Confidence Bounds on the Probability Density of Aftershocks

Niklaus W. Hengartner
and
Philip B. Stark

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Department of Statistics
University of California
Berkeley, California 94720

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Niklaus W. Hengartner
Philip B. Stark
Department of Statistics
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Abstract

We compute confidence bounds on the temporal probability density function (pdf) of aftershocks of the 1992 Joshua Tree event, the 1984 Morgan Hill event, and the 1989 Loma Prieta event using a new statistical technique. The confidence bounds assume that the aftershocks are a realization of an inhomogeneous Poisson process, with an arbitrary monotone decreasing intensity. No functional form for the intensity is needed. Under the usual Poisson assumption, the pdf is equivalent to the intensity of aftershocks, normalized to unit area over the observation time interval. The width of the bounds depends nonlinearly on the data and is quite different for the three events. The maximum likelihood estimates of modified Omori intensity laws fit the Joshua Tree and Morgan Hill data extremely well, but not the Loma Prieta data (the modified Omori law would be rejected at significance levels less than 0.038).

Introduction.
It is generally believed that following a large Earthquake, the chance of aftershocks is large, and decreases as time goes by (at least for a while until stresses build up again). Omori's law [*Omori*, 1895] and its modification [e.g. *Utsu*, 1961] are decreasing intensity functions often fit to aftershock sequences. The modified Omori law is that the number of aftershocks by time t, $N(t)$, has a Poisson distribution with intensity $\lambda(t)$ of the form

$$\lambda(t) = \frac{A(M)}{(c+t)^p}. \tag{1}$$

Davis and Frohlich [1991] verified that the modified Omori law provides a probabilistically adequate fit to many small aftershock sequences with $p \approx .87$; other investigators have generally (though not invariably) found $p > 1$ for large Earthquakes with many aftershocks (e.g. *Utsu* [1961]). A number of theoretical studies using different physical models of Earthquakes predict a modified Omori law for some ranges of time, with values of p between 1 and 1.5 [*Utsu*, 1961; *Mikumo and Miyatake*, 1979; *Kagan and Knopoff*, 1987; *Yamashita* *and Knopoff*, 1987]. Nonetheless, the assumption of a particular parametric form for the intensity of aftershocks is extremely restrictive, and the theoretical support of a modified Omori law is suggestive, rather than conclusive. Furthermore, researchers are wont to draw conclusions about physical differences between events using uncertainty estimates for the parameters in the modified Omori law, and to project Earthquake hazard after main shocks using the law (see, e.g., [*Davis and Frohlich*, 1991; *Reasenberg and Jones*, 1989]). The uncertainty estimates are suspect, since they are conditional on the truth of the parametric model. Estimates of p are also sensitive to the time after the main shock at which the aftershock observations begin (C. Frohlich, personal communication, 1992).

Below we find confidence bounds on the "shape" of the intensity of aftershocks of the 1992 Joshua Tree event (figure 1), the 1984 Morgan Hill event (figure 2) and the 1989 Loma Prieta event (figure 3) without assuming a functional form for the intensity. We assume only that aftershocks are an inhomogeneous Poisson point process whose intensity decreases with time. The "shape" of the intensity is the intensity normalized to have unit area on the interval of observation; under the Poisson assumption it is also the conditional probability density function (pdf) of the times of aftershocks given the total number of observed aftershocks (see equation (7) below). The confidence bounds contain every monotonic pdf, parametric or otherwise, consistent with the data at a specified confidence level. The confidence bounds use a new statistical method introduced by *Hengartner and Stark* [1992] for finding confidence bounds on monotone and unimodal pdf's. The technique is rigorous and conservative: the bounds in figures 1-3 have at least 95% coverage probability, even though the sample is finite.

Models and the Method.

