A relation between Lévy's stochastic area formula, Legendre polynomials, and some continued fractions of Gauss.

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1. Introduction.

(1.1) The recurrence relation:

$$\frac{2(\nu+1)}{x} I_{\nu+1}(x) = I_\nu(x) - I_{\nu+2}(x) \quad (\nu > -1;\ x \geq 0)$$

between modified Bessel functions implies

$$\text{(1.a)} \qquad x\frac{I_{\nu+1}}{I_\nu}(x) = \cfrac{x^2}{2(\nu+1) + x\dfrac{I_{\nu+2}}{I_{\nu+1}}(x)}$$

and leads to the continued fraction expansion:

$$\text{(1.b)} \qquad x\frac{I_{\nu+1}}{I_\nu}(x) = \frac{x^2}{2(\nu+1)} \,+\, \frac{x^2}{2(\nu+2)} \,+\, \frac{x^2}{2(\nu+3)} \,+\, \cdots,$$

a particular case of Gauss's continued fractions for ratios of hypergeometric functions (see Jones and Thron [2], p.211, for example). Formulae (1.a) and (1.b) in the case $\nu = 1/2$ are of special interest since:

$$x \coth x - 1 = x\frac{I_{3/2}}{I_{1/2}}(x)$$

and therefore:

$$\text{(1.c)} \qquad x \coth x - 1 = \frac{x^2}{3} \,+\, \frac{x^2}{5} \,+\, \frac{x^2}{7} \,+\, \cdots.$$

(1.2) Let $k_0(x) = x \coth x - 1$, and $h_0(x) = \dfrac{x}{\operatorname{sh} x}$ $(x \in \mathbb{R})$. The functions h_0 and k_0 appear in Lévy's formula:

$$\text{(1.d)} \qquad E[\exp(ixS) \mid B(1) = m] = h_0(x)\exp\left(-\frac{|m|^2}{2}k_0(x)\right)$$

expressing the conditional characteristic function of the stochastic area

$$S \equiv \int_0^1 (B^{(1)}(s)dB^{(2)}(s) - B^{(2)}(s)dB^{(1)}(s))$$

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of 2-dimensional Brownian motion $B = (B^{(1)}, B^{(2)})$ started at 0, given its position at time 1.

This formula (1.d) plays an important role in various questions, including Bismut's approach [1] to the Atiyah-Singer theorem, and also the asymptotics of the winding numbers for 2-dimensional Brownian motion (Pitman-Yor [6]). Several proofs of formula (1.d) are known, among which:

- Lévy's original proof using the development of Brownian motion along the trigonometric orthogonal basis of $L^2([0,2\pi], ds)$ ([4]);
- an application of Girsanov's theorem, which reduces the problem to determining the semi-group of an Ornstein-Uhlenbeck process;
- an application of Ray-Knight theorem for linear Brownian local times.

These two last proofs are presented in D. Williams [7] (see also Yor [9]), and hinge upon the identity:

$$E[\exp(ixS) \mid B(1) = m] = E[\exp -\frac{x^2}{2}\int_0^1 ds\, |B(s)|^2 \mid |B(1)| = m].$$

(1.3) In this paper, we show the following extension of Lévy's formula (1.d).

Theorem

Consider the orthogonal decomposition of Brownian motion

$$(1.e) \qquad B(t) = \sum_{p=0}^{\infty}((2p+1)\int_0^t ds P_p(2s-1))\beta_p \qquad (t \leq 1)$$

where: $\beta_p = \int_0^1 dB(s)P_p(2s-1)$

and $(P_p\ ;\ p = 0,1,\dots)$ *is the sequence of Legendre polynomials.*

Then:

(i) *With the notation*: $\xi \times \eta = \mathrm{Im}(\bar{\xi}\eta)$, for $\xi,\eta \in \mathbb{C}$, *the stochastic area* S *can be represented as*:

$$S = \sum_{p=0}^{\infty}\beta_p \times \beta_{p+1}\ ,$$

where the convergence holds both in L^2 *and a.s;*

(ii) *For any* $p \in \mathbb{N}$, *we have*:

$$E[\exp(ixS) \mid \beta_k = m_k\ ;\ 0 \leq k \leq p] = \exp(ix\sum_{k=0}^{p-1} m_k \times m_{k+1})h_p(x)\exp -\frac{|m_p|^2}{2}k_p(x)$$

where

$$(1.f) \qquad h_p(x) = \frac{x^\nu}{2^\nu\Gamma(\nu+1)I_\nu(x)}\ ;\ k_p(x) = x\frac{I_{\nu+1}}{I_\nu}(x)\ ;\ \nu = p+\frac{1}{2}.$$

