Signed-Rank Tests for Censored Matched Pairs.*

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Abstract

We consider the problem of testing bivariate symmetry in matched pair experiments where the observations are subject to univariate censoring. Thus the observable random variables are given by (Y_1, Y_2) and (δ_1, δ_2) where $Y_j = \min(X_j, C)$ and $\delta_j = I(X_j \le C)$, $j = 1,2$. Here (X_1, X_2) is a random pair of partially observable lifetimes and C is a fixed or random censoring variable. The hypothesis to be tested is that (X_1, X_2) and (X_2, X_1) have the same distribution. Following Woolson and Lachenbruch (1980), we consider censored data generalizations of signed rank tests such as the sign, signed Wilcoxon and signed normal scores tests. Using counting processes techniques, we derive the asymptotic distribution of the test statistics under fixed and contiguous alternatives. The efficiencies of the signed rank tests are considered in a bivariate exponential model and compared with efficiencies of the paired Prentice-Wilcoxon and log-rank tests.

Key words: bivariate symmetry, censored data, counting processes.

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1. Introduction.

We consider the problem of testing whether (X_{1i}, X_{2i}) has the same distribution as (X_{2i}, X_{1i}) $i = 1, \ldots, n$ where (X_{1i}, X_{2i}) are independent identically distributed nonnegative bivariate random vectors representing failure or survival times of paired subjects. Throughout the failure times (X_{1i}, X_{2i}) are subject to univariate right censoring so that the observable random variables are given by (Y_{1i}, Y_{2i}) and $(\delta_{1i}, \delta_{2i})$, where $Y_{ji} = \min(X_{ji}, C_i)$ and $\delta_{ji} = I(X_{ji} \le C_i)$, $j = 1, 2$, $i = 1 \cdots n$. Here C_i's are independent random variables representing withdrawal times from the study due to reasons unrelated to the study itself. It is assumed that the C's are independent of the X's. The censoring mechanism assumes that for both members of the pair the two time measurements are made on the same time clock. This will occur in the case of matched pair experiments or twin studies when the subjects undergo the study simultaneously and are censored only if failure does not occur by the end of the study. Batchelor and Hackett (1970), Holt and Prenctice (1974) and Woolson and Lachenbruch (1980) for instance report data on survival of skin grafts on burn patients each of whom received two grafts. The donor and the recipient were matched for blood groups and closely or poorly matched for the transplantation antigen system. Censoring occurred at the termination of the study. This censoring mechanism is also applicable when two time measurements are made successively on the same individual.

Thus (X_1, X_2) may represent times from remission to relapse and from relapse to death in cancer patients or time from initiation of a treatment until first response in two successive courses of a treatment in the same patient.

For uncensored data tests for bivariate symmetry can be based on signed rank statistics, see Doksum (1980), Lehmann (1975) and Woolson and Lachenbruch (1980). In the presence of censoring define $Z_i = Y_{2i} - Y_{1i}$ and let ε_i be the sign of Z_i. Note that the censoring mechanism implies $\varepsilon_i = 0$ and $Z_i = 0$ whenever $\delta_{1i} = \delta_{2i} = 0$, $\varepsilon_i = 1$ whenever $\delta_{1i} = 1$ and $\delta_{2i} = 1$ and $\varepsilon_i = -1$ whenever $\delta_{1i} = 0$ and $\delta_{2i} = 1$. Define sets

$$B_1 = \{i: \varepsilon_i = 1, \ \delta_{1i}\delta_{2i} = 1\}$$
$$B_2 = \{i: \varepsilon_i = -1, \ \delta_{1i}\delta_{2i} = 1\}$$
$$B_3 = \{i: \delta_{1i} = 1, \ \delta_{2i} = 0\}$$
$$B_4 = \{i: \delta_{1i} = 0, \ \delta_{2i} = 1\}.$$

For $j = 1, \ldots, 4$ introduce counting processes $N_j(t)$ and $U_j(t)$ where $N_j(t) = \Sigma_{i=1}^n N_{ji}(t)$ and $U_j(t) = \Sigma_{j=1}^n U_{ji}(t)$ with

$$N_{ji}(t) = I(|Z_i| \le t \ \ i \in B_j)$$
$$U_{ji}(t) = I(|Z_i| \ge t \ \ i \in B_j)$$

To test the hypothesis of bivariate symmetry we consider statistics $T = T(\infty)$ where

$$T(t) = \int_0^t K_u d(N_1 - N_2) + \int_0^t K_c d(N_3 - N_4)$$

for some predictable scoring processes K_u and K_c. In particular the following special cases will be of interest.