Models of the Intensity of Aftershocks. Aftershocks are usually modeled as an inhomogeneous Poisson process, assuming: (a) the numbers of aftershocks in disjoint time intervals are independent, (b) the chance of an aftershock in an interval of time of length h is approximately proportional to h, with a remainder that is $o(h)$, and (c) the chance of more than one aftershock in an interval of length h is $o(h)$. (A function $g(h)$ is $o(h)$ if $\lim_{h\to 0} g(h)/h = 0$.) Let $N(t)$ be the (random) number of aftershocks that occur by time t, where $t = 0$ is the time of the main event. The intensity $\lambda(t)$ is

$$\lambda(t) \equiv \lim_{\epsilon \to 0^+} \frac{N(t+\epsilon) - N(t)}{\epsilon}. \tag{2}$$

The modified Omori law says $\lambda(t)$ has the functional form (1). The probability that by time T there have been exactly n aftershocks is

$$P\{N(T) = n\} = \frac{\Lambda^n(T)}{n!} e^{-\Lambda(T)}, \tag{3}$$

where

$$\Lambda(T) \equiv \int_0^T \lambda(t)dt. \tag{4}$$

The expected number of aftershocks by time T is

$$EN(T) = \sum_{j=0}^{\infty} j \frac{\Lambda^j(T)}{j!} e^{-\Lambda(T)} = \Lambda(T). \tag{5}$$

The expected total number of aftershocks is $\Lambda(\infty)$. For the modified Omori law with $p \leq 1$, the expected number of aftershocks is infinite. This strongly suggests that modified Omori laws with $p \approx 0.87$ as found by Davis and Frohlich [1991] can not hold for all t. Theoretical justifications of modified Omori laws typically find $p > 1$ [*Utsu*, 1961; *Mikumo and Miyatake*, 1979; *Kagan and Knopoff*, 1987; *Yamashita and Knopoff*, 1987]. *Kagan and Knopoff* [1987] argue that a modified Omori law should hold initially, followed by a transition to an exponentially decaying intensity function. In these situations, the expected number of aftershocks is finite.

Temporal Probability of Aftershocks. We need to relate the intensity $\lambda(t)$ of aftershocks to the pdf $f(t)$ of the times of aftershocks. Suppose we observe n aftershocks by time T. What is the conditional distribution of the times of those aftershocks, given that $N(t) = n$? Let $\{X_j\}_{j=1}^n$ be the (random) times of the n aftershocks, in no particular order. Then

$$P\{X_j \leq t | N(T) = n\} = \frac{\Lambda(t)}{\Lambda(T)}, \quad 0 \leq t \leq T. \tag{6}$$

The pdf of aftershock times is thus

$$f(t) = \frac{\lambda(t)}{\Lambda(T)}, \quad 0 \leq t \leq T. \tag{7}$$

The Poisson assumption implies that the times are independent, so conditional on $N(T)$, $\{X_j\}$ are independent and identically distributed with density $f(t)$. Equation (7) relates the intensity λ to f, which is the function we find confidence bounds for. If $\lambda(t)$ is monotonic, so is $f(t)$. The pdf $f(t)$ characterizes the "shape" of the intensity, since it is the intensity normalized to unit area on $[0, T]$.

Computational Method. *Hengartner and Stark* [1992] show that confidence intervals for the pdf at a fixed time t can be found by solving two linear programs. The basic idea of the method is to define a set of densities that, with specified probability, are consistent with the observations. Restricting attention to monotone densities within that set, we ask how large and how small those densities can be at a specified time. The solution to those optimization problems gives a confidence interval for the pdf at that time. The computations can be repeated at many points. Monotonicity of the pdf allows us to interpolate the ends of the confidence intervals between those times conservatively to get a confidence envelope. See *Hengartner and Stark* [1992] for the complete theory, which is closely related to the "strict bounds" approach to inverse problems [*Stark*, 1992].

The confidence region for the distribution is defined using the Kolmogorov-Smirnov distance from the empirical distribution. Let $\{X_j\}_{j=1}^n$ be the random times of n aftershocks. The empirical distribution function is

$$\hat{F}_n(t) \equiv \sum_{j=1}^{n} 1_{t \geq X_j}, \tag{8}$$

where

$$1_{t \geq X_j} \equiv \begin{cases} 1, & t \geq X_j \\ 0, & t < X_j. \end{cases} \tag{9}$$

The Kolmogorov-Smirnov distance between two measures F and G is

$$\|F - G\| \equiv \sup_t |F(t) - G(t)|. \tag{10}$$

Massart [1990], sharpening a result of Dvoretzky, Kiefer and Wolfowitz, shows that if F is the true probability distribution of the data, then

$$Pr\{\|F - \hat{F}_n\| \leq \chi\} \geq 1 - \alpha, \tag{11}$$

where

$$\chi = \chi_n(\alpha) \equiv \sqrt{\frac{\ln \frac{2}{\alpha}}{2n}}. \tag{12}$$

Thus $\{G : \|G - \hat{F}_n\| \leq \chi\}$ is a $1 - \alpha$ confidence region for F.