(1.4) In order to show more naturally how the Legendre polynomials are linked with Lévy's stochastic area, we have organized the proof as follows:

- in chapter 2, we prove that, if we represent $(B(t), t \leq 1)$ as:

(1.g) $$B(t) = \rho(t) + tB(1), \quad t \leq 1$$

with $(\rho(t), t \leq 1)$ a Brownian bridge independent of $B(1)$, and more generally, if this orthogonalization procedure is adequately iterated, then Lévy's formula (1.d) yields a sequence of analogous identities, whose right-hand sides are:

$$h_p(x)\exp\left(-\frac{|m|^2}{2}k_p(x)\right)$$

where h_p and k_p are defined in (1.f);

- in chapter 3, we identify the orthogonal expansion

$$B(t) = \sum_{p=0}^{\infty} u_{p+1}(t)\beta_p \quad (t \leq 1)$$

which is obtained in our orthogonalization procedure as the decomposition (1.e).

2. Lévy's formula and some continued fractions of Gauss.

(2.0) NOTATION.

• If $Z(t) = X(t) + iY(t)$, $t \leq 1$, is a complex valued continuous semi-martingale, we write:

$$S_Z = \int_0^1 X(s)dY(s) - Y(s)dX(s)$$

• If $m = m^{(1)} + im^{(2)}$, and $n = n^{(1)} + in^{(2)}$ are two complex numbers, we write $m \times n$ for $\mathrm{Im}(\bar{m}n) = m^{(1)}n^{(2)} - n^{(1)}m^{(2)}$, and $m \cdot n$ for $\mathrm{Re}(\bar{m}n) = m^{(1)}n^{(1)} + m^{(2)}n^{(2)}$.

• For $\nu > -1$, we note: $\tilde{I}_\nu(x) = \dfrac{2^\nu\Gamma(\nu+1)}{x^\nu}I_\nu(x)$

(2.1) We first reinterpret formula (1.d) in terms of the Brownian bridge ρ defined in (1.g). Developing S, we obtain:

$$S = S_\rho + B(1)\times\beta_1, \quad \text{where: } \beta_1 = -2\int_0^1 ds\rho(s),$$

and formula (1.d) becomes:

$$E[\exp(ixS_\rho + ixm\times\beta_1)] = h_0(x)\exp\left(-\frac{|m|^2}{2}k_0(x)\right)$$

so that:

(2.a) $$E[\exp(ixS_\rho + in\cdot\beta_1)] = h_0(x)\exp\left(-\frac{|n|^2}{2}\frac{k_0(x)}{x^2}\right).$$

This formula confirms that β_1 is a centered 2-dimensional Gaussian variable, with the additional information that:

$$\frac{1}{2}E(|\beta_1|^2) = \lim_{x\to 0}\frac{k_0(x)}{x^2} \equiv \frac{1}{c_0}.$$

Moreover, we deduce from (2.a) that:

$$E[\exp(ixS_\rho) \mid \beta_1 = m] = h_1(x)\exp\left(-\frac{|m|^2}{2}k_1(x)\right)$$

with:

$$h_1(x) = \frac{h_0(x)x^2}{k_0(x)c_0};\ k_1(x) = \frac{x^2}{k_0(x)} - c_0.$$

From the recurrence relation (1.a), we get:

$$h_1(x) = \frac{1}{I_{3/2}(x)};\ k_1(x) = x\frac{I_{5/2}}{I_{3/2}}(x);\ c_0 = 3.$$

