Local Times and Almost Sure Convergence of Semi-Martingales

By

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Technical Report No. 316
August 1991
(revised August 1992)

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*Research partially supported by NSF Grant No. DMS91-07531

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Introduction. In this paper, we study the connection between the almost sure convergence of semi-martingales and the asymptotic behaviour of their local times. Our study is motivated by the following two examples.

Example 1. Let $h(t)$ be a non-negative decreasing function and (B_t) a Brownian motion. Let $X_t = h(t) B_t$. Integrating by parts, we get

$$X_t = X_0 + \int_0^t h(s)\, dB_s + \int_0^t B_s\, dh(s).$$

If X_t converges as $t \to \infty$, then it must converge to zero, because $\{t : B_t = 0\}$ is unbounded. We expect that this property gets reflected in the behavior of the local time of X at zero, which is easily seen to be equal to the process $(\int_0^t h(s)\, dl(s))$, where l_t is the local time of B at zero. In section 1, we state necessary and sufficient conditions so that $\lim_{t\to\infty} h(t) B_t = 0$ and $\lim_{t\to\infty} \int_0^t h(s)\, dl_s < \infty$, in the case when h is a deterministic function. These results are due to Jeulin and Yor {see [6], Proposition 15) and C. Donati-Martin (see [3], p.150) respectively.

Example 2. Let $\varepsilon(t)$ be a non-negative decreasing function, $F : (a_1, a_2) \to \mathbb{R}$, F strictly increasing, $a_1 < 0 < a_2$, $F(0) = 0$. The process (Y_t), studied in Chan and Williams [2], satisfies the stochastic differential equation

$$dY_t = -F(Y_t)\, dt + \varepsilon(t)\, dB_t.$$

In [2], the authors show that $Y_t \to 0$ a.s. iff $\int_0^\infty \exp\left(\frac{-c}{(\varepsilon(u))^2}\right) du < \infty$ for all $c > 0$.

We expect that the local times at the origin play some role in the convergence of the processes X and Y in Examples 1 and 2 respectively. A closer examination of the processes X and Y strengthens this belief: in both cases, the effect of the drift term is to pull the process towards the origin. We make this notion precise below in Section 2, where we study subsets of the sample space Ω on which the drift term of a given semi-martingale has this property. In the case of a continuous martingale, the almost sure convergence of the martingale, its local time and its quadratic variation processes are all equivalent. In the case of the semi-martingales which we study, the situation is similar and yet there are some subtle differences.

1. **Brownian Asymptotics.**

Consider a continuous local martingale $(M_t)_{t\geq 0}$, $M_0 = 0$ a.s. Let $(L_t^x)_{t\geq 0; x \in \mathbb{R}}$ denote a jointly continuous version of its local times and $(<M>_t)$ its quadratic variation process. The following lemma is well known and follows by writing M as a time change of Brownian motion.

Lemma 1.1. *For every* $x \in \mathbb{R}$, *the following sets are almost surely equal:*

(i) $\{\omega : \lim_{t\to\infty} M_t \text{ exists}\}$ ii) $\{\omega : \lim_{t\to\infty} M_t \text{ exists and is finite}\}$

(iii) $\{\omega : <M>_\infty < \infty\}$ iv) $\{\omega : L_\infty^x < \infty\}$.

Now, consider a Brownian motion $(B_t)_{t\geq 0}$ starting from zero. Let $g_t = \sup\{s \leq t : B_s = 0\}$ and $\{H_u, u \geq 0\}$ a locally bounded previsible process; let $M_t^H = H_{g_t} B_t$. It follows from the 'Balayage formula' (Azéma-Yor [1], Yor [8]) that (M_t^H) is a continuous local martingale and it is easy to see that its local time at 0 is $(\int_0^t |H_s|\, dl_s)_{t\geq 0}$ where $(l_s)_{s\geq 0}$ is the local time of (B_t) at the origin. We have the following consequence of lemma 1.1.

Proposition 1.2. *The following are equivalent.*

(i) $\lim_{t\to\infty} (H_{g_t} B_t) = 0$ a.s., (ii) $\int_0^\infty H_{g_s}^2\, ds < \infty$ a.s., (iii) $\int_0^\infty |H_s|\, dl_s < \infty$ a.s.

We now specialise to the case when $H_s = h(s)$, where h is a non-negative decreasing deterministic function. It is easy to see that the process $(\int_0^t h(s)\, dl_s)_{t\geq 0}$ is also the local time at the origin of the semi-martingale $X_t = h(t) B_t$. It is well known that the measure (induced by the increasing process $(l_t)_{t\geq 0}$ is, almost surely, singular w.r.t. the Lebesgue measure on $[0, \infty)$. The following result (first proved by Donati-Martin [3], when h is smooth) is therefore surprising.