Hengartner and Stark [1992] show that finding the largest and smallest values at a point among monotonic densities of measures in that confidence region can be reduced to finite-dimensional linear programs: Let $\{x_j\}_{j=1}^n$ be the observed aftershock times, and let $\{z_j\}_{j=1}^{n+1}$ be the x's augmented by the extra point $t > x_1$, and sorted in increasing order. Suppose $t = z_k$. Define

$$\omega_j \equiv z_{j+1} - z_j, j = 1, \cdots, n-1. \tag{13}$$

Hengartner and Stark [1992] show that if F has a monotone density $f(t)$ then the two linear programs (1) maximize β_{k-1} and (2) minimize β_k, subject to the constraints:

C1 $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \geq 0$,

C2 $\sum_{j=1}^n \omega_j \beta_j = 1$

C3 $-\chi + \frac{m}{n} \leq \sum_{\{j: z_j \leq x_m\}} \beta_j \omega_j \leq \chi + \frac{m-1}{n}, \quad m = 1, \cdots, n$

give the endpoints of a $1 - \alpha$ confidence interval for $f(t)$. The constraint C1 imposes monotonicity and positivity; C2 ensures the density integrates to unity, and C3 forces the density to be in the confidence set. Intervals at any finite set of points can be interpolated conservatively using the monotonicity of f to get a piecewise constant simultaneous $1 - \alpha$ confidence envelope for f.

Changing variables to $\gamma_j \equiv \beta_{j+1} - \beta_j$ facilitates implementation since positivity of each γ_j replaces C1 without using slack variables. The other constraints are easily rewritten in terms of $\{\gamma_j\}$. We used Numerical Algorithms Group, Inc. (NAG) subroutines to solve these linear programming problems for hundreds of confidence intervals for each data set.

Data and Results.

1992 Joshua Tree event. Lucy Jones provided us with Southern California Seismographic Network (SCSN) aftershock data for this magnitude 6.1 event, which occurred at 4:50:22 on April 23 of this year. At the time of writing (May 12, 1992), less than three weeks after the event, the SCSN Real-Time Processor had identified approximately 3460 putative aftershocks within a 20km radius of the main shock, located near 33° 57′N latitude, 116° 19′W longitude. We culled the aftershock sequence for events larger than magnitude 2.0, of which there were 1820. Figure 1 plots the confidence bounds on the probability of aftershocks for this event, and the density corresponding to the maximum likelihood estimate (MLE) of a modified Omori law for this data [*Ogata*, 1983]. (The MLE of the parameters of the modified Omori law fit as an intensity function is the same as the MLE of the parameters when the corresponding Omori-based density is fit to the empirical distribution of times; see also equation (7).)

The bounds are a 95% confidence envelope for f: every monotonic pdf that agrees adequately with the observed data in the Kolmogorov-Smirnov sense lies between the bounds. The parameters of the MLE were $\hat{p} = 1.453$ and $\hat{c} = 1.540$. This is an unusually large estimated value for c (Lucy Jones, personal communication, 1992), reflecting the slow decay of the intensity of aftershocks with time. The MLE modified Omori law lies between the bounds, but this does not guarantee that it agrees adequately with the observations. In fact, the density derived from the MLE Omori law has a KS misfit to the empirical distribution of 0.0176, which is extremely small. (A hypothesis test based on Massart's inequality would reject the null-hypothesis that the data come from the MLE Omori law at significance levels above 0.646.)