(2.2) We now iterate the above procedure in defining a sequence of processes $(B_p(t), t \leq 1)$, and of Gaussian variables (β_p) via the recurrence relation:

$$\text{(2.b)}\qquad \begin{cases} B_p(t) = B_{p+1}(t) + u_{p+1}(t)\beta_p \\ \beta_p = -2\int_0^1 du_p(s)B_p(s) \end{cases}$$

with original conditions: $B_0(t) = B(t)$, and $\beta_0 = B(1)$, and the additional requirement that $B_{p+1}(t)$ is orthogonal to β_p. In order that this recurrence relation be meaningful, we must verify recursively that the functions (u_p) are of bounded variation. Suppose this is so for $u_1, \cdots, u_p$. Then, from the first half of (2.b), using the orthogonality of $\beta_p, \beta_{p-1}, \cdots, \beta_0$, we obtain:

$$u_{p+1}(t)E[\beta_p^2] = E[B(t)\beta_p] = \int_0^t ds\phi_p(s)$$

where $\phi_p(\epsilon L^2([0,1],ds))$ is the function appearing in the Wiener representation of $\beta_p \equiv \int_0^1 dB(s)\phi_p(s)$. Therefore, u_{p+1} is absolutely continuous, and the recurrence is meaningful. Now, from (2.b), we obtain:

$$S_p = S_{p+1} + \beta_p \times \beta_{p+1},$$

where, for simplicity, we have written S_k for S_{B_k} $(k = p, p+1)$. Consequently, the functions h_p and k_p being defined via the formula:

$$E[\exp(ixS_p) \mid \beta_p = m] = h_p(x)\exp\left(-\frac{|m|^2}{2}k_p(x)\right)$$

we obtain, much as in (2.1) above, the recurrence formulae:

(2.c) $\quad$ (i) $h_{p+1} = \dfrac{h_p(x)x^2}{k_p(x)c_p}$; $\quad$ (ii) $k_{p+1}(x) = \dfrac{x^2}{k_p(x)} - c_p$

where $c_p = \lim\limits_{x\to 0} \dfrac{x^2}{k_p(x)}$. Moreover, we also have:

(2.d) $\quad$ $\frac{1}{2}E(|\beta_{p+1}|^2) = 1/c_p$.

We now deduce from the recurrence formula (1.a) that:

$$h_p(x) = \frac{1}{\tilde{I}_\nu(x)};\ k_p(x) = x\frac{I_{\nu+1}}{I_\nu}(x);\ c_p = 2(\nu+1), \text{ with } \nu = p + 1/2.$$

(2.3) For $p > 0$, we introduce the process V_p defined by:

$$V_p(t) = \frac{1}{t^p}\int_0^t dB(s)s^p \ \ (t > 0), \text{ and } V_p(0) = 0.$$

This is a continuous semimartingale with decomposition:

$$V_p(t) = B(t) - p\int_0^t \frac{ds}{s}V_p(s).$$

Our interest in the process V_p comes from the fact that, if $(\ell^a;\ a \geq 0)$ denotes the family of local times over the whole of R_+ for the Bessel process, call it R_p, with dimension $c_p = 2p + 3$, then:

(2.e) $\quad$ $(\ell^a;\ a \geq 0) \stackrel{(d)}{=} (|V_p(a)|^2;\ a \geq 0)$.

This is easily deduced from the particular case $p = 0$, which is due to D. Williams [8], and is in agreement with Le Gall [3], using deterministic time change, and time-inversion.

We have the following

Theorem 1: *Let* $p \in N$, and $\nu = p+\frac{1}{2}$. *Then:*

$$E[\exp(ixS_p) \mid \beta_p = m] = E[\exp(ixS_{V_p}) \mid V_p(1) = m]$$
$$= \frac{1}{\tilde{I}_\nu(x)}\exp\left(-\frac{|m|^2}{2}x\frac{I_{\nu+1}}{I_\nu}(x)\right).$$

Proof: We have already shown the equality between the first and the last expressions. To prove that the second and the last expressions are equal, we remark that:

(2.f) $\quad$ $E[\exp(ixS_{V_p}) \mid V_p(1) = m] = E[\exp - \frac{x^2}{2}\int_0^1 |V_p(s)|^2\, ds \mid |V_p(1)|^2 = |m|^2]$

by a classical skew-product argument.