(i) The sign test: $K_u = K_c = 1$

(ii) The signed Wilcoxon test: $K_u = 1 - \hat{F}_-$ and $K_c = 1 - \hat{F}_-/2$.

(iii) The signed normal scores test: $K_u = \Phi^{-1}(1 - \hat{F}_-/2)$ and $K_c = 2\hat{F}_-^{-1}\phi\{\Phi^{-1}(1 - \hat{F}_-/2)\}$ where ϕ and Φ are the density and the distribution function of the standard normal distribution.

Here $\hat{F}_-$ is the left continuous version of the product integral

$$\hat{F}(t) = \prod_{s \le t}\{1 - \Delta\hat{\Lambda}(s)\}$$

with

$$\hat{\Lambda}(t) = \int_0^t U^{-1} I(U > 0)\, d(N_1 + N_2)$$

where $U = \Sigma_{j=1}^4 U_j$. Under the null hypothesis and in the absence of censoring, $\hat{\Lambda}(t)$ is the Aalen-Nelson estimator of the cumulative hazard function of $|X_{2i} - X_{1i}|$ while $\hat{F}(t)$ is the corresponding empirical survival function.

In general we assume that $K_u = J_u(1 - \hat{F}_-)$ and $K_c = J_c(1 - \hat{F}_-)$ where the score generating functions J_u and J_c satisfy the relationship

$$J_u(v) = -\{(1 - v) J_c(v)\}' \tag{1.1}$$

This choice of the scoring functions is motivated by the censored data signed rank statistics considered by Woolson and Lachenbruch (1980) who discussed these tests in the case of log-linear models. More precisely, if $\log X_{1i} = \theta + \eta_{1i} + \varepsilon_i$ $\log X_{2i} = \eta_{2i} + \varepsilon_i$ where $\{\eta_{1i}\}_{i=1}^n$ and $\{\eta_{2i}\}_{i=1}^n$ are mutually independent samples from a distribution with density ϕ and $\{\varepsilon_i\}_{i=1}^n$ is a sample independent of η_{1i}'s and η_{2i}'s then locally most powerful signed rank test for testing $\theta = 0$ against $\theta > 0$ is based on statistic T with score functions $J_u(v) = -\phi'(z)/\phi(z)$ and $J_c(v) = 2\phi(z)/(1-v)$, where $z = \Phi^{-1}(1/2 + v/2)$ and Φ is the distribution function corresponding to ϕ. In particular the sign, signed Wilcoxon and signed normal scores test correspond to double exponential, logistic and normal densities ϕ, respectively. The term "locally most powerful test" refers here to the signed rank test that is locally most powerful in the uncensored version of the experiment, given the observed pattern of deaths and withdrawals.

In this paper we consider asymptotic distributions of censored data signed rank statistics. Using counting processes techniques of Aalen (1978), Gill (1980) and Andersen et al. (1982), in Section 2 we derive the asymptotic null distribution of the test statistics. The form of the asymptotic null variance was first derived heuristically by Woolson and Lachenbruch (1980). In Section 3 we consider contiguous alternatives, discuss the loglikelihood expansion of $(|Z_i|, \varepsilon_i, \delta_{1i}, \delta_{2i})$ and efficacies of the tests. In contrast to the two-sample comparisons with censored data (Gill (1980), Harrington and Fleming (1982)), the signed rank tests are in general inefficient. This is caused by two reasons. Firstly, the scores of asymptotically optimal tests depend on the distribution of the censoring variable however the signed rank tests derived from marginal likelihood do not take into account this distribution. Furthermore, the marginal likelihood and tests derived from it omit information carried by doubly censored observations. In Section 4 we consider the efficiencies of the signed Wilcoxon, sign and signed normal scores tests in the exponential Farlie-Gumbel-Morgenstern family subject to exponential and fixed censoring and compare them with efficiencies of the paired Prentice-Wilcoxon and logrank tests (see O'Brien and Fleming (1987), Dabrowska (1988) and Albers (1988)).