Theorem 1.3. *The following are equivalent.*

(i) $\int_0^\infty h(s)\,dl_s < \infty$ a.s.; (ii) $\int_0^\infty h(s)\,\frac{ds}{\sqrt{s}} < \infty$

The proof of this theorem depends on a result of Jeulin [4] (see also Jeulin [5] and, for some applications, Xue [7]). We state and prove it for completeness.

Lemma 1.4. *Let* $(R_t)_{t\geq 0}$ *be a positive measurable process such that*

1) The law ν *of* R_t *does not depend on* t.

2) $\nu(\{0\}) = 0$

3) $E(R_t) < \infty$.

Then, for any positive Radon measure μ *on* $\mathbb{R}$, $\int_0^\infty d\mu(t) < \infty$ *iff* $\int_0^\infty R_t\,d\mu(t) < \infty$ *a.s.*

Proof. The 'if' part is clear: If $\int_0^\infty d\mu(t) < \infty$, then in fact by condition 3), $E\int_0^\infty R_t\,d\mu(t) < \infty$.

Conversely, let $\int_0^\infty R_t\,d\mu(t) < \infty$ a.s. and let n be such that $P(J_n) > 0$ where $J_n = \{\omega : \int_0^\infty R_t\,d\mu(t) \leq n\}$. Then,

$$\begin{aligned} E(I_{J_n} R_t) &= \int_0^\infty du\,E[I_{J_n} I_{\{R_t>u\}}] = \int_0^\infty du\,E([I_{J_n} - I_{\{R_t\leq u\}}]^+) \\ &\geq \int_0^\infty du\,(P(J_n) - \nu([0,u]))^+ . \end{aligned}$$

Since $P(J_n) > 0$ and $\nu[0,u] \to 0$ as $u \to 0$ by Condition 2), the last integral is in fact strictly positive, say equal to a_n. It follows that:

$\infty > nP(J_n) \geq \int_0^\infty E(J_n R_t)\,d\mu(t) \geq a_n \int_0^\infty d\mu(t)$ and the proof is complete.

Proof of Theorem 1.3. Since $E(l_s) = c\sqrt{s}$ where c is a constant not depending on s, it is easy to see (since h is deterministic) that $E\int_0^\infty h(s)\,dl_s = \frac{c}{2}\int_0^\infty h(s)\,\frac{ds}{\sqrt{s}}$. Clearly,

(ii) $\Rightarrow$ (i).

To go the other way, suppose that (i) holds. We will show that the two functions $h(t)\sqrt{t}$ and $\int_0^t \sqrt{s}\, dh(s)$ converge to a finite limit as $t \to \infty$. It follows from the equation

$$\sqrt{t}\, h(t) = \int_0^t \sqrt{s}\, dh(s) + \int_0^t h(s) \frac{ds}{2\sqrt{s}}$$

that $\int_0^\infty h(s) \frac{ds}{\sqrt{s}} < \infty$.

We now show: $\lim_{t\to\infty} \int_0^t \sqrt{s}\,(-dh(s)) < \infty$. We have:

$$h(t)\, l_t + \int_0^t l_s\,(-dh_s) = \int_0^t h(s)\, dl_s .$$

Since both terms in the LHS are non negative and since the RHS converges by assumption, it follows that $\int_0^\infty l_s(-dh_s) < \infty$ a.s. Now, Lemma 1.4 applied to $R_t = \frac{l_t}{\sqrt{t}}$ and $d\mu(s) = -\sqrt{s}\, dh(s)$ shows that $\int_0^\infty \sqrt{s}\,(-dh(s)) < \infty$.

We then show: $\lim_{t\to\infty} h(t)\sqrt{t} < \infty$. Since h is decreasing, $h(t)\,|B_t| \le h(g_t)\,|B_t|$, where $g_t = \max\{s \le t : B_s = 0\}$. From Proposition 1.2, it follows that $h(t)\, B_t \to 0$ a.s. Since $(h_t B_t)$ are gaussian random variables, this implies that $h_t B_t \to 0$ in L^1. i.e. $h(t)\sqrt{t} \to 0$ and the proof is complete.

Remark 1.5. Let H be a bounded previsible process. From Proposition 1.2, $(H_{g_t} B_t)_{t\ge 0}$ is a martingale which is not uniformly integrable, unless it is identically zero. Consequently, if $\int_0^\infty |H_s|\, dl_s < \infty$ a.s. then by Proposition 1.2 $\int_0^\infty H_{g_s}^2\, ds < \infty$ a.s. but $E(\int_0^\infty H_{g_s}^2\, ds)^{1/2} = \infty$.

