = 8$, the initial distribution of the credit index process Y was taken to be normal ($Y_0 \sim N(0, \sigma^2)$ with standard deviations $\sigma = 0.0625, 0.125, 0.25, 0.5$), and the time to default was taken to have an exponential distribution ($G(t) = e^{-\nu t}$ with rates $\nu = 0.0625, 0.125, 0.25, 0.5$). We show the resulting barrier b in Figure 1. We also show the relative error between the survival function $G(t)$ and the numerically computed value of $\int_{\mathbb{R}} u(x, t)\, dx$ (recall (2.3)), and the relative error between the hazard rate $-g(t)/G(t)$ and the numerically computed value of $\int_{\mathbb{R}} \psi(x - b(t))u(x, t)\, dx / \int_{\mathbb{R}} u(x, t)\, dx$ (recall (2.4)).
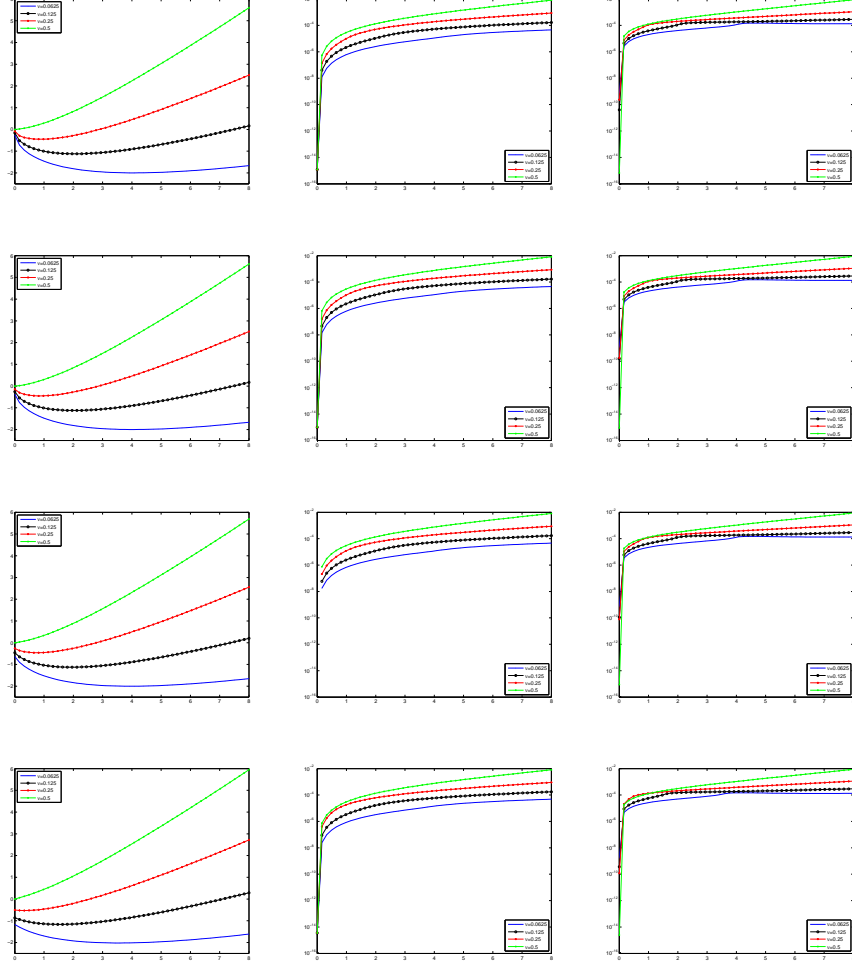
FIGURE 1. This figure displays the results of the numerical experiments described in Section 6. Successive rows represent the values $\sigma = 0.0625, 0.125, 0.25, 0.5$ of the standard deviation for the initial distribution of the credit index process Y. The first panel in each row gives the barriers for the rate parameters $\nu = 0.0625, 0.125, 0.25, 0.5$ of the exponential default time distribution. The second (resp. third) panel in each row gives the relative errors between the actual survival function values $G(t)$ (resp. the actual hazard function values $-g(t)/G(t)$) and the numerically computed ones – see the text for details.

7. Calibrating the default distribution using CDS rates

For the sake of completeness, we review briefly the scheme proposed in [DP11] for determining the distribution of the time to default.