Using the identity in law (2.e), the right-hand side of (2.f) equals:

$$E[\exp - \frac{x^2}{2}\int_0^1 ds 1_{(R_p(s) \leq 1)} | t^1 = |m|^2]$$

and, from Pitman-Yor [5], for example, this quantity is equal to the closed form expression presented in Theorem 1. ●

Theorem 1 may be extended, with no more difficulty, as follows: for any $p > 0$, and $q \in N$, we denote $S^{(p)}$ for S_{V_p}, and $S_q^{(p)}$ for $S_{(V_p)_q}$, where $((V_p)_q;\ q \in N)$ is the sequence of processes appearing in the orthogonalization procedure detailed in (2.2), but now applied to the process V_p, instead of $B \equiv V_0$.

The identities stated in theorem 1 now become:

$$E[\exp(ixS_q^{(p)}) \mid \beta_q^{(p)} = m] = E[\exp(ixS_{V_{p+q}}) \mid V_{p+q}(1) = m]$$

$$= \frac{1}{I_\nu(x)}\exp\left(-\frac{|m|^2}{2}x\frac{I_{\nu+1}}{I_\nu}(x)\right), \text{ where } \nu = p+q+\frac{1}{2}.$$

3. Lévy's formula and Legendre polynomials.

We shall now determine explicitly the functions (u_p) which appear in the recurrence relation (2.b).

Obviously we may, and we shall, assume here that $(B(t), t \leq 1)$ is real-valued. We need to introduce the Legendre polynomials (P_n) which may be defined by the Rodrigues formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n],$$

and constitute an orthogonal basis of $L^2([-1, +1], dx)$.

We now have the following

Theorem 2: *Let* $p \in N$; *then:*

$$\text{(i)}\ E(\beta_p^2) = \frac{1}{2p+1}; \qquad \text{(ii)}\ u_{p+1}(t) = (2p+1)\int_0^t ds P_p(2s-1).$$

Proof: a) In our proof of Theorem 1, we have already shown that

$$\lambda_p \equiv E(\beta_p^2) = \frac{1}{2p+1}.$$

(The difference of (1/2) with formula (2.d) comes from changing dimension 2 to 1). We shall give a direct proof of this below.

b) We now prove that $(u'_{k+1}, k \geq 0)$ is a sequence of orthogonal functions in $L^2([0,1], ds)$.

The Gaussian variable β_k admits a Wiener representation:

$$\beta_k = \int_0^1 dB(s)\phi_k(s), \text{ with } \phi_k \in L^2([0,1], ds).$$

For any k, we deduce from the orthogonal development:

$$B(t) = B_{k+1}(t) + \sum_{p=0}^{k} u_{p+1}(t)\beta_p,$$

that:

$$u_{k+1}(t)\lambda_k = E[B(t)\beta_k] = \int_0^t ds\phi_k(s)$$

a formula we already obtained in showing that (2.b) is meaningful. Therefore, $(u'_{k+1} = \frac{1}{\lambda_k}\phi_k;\ k \geq 0)$ is an orthogonal sequence in $L^2([0,1], ds)$.

c) We now show the following relations:

$$\text{(3.a)} \qquad \text{(i)} \int_0^1 du_p(s)u_{p+1}(s) = -1/2; \qquad \text{(ii)} \int_0^1 du_p(s)u_{k+1}(s) = 0 \quad (k > p)$$

which, by integration by parts, may also be written as:

$$\text{(3.a')} \qquad \text{(i')} \int_0^1 du_{p+1}(s)u_p(s) = 1/2; \qquad \text{(ii')} \int_0^1 du_{k+1}(s)u_p(s) = 0 \quad (k > p).$$

These relations are obtained by writing:

$$B_p(t) = B_{q+1}(t) + \sum_{k=p}^{q} u_{k+1}(t)\beta_k \quad (q > p);$$

Thus:

$$\beta_p \equiv -2\int_0^1 du_p(s)\{B_{q+1}(s) + \sum_{k=p}^{q} u_{k+1}(s)\beta_k\}$$

which implies (3.a), since $\beta_p, \beta_{p+1}, \cdots, \beta_q, B_{q+1}$ are orthogonal.

d) Next, we remark that the covariance of the process B_p may be deduced from the orthogonal development: $B(t) = B_p(t) + \sum_{k=0}^{p-1} u_{k+1}(t)\beta_k$.