Remark 1.6. There are situations where the conclusions of Lemma 1.4 are true, but $ER_t = \infty$. Let $g_s = \sup\{u \le s : B_u = 0\}$. Consider the pair $(R_s, \mu(ds))$ where $R_s = \frac{s^{2\alpha}}{g_s^{2\alpha}}$, $\alpha > 1/2$ and $\mu(ds) = \frac{ds}{s^{2\alpha}} 1_{[1,\infty)}(s)$. Then $\int_0^\infty \mu(ds) < \infty$ and from Proposition 1.2 applied to $H_s = \frac{1}{s^\alpha}$, it follows that $\int_0^\infty R_s\, \mu(ds) < \infty$. But this conclusion can-not be obtained from Lemma 1.4 because $ER_1 = \infty$, since g_1 is arcsine distributed.

In the next Theorem, we reproduce a result of Jeulin and Yor [6] which gives a necessary and sufficient condition for $h(t)\, B_t$ to converge to zero almost surely when h is a deterministic function. Here, we consider the particular case where h is decreasing.

Theorem 1.7. *Let* $h(t)$ *be a non negative decreasing deterministic function. Then, the following are equivalent*

i) $\lim_{t\to\infty} h(t)\, B_t = 0$ a.s.; ii) *for every* $\varepsilon > 0$, $\int_1^\infty \exp\left(\frac{-\varepsilon}{th^2(t)}\right) \frac{dt}{t} < \infty$.

Proof. Let $B_1^* = \sup_{0 \le t \le 1} |B_t|$. Then, condition ii) is equivalent to

ii′) $\sum_{n=1}^{\infty} P\left(B_1^* > \frac{x 2^{-n/2}}{h(2^n)}\right) < \infty$. for every $x > 0$.

This follows from the inequalities: for $\varepsilon > 0$,

$$\sum_{k \ge 0} \int_{2^k}^{2^{k+1}} \exp\left(\frac{-2\varepsilon 2^{-(k+1)}}{h^2(2^{k+1})}\right) \frac{dt}{t} \le \int_1^\infty \exp\left(-\frac{\varepsilon}{th^2(t)}\right) \frac{dt}{t} \le \sum_{k \ge 0} \int_{2^k}^{2^{k+1}} \exp\left(-\frac{\varepsilon}{2} \cdot \frac{2^{-k}}{h^2(2^k)}\right) \frac{dt}{t}$$

and $\left(\frac{2}{\pi}\right)^{1/2} \int_\varepsilon^\infty \exp\left(-\frac{1}{2} u^2\right) du \le P(B_1^* > \varepsilon) \le 2 \exp\left(-\frac{1}{2} \varepsilon^2\right)$. We now show that ii′) holds iff i) holds. Let $V_n = 2^{-n/2} \sup_{2^n \le t \le 2^{n+1}} |B_t - B_{2^n}|$. It is easy to see that ii′) is equivalent to $\lim_{n\to\infty} 2^{n/2} V_n h(2^n) = 0$ and that ii′) implies $\lim_{n\to\infty} |B_{2^n}|\, h(2^n) = 0$. Now ii′) $\Rightarrow$ i) follows from the above observations and from the inequalities

$$\sup_{2^n \le t \le 2^{n+1}} h(t)\, |B_t| \le 2^{n/2} V_n h(2^n) + |B_{2^n}|\, h(2^n)$$

Conversely, if i) holds, then $\lim_{t\to\infty} \sqrt{t}\, h(t) = 0$. Moreover, (V_n) are independent and have the same law as $B_1^* = \sup_{0 \le t \le 1} |B_t|$. Hence $\lim_{n\to\infty} 2^{n/2} V_n h(2^n) = 0$ and ii′) holds.

Section 2. In this section, we prove a version of Lemma 1.1 for semimartingales. This is Theorem 2.4. Our analysis will be restricted to the class of semi-martingales (X_t) with $\sum_{s \le t} |\Delta X_s| < \infty$, a.s. for all $t \ge 0$. These can be written in the form $X_t = X_0 + M_t + V_t$ where (M_t) is a *continuous* local martingale and (V_t) a process of finite variation. For a semi-martingale X, this property will be assumed to hold for the rest of the paper.