A credit default swap (CDS) is a contract between two parties. The buyer of the swap makes a number of predetermined payments until the moment of default. The seller is liable to pay the unrecovered value of the underlying bond in the event of a default before maturity. Normalizing the notional value of the bond to 1, the seller's contingent payment is $1 - R$, where $R \in (0, 1)$ is the recovery rate, which we take to be constant. The premium payments are made at a set of times $\{t_i\}$. The maturities are a subset of the premium payment times; that is, they are of the form $T_0 = 0$, $T_j = t_{k(j)}$, $j = 1, \ldots, n$. For $j = 1, \ldots, n$ there is an upfront premium π_j^0 and a running premium rate π_j^1 (having accrual factors δ_i). Denote the price at time zero of a zero coupon risk-free bond with maturity t_j by $p_0(t_j)$. It follows from standard non-arbitrage arguments that

$$(7.1) \quad \pi_j^0 + \pi_j^1 \sum_{i=k(j-1)}^{k(j)-1} \delta_i p_0(t_i) G(t_i) = (1 - R) \sum_{i=k(j-1)+1}^{k(j)} p_0(t_i)(G(t_{i-1}) - G(t_i)),$$

where $G(t) = \mathbb{P}\{\tau > t\}$ is the tail of the distribution of the time to default.

Suppose now that the default distribution has piecewise constant hazard rate; that is, that

$$G(t) = \exp\left(-\int_0^t h(s)\, ds\right), \quad t \geq 0,$$

where $h(s) = h_i$ for $s \in [T_i, T_{i+1})$. Given the market data $(\pi_1^0, \pi_1^1), (\pi_2^0, \pi_2^1), \ldots$ we can find, using equation (7.1), the constants $h_0, h_1, \ldots$.

We use the following procedure to find the barrier b. Set $\nu = h_0$ and $G(t) = e^{-\nu t}$. Given the initial density f, which we can choose to be any strictly positive function f that is twice continuously differentiable with bounded f, f' and f'', we want to find a barrier such that for $0 \leq t \leq T = T_1$ we have

$$e^{-\nu t} = \mathbb{E}\left[\int_{\mathbb{R}} f(x) \exp\left(-\lambda \int_0^t \psi(x + B_s - b(s))\, ds\right)\, dx\right].$$

This can be achieved by solving the ODE/PDE system (2.12). Next, set $\nu_1 = h_1$, $T = T_2 - T_1$, $f_1(x) = \mathbb{E}\left[f(x) \exp\left(-\lambda \int_0^{T_1} \psi(x + B_s - b(s))\, ds\right)\right]$ and find a barrier with $b_1(0) = b(T_1)$ such that on $0 \leq t < T = T_2 - T_1$ we have

$$e^{-\nu_1 t} = \mathbb{E}\left[\int_{\mathbb{R}} f_1(x) \exp\left(-\lambda \int_0^t \psi(x + B_s - b_1(s))\, ds\right)\, dx\right].$$

This procedure can be repeated until we find a function b on $[0, T_n]$ that is continuously differentiable everywhere, except perhaps the finite number of points $T_1, \ldots, T_n$.

8. Duhamel's formula

For the sake of reference, we provide a statement of Duhamel's formula. Given functions $v : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ and $b : \mathbb{R}_+ \to \mathbb{R}$, the solution of

$$(8.1) \qquad \begin{cases} \left(\partial_t - \dfrac{\partial_{xx}}{2}\right) u = -\psi_b v, & x \in \mathbb{R}, t > 0, \\ \qquad\qquad u(x, 0) = f(x) & x \in \mathbb{R}, \end{cases}$$

is given by

$$u(x, t) \quad = \quad [G * (f\delta_{t=0})](x, t) + [G * (-\psi_b v)](x, t)$$

$$(8.2) \qquad = \int_{\mathbb{R}} G(x-y,t)f(y)\,dy - \int_0^t \int_{\mathbb{R}} G(x-y,t-s)\psi_{b(s)}(y)v(y,s)\,dy\,ds,$$

where

$$(8.3) \qquad\qquad G(x,t) := \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}, \quad x \in \mathbb{R},\, t > 0.$$

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