1989 Loma Prieta Event. Robert Uhrhammer provided us with University of California at Berkeley Seismographic Stations aftershock data for this event, which occurred at 4:15:43 on October 18 1989. The data, spanning October 18, 1989–January 1, 1992, comprise events within 40km radius of the epicenter (37.0N latitude, 121.8W longitude) at depths of 0-20km. There were 221 events with magnitudes at least 3.0, which is the level at which the catalog is thought to be complete (R. Uhrhammer, personal communication, 1992). Figure 2 shows 95% confidence bounds on the probability of aftershocks, together with the density derived from the MLE modified Omori law. The MLE of p was $\hat{p} = 0.963$, and the MLE of c was $\hat{c} = 7.71 \times 10^{-3}$. Since $\hat{p} < 1$, this corresponds to a nonphysical situation where the expected number of aftershocks is infinite. The KS misfit of the MLE Omori law to the data is 0.095; a hypothesis test based on Massart's inequality would reject the hypothesis that the data come from this modified Omori law at significance levels below 0.038.

1984 Morgan Hill Event. Robert Uhrhammer provided us with USGS identified aftershocks of the 21:15:18 24 April 1984 magnitude 5.9 Morgan Hill event, located near 37° 18.58′N latitude, 121° 40.60′ W longitude, for April 24, 1984-December 31, 1984. The aftershocks include all events located between 37.0 and 37.5 degrees North latitude and between 121.5 and 121.83 degrees West longitude. We culled the file for events with magnitudes at least 2.0; there were 766 such events. Figure 3 shows 95% confidence bounds and the MLE Omori density for this event. The MLE was $\hat{p} = 0.554$ and $\hat{c} = 0.0018$. (Again, if this value of p held for all time, the expected number of aftershocks would be infinite.) The KS misfit of the MLE modified Omori density to the empirical distribution was 0.0345: Massart's inequality assigns a (conservative) p-value of 0.324 to the null hypothesis that the data come from the MLE Omori law.

Conclusions.

If aftershocks are a Poisson process, the conditional pdf of aftershock times given the number of aftershocks observed is just a scaled version of the intensity function. Nonparametric confidence bounds on the conditional pdf can be computed by linear programming using the additional assumption that the intensity decreases with time. No functional form for the intensity need be specified. This gives hope that one could detect real differences between statistical properties of aftershocks of different events and rigorously constrain physical theories of aftershock generation, without resorting to parametric estimates and parametric uncertainties.

The linear programming approach of *Hengartner and Stark* [1992] is computationally practical for sequences of several thousand aftershocks using off-the-shelf linear programming subroutines; more efficient code would allow tens of thousands of events to be used. Based on the SCSN data for the Joshua Tree event, BSS data for the Loma Prieta event and USGS data for the Morgan Hill event, the maximum likelihood estimates of modified Omori Laws provide probabilistically excellent fits to the 1992 Joshua Tree event and 1984 Morgan Hill event, but a mediocre fit to the 1989 Loma Prieta event. In all three cases, the MLE Omori densities are not rejected at significance level 0.05, and lie within the confidence envelopes.

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Figure Captions.

Fig. 1. Confidence bounds on the pdf of aftershocks of the April 23, 1992 Joshua Tree event. Solid lines are 95% simultaneous confidence bounds on the pdf of aftershocks, using only the assumption that the probability of aftershocks decreases with time. The dashed line is the pdf derived from the maximum likelihood estimate (MLE) of the modified Omori law for the intensity of aftershocks (see equation 7). The MLE parameter estimates were $\hat{p} = 1.453$, $\hat{c} = 1.540$. The results are based on 1820 aftershocks with magnitudes at least 2.0 occurring between April 23 and May 12, 1992, as identified by the Real-Time Processor of the Southern California Seismographic Network.

Fig. 2. 95% confidence bounds on the pdf of aftershocks of the October 18, 1989 Loma Prieta event. The dashed line is the pdf derived from the MLE modified Omori law, which had $\hat{p} = 0.963$, $\hat{c} = 7.71 \times 10^{-3}$. The figure is based on 221 aftershocks with magnitudes at least 3.0, as identified by the University of California at Berkeley Seismographic Stations.