We recall the Tanaka formula for the semi-martingale X: for $a \in \mathbb{R}$,

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t I_{\{X_{s-}>a\}}\, dX_s + \frac{1}{2}\mathbf{L}_t^a \qquad (1)$$

$$(X_t - a)^- = (X_0 - a)^- - \int_0^t I_{\{X_{s-}\le a\}}\, dX_s + \frac{1}{2}\mathbf{L}_t^a \qquad (2)$$

where

$$\mathbf{L}_t^a = l_t^a + L_t^a$$

$$l_t^a = 2\left(\sum_{s\le t} \left(I_{\{X_{s-}<a\}}(X_s - a)^- + I_{\{X_{s-}\le a\}}(X_s - a)^+\right)\right)$$

and $(L_t^a)_{t\ge 0}$ is the local time at a of X, which is an increasing continuous process, supported on the set $\{s : X_{s-} = X_s = a\}$. For the semi-martingales which we consider, there is a version of the process $(\mathbf{L}_t^a, t \ge 0, a \in \mathbb{R})$ which is jointly right continuous in (t, a) and has left limits (see Yor [9]). This property is important in what follows and we will always take equations (1) and (2) to be true outside a null set, for all $t \ge 0$ and for all $a \in \mathbb{R}$.

Let $V_+(t) = \int_0^t I_{\{X_{s-}>0\}}\, dV_s$ and $V_-(t) = \int_0^t I_{\{X_{s-}\le 0\}}\, dV_s$. The finite variation part V of the processes considered in Examples 1 and 2 have the property that V_+ is decreasing and V_- is increasing. For a semi-martingale $X = X_0 + M + V$, we define the following subsets of Ω:

$$\nu_+^- = \{\omega : V_+ \text{ is decreasing and } V_- \text{ is increasing}\}$$

$$\nu_-^+ = \{\omega : V_+ \text{ is increasing and } V_- \text{ is decreasing}\}$$

$$\nu_+^+ = \{\omega : V_+, V_- \text{ are increasing}\}$$

$$\nu_-^- = \{\omega : V_+, V_- \text{ are decreasing}\}.$$

Of course, these sets do not necessarily partition Ω. We shall use the notation $\underline{X}_\infty$ (resp. $\overline{X}_\infty$) for $\liminf_{t\to\infty} X_t$ (resp. $\limsup_{t\to\infty} X_t$). We shall denote $\lim_{t\to\infty} X_t$ by X_∞ whenever it exists. In this notation, $\{\omega : X_\infty \in (a, b)\}$ means $\{\omega : \lim_{t\to\infty} X_t$ exists and belongs to $(a, b)\}$. The following lemma will be used in the proof of our main result (Theorem 2.4).

Lemma 2.1. *Let* $-\infty < a < b < \infty$. *Then:*

(i) On the set $\{\omega : \int_0^1 I_{(a,b]}(X_{s-})\, dV_s$ *is decreasing* $\}$, *we have, for all* $c \in (a, b]$, *almost surely,*

$$\{\omega : L^a_\infty < \infty\} \subset \{\omega : L^c_\infty < \infty\}$$

and $\{\omega : L^a_\infty < \infty\} \subset \{\omega : \underline{X}_\infty \geq b\} \cup \{\omega : \overline{X}_\infty \leq a\} \cup \{\omega : X_\infty \in (a, b)\}$.

(ii) On the set $\{\omega : \int_0^1 I_{(a,b]}(X_{s-})\, dV_s$ *is increasing* $\}$, *we have for all* $c \in [a, b)$, *almost surely,*

$$\{\omega : L^b_\infty < \infty\} \subset \{\omega : L^c_\infty < \infty\}$$

and

$$\{\omega : L^b_\infty < \infty\} \subset \{\omega : \underline{X}_\infty \geq b\} \cup \{\omega : \overline{X}_\infty \leq a\} \cup \{\omega : X_\infty \in (a, b)\}$$

Proof. (i) Suppose that $\int_0^t I_{(a,b]}(X_{s-})\, dV_s$ is decreasing. Then, for $c \in (a, b]$ we have, from equation (1), that

$$\begin{aligned} &(X_t - a)^+ - (X_0 - a)^+ - (X_t - c)^+ + (X_0 - c)^+ \\ &= \int_0^t I_{(a,c]}(X_{s-})\, dM_s + \frac{1}{2} L^a_t - \frac{1}{2} L^c_t + \int_0^t I_{(a,c]}(X_{s-})\, dV_s . \end{aligned} \tag{3}$$

Note that the LHS of equation (3) is bounded by 2(b-a). Now if $L^a_\infty < \infty$, it follows that the decreasing function $\int_0^t I_{(a,c]}(X_{s-})\, dV_s - \frac{1}{2} L^c_t$ has a finite limit, as also $\int_0^t I_{(a,c]}(X_{s-})\, dM_s$ (this follows from Lemma 1.1). Thus $L^c_\infty < \infty$, proving the first inclusion in (i). Now the LHS of equation (3) has a finite limit as $t \to \infty$. In particular, it has a finite limit when $c = b$. But this means precisely that $\underline{X}_\infty \geq b$ or $\overline{X}_\infty \leq a$ or $X_\infty \in (a, b)$.