We obtain:

$$E[B_p(t)B_p(s)] = (t \wedge s) - \sum_{k=1}^{p} u_k(t)u_k(s)\lambda_{k-1}.$$

e) Using our previous remarks, we shall now obtain a simple recurrence formula between u_{p-1}, u_p and u_{p+1}. We deduce from the equality:
$B_p(t) = B_{p+1}(t) + u_{p+1}(t)\beta_p$ that:

$$u_{p+1}(t)\lambda_p = E[B_p(t)\beta_p] = -2\int_0^1 du_p(s)E[B_p(t)B_p(s)]$$

which, using d), and then c), gives:

$$u_{p+1}(t)\lambda_p = -2\int_0^t du_p(s)s + 2tu_p(t) + u_{p-1}(t)\lambda_{p-2} \quad (p > 1). \tag{3.b}$$

For $p = 1$, we have:

$$u_2(t)\lambda_1 = -2\int_0^1 ds\{t \wedge s - st\} = -t(1-t).$$

In particular, a recurrence argument shows that for every $p \in N$, u_p is a polynomial of degree $(p + 1)$.
Consequently, using b), we have: $u_{p+1}(t) = \alpha_p P_p^*(t)$, where α_p is a constant to be determined, and

$$P_p^*(t) = (2p+1)^{1/2} P_p(2t-1) \quad (p \in N)$$

is the orthonormal family in $L^2([0,1], dt)$ which is deduced from the Legendre polynomials (P_p).

f) It remains to determine the two sequences (α_p) and (λ_p). Writing (3.b) again in terms of (α_p), (λ_p) and (P_p), gives the following relation:

$$\lambda_{n+1}\alpha_{n+1}(2n+3)^{1/2}P'_{n+1}(x) = \alpha_n(2n+1)^{1/2}P_n(x) + \lambda_{n-1}\alpha_{n-1}(2n-1)^{1/2}P'_{n-1}(x)$$

which, when compared with the classical relation:

$$P'_{n+1} = (2n+1)P_n + P'_{n-1}$$

implies:

$$\lambda_p = \frac{1}{2p+1}, \text{ and } \alpha_p = (2p+1)^{1/2} \quad \bullet$$

4. Concluding remarks.

(4.1) The proof of the theorem stated in the Introduction is obtained by putting together Theorem 1 and Theorem 2. Indeed, since $(P_p\, ;\, p \in \mathbb{N})$ is an orthogonal basis of $L^2([-1,1], ds)$, we now know that $(\beta_p\, ;\, p \in \mathbb{N})$ is an orthogonal basis of the Gaussian space generated by $(B(t), t \le 1)$. Hence, the formula

$$B(t) = B_{k+1}(t) + \sum_{p=0}^{k} u_{p+1}(t)\beta_p$$

implies (1.e), as $k \to \infty$.

Likewise, the formula:

$$S = S_{k+1} + \sum_{p=0}^{k} \beta_p \times \beta_{p+1}$$

implies

$$S = \sum_{p=0}^{\infty} \beta_p \times \beta_{p+1}$$

and the convergence holds both in L^2 and a.s, since:

$$\left(\sum_{p=0}^{k} \beta_p \times \beta_{p+1} \, ; \, k \in \mathbb{N} \right) \text{ is a} (\mathbf{F}_k) \text{ martingale,}$$

where $\mathbf{F}_k$ is the σ-field generated by $(\beta_0, \beta_1, \cdots, \beta_{k+1})$. This proves part (i) of the theorem. Part (ii) is then an immediate consequence of theorem 1.

(4.2) To prove formula (1.d), P. Lévy [4] develops Brownian motion along the trigonometric basis of $L^2([0,1],ds)$, and obtains $h_0(x) \equiv \frac{x}{\mathrm{sh} x}$, and $k_0(x) \equiv x \coth x - 1$ in their classical infinite product representations. On the other hand, we have shown in this paper that, when developing Brownian motion along the Legendre basis, one obtains $k_0(x)$ in its continued fraction representation (1.c).

(4.3) A number of variants of theorems 1 and 2 can be obtained if we replace the Brownian functional S by

$$S^{(\phi)} = \int_0^1 dS_s \phi(s), \text{ or by } A^{(\phi)} = \int_0^1 ds \phi(s) \, |B(s)|^2,$$

with $\phi : [0,1] \to R_+$ a nice function, and in particular $\phi(s) = s^k (k \geq 0)$.

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