Fig. 3. 95% confidence bounds on the pdf of aftershocks of the April 24, 1984 Morgan Hill event using 766 aftershocks with magnitudes at least 2, identified by the US Geological Survey between April 24 and December 31, 1984. The dashed line is the pdf implied by the MLE modified Omori law, which had $\hat{p} = 0.554$, $\hat{c} = 0.0018$.

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Fig. 1: Joshua Tree Density Bounds; MLE Omori's Law

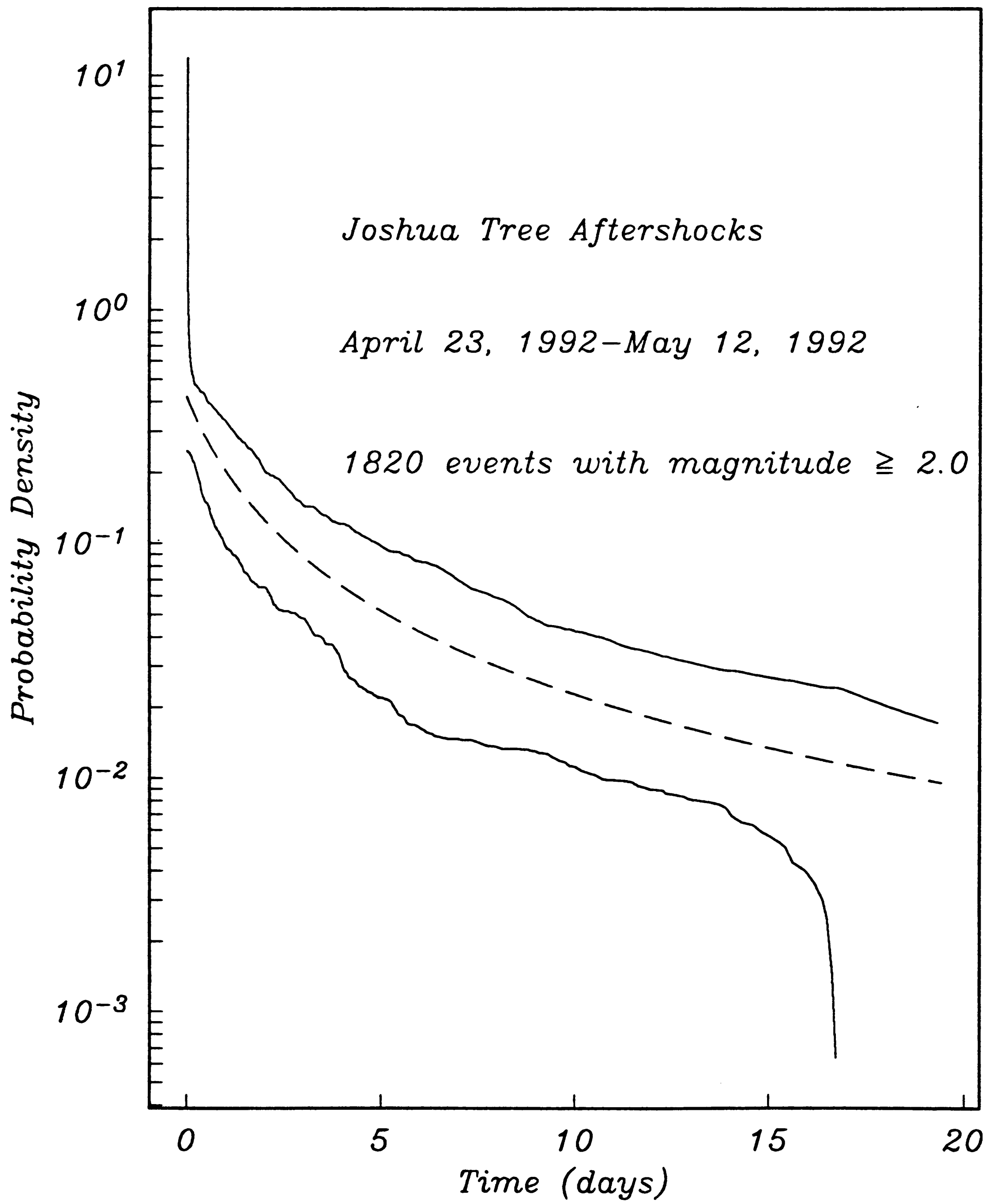

Fig. 2: Loma Prieta Density Bounds; MLE Omori's Law

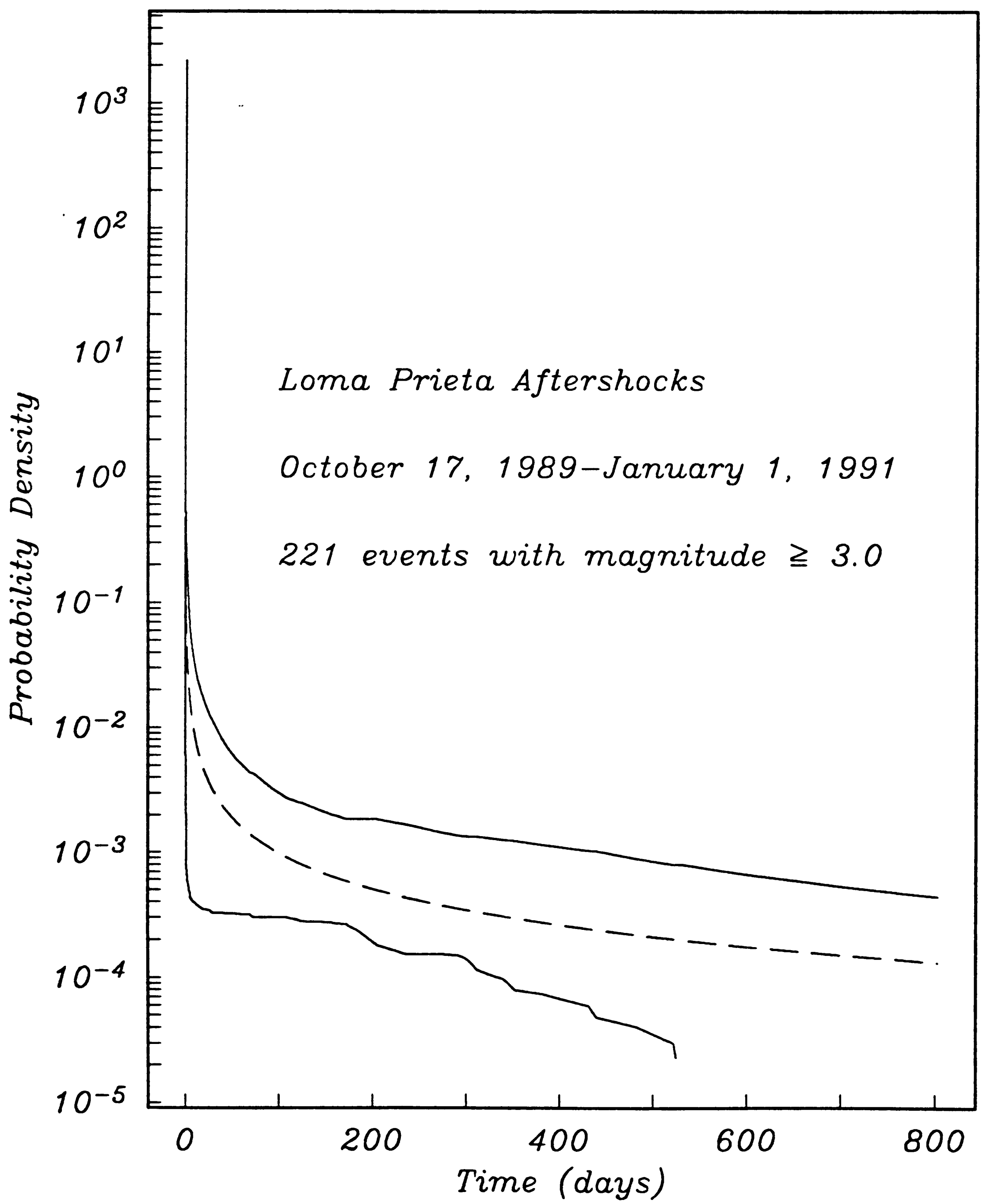

Fig. 3: Morgan Hill Density Bounds; MLE Omori's Law