(ii) The proof is similar to that of (i) using the equation

$$\begin{aligned} &(X_t - b)^- - (X_0 - b)^- - (X_t - c)^- + (X_0 - c)^- \\ &= -\int_0^t I_{(c,b]}(X_{s-})\, dM_s + \frac{1}{2} L^b_t - \frac{1}{2} L^c_t - \int_0^t I_{(c,b]}(X_{s-})\, dV_s . \end{aligned} \tag{4}$$

for all $c \in (a, b]$. □

Corollary 2.2. $\{\omega : \int_0^t I_{(a,b]}(X_{s-})\, dV_s$ *is monotonic*$\} \cap \{\omega : [X, X]_\infty < \infty\}$

$$\overset{\text{a.s.}}{\subset} \{\omega : \underline{X}_\infty \geq b\} \cup \{\omega : \overline{X}_\infty \leq a\} \cup \{\omega : X_\infty \in (a, b)\} .$$

Proof. It follows from the results in Yoeurp [10] that

$$[X,X]_\infty = \int_{-\infty}^{\infty} L_\infty^x \, dx .$$

Hence if $[X,X]_\infty(\omega) < \infty$, then $L_\infty^x(\omega) < \infty$ for all $x \notin N(\omega) \subset \mathbb{R}$, $\lambda(N(\omega)) = 0$, where λ is the Lebesgue measure. It follows from Fubini's theorem that there exists $N \subset \mathbb{R}$, $\lambda(N) = 0$, such that:

$$\forall x \notin N, \quad \{\omega : [X,X]_\infty < \infty\} \overset{\text{a.s.}}{\subset} \{\omega : L_\infty^x < \infty\} .$$

Choose a_n, $b_n \notin N$, $a_n \downarrow a$, $b_n \uparrow b$. If $\int_0^t I_{(a,b]}(X_{s-})\, dV_s$ is monotone, then so is $\int_0^t I_{(a_n,b_n]}(X_{s-})\, dV_s$ for every n. It follows from Lemma 2.1 that if $[X,X]_\infty(\omega) < \infty$, then $\underline{X}_\infty \geq b_n$ or $\overline{X}_\infty \leq a_n$ or $X_\infty \in (a_n, b_n)$ for all n. This means that $\underline{X}_\infty \geq b$ or $\overline{X}_\infty \leq a$ or $X_\infty \in (a,b)$. □

Corollary 2.3. *On the set* V_+^-,

$$\{\omega : \underline{X}_\infty < \overline{X}_\infty\} \cap \{\omega : L_\infty^a < \infty\} \overset{\text{a.s.}}{\subset} \{\omega : \overline{X}_\infty \leq a\} \text{ for } a > 0 ,$$

and

$$\{\omega : \underline{X}_\infty < \overline{X}_\infty\} \cap \{\omega : L_\infty^a < \infty\} \overset{\text{a.s.}}{\subset} \{\omega : \underline{X}_\infty \geq a\} \text{ for } a < 0 .$$

In particular, on the set V_+^-,

$$\{\omega : \underline{X}_\infty < a < \overline{X}_\infty\} \overset{\text{a.s.}}{\subset} \{\omega : L_\infty^a = \infty\}$$

for every a.

Theorem 2.4. *Let* (X_t) *be a semi-martingale with* $X_t = X_0 + M_t + V_t$ *where* (M_t) *is a continuous local martingale and* (V_t) *a process of finite variation. Then, we have the following:*

a) (i) On the set V_+^-, *for all* $x \neq 0$,

$$\{\omega : \lim_{t\to\infty} X_t \text{ exists}\} \overset{\text{a.s.}}{\subset} \{\omega : L_\infty^x < \infty\}$$

and

$$\{\omega : \lim_{t\to\infty} X_t \text{ exists}\} = \bigcap_{\substack{x \in Q \\ x \neq 0}} \{\omega : L_\infty^x < \infty\}$$

(ii) $\{\omega : \lim_{t\to\infty} X_t = X_\infty$ *exists and* $X_\infty \neq 0, \pm\infty\} \overset{\text{a.s.}}{\subset} \{\omega : [X,X]_\infty < \infty\} \subset \{\omega : \lim_{t\to\infty} X_t$

exists } *on the set* ν_+^-.

b) (i) For all $x \in \mathbb{R}$,

$$\{\omega : \lim_{t\to\infty} X_t \text{ exists}\} \overset{\text{a.s.}}{\subset} \{\omega : L_\infty^x < \infty\}$$

$$\{\omega : \lim_{t\to\infty} X_t \text{ exists}\} \overset{\text{a.s.}}{\subset} \bigcap_{x\in Q} \{\omega : L_\infty^x < \infty\}$$

holds on each of the sets ν_-^+, ν_+^+, ν_-^-.

(ii) $\{\omega : \lim_{t\to\infty} X_t = X_\infty \textit{ exists, } X_\infty \neq \pm\infty\} \overset{\text{a.s.}}{\subset} \{\omega : [X,X]_\infty < \infty\} \overset{\text{a.s.}}{\subset} \{\omega : \lim_{t\to\infty} X_t \textit{ exists}\}$

holds on each of the sets ν_-^+, ν_+^+, ν_-^-.

Proof. a) (i). Suppose $0 < a < b$, $a,b \in \mathbf{Q}$. Then, by Lemma 2.1(i),

$$\bigcap_{\substack{x\in Q\\ x\neq 0}} \{\omega : L_\infty^x < \infty\} \cap \nu_+^- \subset \{\omega : L_\infty^a < \infty\} \cap \{\omega : \int_0^{\cdot} I_{(a,b]}(X_{s-})\, dV_s \text{ is } \downarrow\}$$

$$\overset{\text{a.s.}}{\subset} \{\omega : \underline{X}_\infty \geq b\} \cup \{\omega : \overline{X}_\infty \leq a\} \cup \{\omega : X_\infty \in (a,b)\}$$

Letting $a \downarrow 0$, $b \uparrow \infty$, we get

$$\bigcap_{\substack{x\in Q\\ x\neq 0}} \{\omega : L_\infty^x < \infty\} \cap \nu_+^- \overset{\text{a.s.}}{\subset} \{\omega : \overline{X}_\infty \leq 0\} \cup \{\omega : X_\infty \in (0,\infty]\}$$

Similarly, from Lemma 2.1 (ii), it follows that

$$\bigcap_{\substack{x\in Q\\ x\neq 0}} \{\omega : L_\infty^x < \infty\} \cap \nu_+^- \overset{\text{a.s.}}{\subset} \{\omega : \underline{X}_\infty \geq 0\} \cup \{\omega : X_\infty \in [-\infty, 0)\}.$$

It follows that on the set ν_+^-

$$\bigcap_{\substack{x\in Q\\ x\neq 0}} \{\omega : L_\infty^x < \infty\} \overset{\text{a.s.}}{\subset} \{\omega : \lim_{t\to\infty} X_t \text{ exists}\}.$$

Conversely, suppose $\omega \in \{\lim_{t\to\infty} X_t \text{ exists}\} \cap \nu_+^-$. Since $\{\lim_{t\to\infty} X_t \text{ exists}\} = \{X_\infty \in (0,\infty)\} \cup \{X_\infty \in (-\infty,0)\} \cap \{X_\infty = 0, \pm\infty\}$, it is sufficient to show that on each of the sets in the RHS, $L_\infty^a(\omega) < \infty$ for $\omega \in \nu_+^-$ and $a \neq 0$.

If $\omega \in \{X_\infty \in (0,\infty\} \cap \nu_+^-$, then from the decomposition

$X_t = X_0 + M_t + V_+(t) + V_-(t)$, it follows that V_-, V_+ and hence M converge to a finite limit and hence from equation (1) that $L_\infty^a < \infty$.

$$\text{i.e. } \{\omega : X_\infty \in (0,\infty)\} \cap \nu_+^- \overset{\text{a.s.}}{\subset} \{\omega : X, M, V_+, V_- \text{ converge}\}$$

$$\overset{\text{a.s.}}{\subset} \{\omega : L_\infty^a < \infty\}.$$

Similarly, from equation (2), it follows that

$$\{\omega: X_\infty \in (-\infty, 0)\} \cap \nu_+^- \overset{a.s.}{\subset} \{\omega: L_\infty^a < \infty\}.$$

If $\omega \in \{X_\infty = 0, \pm\infty\} \cap \nu_+^-$, then since $a \neq 0$, the process does not visit a after a finite time and there are no jumps of X which cross a. Then, since $L_t^a = L_t^a + l_t^a$, it follows that L_t^a does not increase after a finite time i.e. $L_\infty^a < \infty$. Thus $\{X_\infty = 0, \pm\infty\} \cap \nu_+^- \overset{a.s.}{\subset} \{\omega: L_\infty^a < \infty\}$ and the proof of a) (i) is complete.

a) (ii) To prove the first inclusion in a) (ii), note that as in the proof of a) (i),

$$\{\omega: X_\infty \neq 0, \pm\infty\} \overset{a.s.}{\subset} \{\omega: M, V_+, V_- \text{ converge to a finite limit}\}.$$

If $\omega \in$ RHS above, then of course $< X^c >_\infty = < M >_\infty < \infty$ and

$$\sum_s \Delta X_s^2 \leq \sup_s |\Delta X_s| \sum_s |\Delta X_s| \leq \sup_s |\Delta X_s| (\sum_s |\Delta V_+(s)| + |\Delta V_-(s)|) < \infty.$$

Thus, $\{\omega: X_\infty \neq 0, \pm\infty\} \cap \nu_+^- \overset{a.s.}{\subset} \{\omega: [X,X]_\infty < \infty\}$.

To prove the second inclusion in a) (ii), let first $0 < a < b$. From Corollary 2.2 we get,

$$\{\omega: [X,X]_\infty < \infty\} \cap \nu_+^- \overset{a.s.}{\subset} \{\omega: \overline{X}_\infty \leq a\} \cup \{\omega: \underline{X}_\infty \geq b\} \cup \{\omega: X_\infty \in (a,b)\}.$$

Now letting $a \downarrow 0$, $b \uparrow \infty$, we get

$$\{\omega: [X,X]_\infty < \infty\} \cap \nu_+^- \overset{a.s.}{\subset} \{\omega: \overline{X}_\infty \leq 0\} \cup \{\omega: X_\infty \in (0,\infty]\}. \tag{5}$$

Similarly, by applying Corollary 2.2 to the case $a < b < 0$, we can prove,

$$\{\omega: [X,X]_\infty < \infty\} \cap \nu_+^- \overset{a.s.}{\subset} \{\omega: \overline{X}_\infty \geq 0\} \cup \{\omega: X_\infty \in [-\infty,0)\}. \tag{6}$$

The second inclusion in a)(ii) follows from (5) and (6) and completes the proof of a).

b)(i). The proof that $\bigcap_{b \in Q} \{\omega: L_\infty^b < \infty\} \overset{a.s.}{\subset} \{\omega: \lim_{t\to\infty} X_t \text{ exists}\}$ on the sets ν_-^+, ν_+^+, ν_-^- is similar to the proof for the set ν_+^-, described in a)(i) using Lemma 2.1. We will prove the reverse inclusion on the set ν_-^+, the other two cases being similar. Fix $b \in \mathbb{R}$.

Since $\{\omega: \lim_{t\to\infty} X_t \text{ exists}\} = \{\omega: X_\infty = 0, \pm\infty\} \cup \{X_\infty \neq 0, \pm\infty\}$, it suffices to show that each of the sets in the RHS is contained in $\{\omega: L_\infty^b < \infty\}$ on ν_-^+. For the set $\{\omega: X_\infty \neq 0, \pm\infty\}$, the proof is similar to the proof given in a)(i). We will show the inclusion $\{\omega: X_\infty = 0, \pm\infty\} \cap \nu_-^+ \overset{a.s.}{\subset} \{\omega: L_\infty^b < \infty\}$ for every $b \in \mathbb{R}$.

Now, if $X_\infty = \pm\infty$ or if $X_\infty = 0, \pm\infty$ and $b \neq 0$, then L_t^b does not increase after a finite time. In particular, $L_\infty^b < \infty$. If $X_\infty = 0$ and $b = 0$, then since on v_-^+, $V_+(t) + L_t^0$ is increasing, it follows, by letting $t \to \infty$ in equation (5) that $L_\infty^0 < \infty$.

b)(ii) It is easy to see, using the decomposition $X_t = X_0 + M_t + V_+(t) + V_-(t)$, that if $X_\infty \neq 0, \pm\infty$ then M, V_+, V_- converge to a finite limit on the set v_-^+. If $X_\infty = 0$, then we argue as in b)(i) and conclude that M, V_+, V_- converge to a finite limit on the set v_-^+. Thus, on v_-^+,

$$\{\omega: X_\infty \neq \pm\infty\} \overset{\text{a.s.}}{\subset} \{\omega: V^+, V^-, M \text{ converge to a finite limit}\}.$$

Now exactly as in case a)(ii), we can show that the RHS set is contained in $\{\omega: [X,X]_\infty < \infty\}$. The inclusion $\{\omega: [X,X]_\infty < \infty\} \cap v_-^+ \overset{\text{a.s.}}{\subset} \{\omega: \lim_{t\to\infty} X_t \text{ exists }\}$ is also exactly as in case a)(ii). This completes the proof of b) and the proof of the theorem. □.

Remark 2.5. The inclusions in Theorem 2.4 viz:

$$\{\omega: X_\infty \neq 0, \pm\infty\} \subset \{\omega: [X,X]_\infty < \infty\} \subset \{\omega: \lim_{t\to\infty} X_t \text{ exists}\}$$

can be strict. Consider for example the process $X_t = h(t) B_t$ where (B_t) is a Brownian motion and h(t) non negative decreasing, $h(t) \sim \frac{1}{\sqrt{t \log t}}$ as $t \to \infty$. Then, by the law of the iterated logarithm, $X_t \to 0$ a.s., but $[X,X]_\infty = \langle X^c \rangle_\infty = \int_0^\infty h^2(s)\,ds = \infty$. On the other hand, if $h(t) \sim \frac{1}{t}$ as $t \to \infty$, then $\{\omega: [X,X]_\infty < \infty\} = \Omega$, $\{\omega: X_\infty \neq 0\} = \phi$.

We now consider the local time L_t^0 on the set v_+^-. The following lemma shows that in this situation, L_t^0 plays a special role.

Lemma 2.6. *On the set* v_+^-,

$$\{\omega: L_\infty^0 < \infty\} \overset{\text{a.s.}}{\subset} \{\omega: \lim_{t\to\infty} X_t \text{ exists and is finite}\} \cap \{\omega: [X,X]_\infty < \infty\}.$$

Proof. From Lemma 2.1, it follows that for all $b \in \mathbb{R}_+$, on the set v_+^- $\{\omega: L_\infty^0 < \infty\} \overset{\text{a.s.}}{\subset} \{\omega: L_\infty^b < \infty\}$. From Theorem 2.4 a)(i), it follows that on v_+^-,

$$\{\omega: L_\infty^0 < \infty\} \overset{\text{a.s.}}{\subset} \{\omega: \lim_{t\to\infty} X_t \text{exists}\}.$$

Now, equations (1) and (2) imply that in fact on v_+^-,

$$\{\omega : L_\infty^0 < \infty\} \overset{a.s.}{\subset} \{\omega : \lim_{t\to\infty} X_t \text{ exists and is finite}\}.$$

But the RHS set is the union of the sets $\{\omega : X_\infty = 0\}$ and $\{\omega : X_\infty \neq 0, \pm\infty\}$. The latter set is already contained in $\{\omega : [X,X]_\infty < \infty\}$, by Theorem 2.4 a)(ii). If $\omega \in \{X_\infty = 0\}$, we argue as follows: firstly, it follows from equations (3) and (4) that $\int_0^\infty I_{(0,a)}(X_{s-}) d < M >_s$ (for $a > 0$) and $\int_0^\infty I_{(a,0)}(X_{s-}) d < M >_s$ (for $a < 0$) are both finite. Since $X_\infty = 0$, it implies in fact that $< M >_\infty < \infty$ and hence that the martingale part M_t converges. Now, from Tanaka's formula, it follows that V_+ and V_- converge to a finite limit and hence that $\sum_s \Delta X_s^2 < \infty$. $[X,X]_\infty < \infty$ now follows.

Remark 2.7. We list below the mutually exclusive possibilities that can occur on the set v_+^- with respect to the variables L_∞^0, $[X,X]_\infty$ and X_∞. Now with the help of Theorems 1.3 and 1.7, we can explicitly compute functions h(t) so that these possibilities are in fact realised by the semimartingale $X_t = h(t) B_t$ where B_t is a Brownian motion.

a) $L_\infty^0 < \infty$, $[X,X]_\infty < \infty$ and $X_\infty \neq 0, \pm\infty$. This case does not arise for the semimartingale $X_t = h(t) B_t$.

b) $L_\infty^0 < \infty$, $[X,X]_\infty < \infty$ and $X_\infty = 0$ $\quad (h(t) \sim \frac{1}{t^\alpha}, \alpha > \frac{1}{2})$

c) $L_\infty^0 = \infty$, $[X,X]_\infty < \infty$ and $X_\infty = 0$ $\quad (h(t) \sim \frac{1}{\sqrt{t}(\log t)^\beta}$ for $\frac{1}{2} < \beta \leq 1)$

d) $L_\infty^0 = \infty$, $[X,X]_\infty = \infty$ and $X_\infty = 0$ $\quad (h(t) \sim \frac{1}{\sqrt{t}(\log t)^\beta}$ for $0 < \beta \leq \frac{1}{2})$

e) $L_\infty^0 = \infty$, $[X,X]_\infty = \infty$ and $-\infty < \underline{X}_\infty < \overline{X}_\infty < \infty$ $\quad (h(t) \sim \frac{1}{\sqrt{t(\log\log t)^\gamma}}$, for $0 < \gamma \leq \frac{1}{2})$.

f) $L_\infty^0 = \infty$, $[X,X]_\infty = \infty$, $\overline{X}_\infty = -\underline{X}_\infty = \infty$ $\quad (h(t) \sim \frac{1}{\sqrt{t}})$

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Acknowledgments:

1) This work was done while the first author was visiting the Laboratoire de Probabilités, Université de Paris VI, under an exchange programme between CNRS (France) and NBHM (India). He would like to thank Prof. P.A. Meyer for making the visit possible.

2) Partial support of the research of the second author by NSF Grant No. DMS 8801808 is gratefully acknowledged.