2. Asymptotic distributions under the null hypothesis.

We consider first the counting process $N = [\{N_{ji}(t) : j = 1, \dots, 4, i = 1, \dots, n\} : 0 < t < \infty]$. Clearly, each of the component processes has jumps of size 1 and no two processes jump at the same time. The behaviour of the process N is determined by its intensity $\alpha(t) = [\{\alpha_{ji}(t) : j = 1, \dots, 4, i = 1, \dots, n\} : 0 < t < \infty]$ where

$$\alpha_{ji}(t)\,dt \;=\; P\{dN_{ji}(t) = 1 \mid \mathcal{F}_{t-}\} \quad j = 1,2.$$

Here $dN_{ji}(t)$ stands for the increment of N_{ji} over the interval $[t, t+dt]$, whereas $\{\mathcal{F}_t\}$ is the self-exciting filtration generated by the null sets and processes $N_{ji}(t)$ $j = 1, \dots, 4$, $i = 1, \dots, n$. Thus $\alpha_{ji}(t)\,dt$ is the conditional probability that N_{ji} jumps in an infinitesimal interval of length dt around time t given the history $\mathcal{F}_{t-}$. It can be easily verified that in our case $\alpha_{ji}(t) = U_{ji}(t)\lambda_j(t)$ $j = 1, \dots, 4$, where $U_{ji}(t)$ are as in Section 1 while $\lambda_j(t)$ are given by

$$\lambda_j(t) = \lim_{h \downarrow 0} h^{-1} P(t \le |Z_i| \le t + h\ i \in B_j \,|\, |Z_i| \ge t\ i \in B_j)$$

Suppose that the joint distribution of (X_{1i}, X_{2i}) has density $\psi(x,y)$ and the censoring times C_i have distribution function G and survival function $\overline{G} = 1 - G$. Introduce sub-survival functions $\overline{F}_j(t) = P(|Z_i| \ge t,\ i \in B_j)$ which are explicitly given by

$$\overline{F}_1(t) = \int \overline{G}(u) \int_{-\infty}^{u-t} \psi(x,u)\,dx\,du$$

$$\overline{F}_2(t) = \int \overline{G}(u) \int_{-\infty}^{u-t} \psi(u,y)\,dy\,du$$

$$\overline{F}_3(t) = \int \{ \int_{-\infty}^{u-t} \int_{u}^{\infty} \psi(x,y)\,dx\,dy \}\, dG(u)$$

$$\overline{F}_4(t) = \int \{ \int_{u}^{\infty} \int_{-\infty}^{u-t} \psi(x,y)\,dx\,dy \}\, dG(u)$$

An easy calculation shows that the hazard functions $\lambda_j(t)$, $j = 1, \ldots, 4$ are given by $\lambda_j(t) = f_j(t)/\overline{F}_j(t)$ where f_j is the density corresponding to $\overline{F}_j(t)$, i.e.

$$\begin{aligned} f_1(t) &= \int \overline{G}(u)\,\psi(u-t,u)\,du \\ f_2(t) &= \int \overline{G}(u)\,\psi(u,u-t)\,du \\ f_3(t) &= \int \{ \int_{u}^{\infty} \psi(u-t,y)\,dy \}\, dG(u) \\ f_4(t) &= \int \{ \int_{u}^{\infty} \psi(x,u-t)\,dx \}\, dG(u). \end{aligned} \tag{2.1}$$

Let $\Lambda_j(t)$ be the cumulative hazard function corresponding to $\lambda_j(t)$. It follows from the theory of counting processes (see e.g. Andersen and Borgan (1985)) that

$$M_{ji}(t) = N_{ji}(t) - \int_0^t U_{ji}\, d\Lambda_j \quad j = 1, \ldots, 4$$

are mean zero square integrable local martingales with predictable variation processes

$$< M_{ji} > (t) \; = \; \int_0^t U_{ji} \, d\Lambda_j .$$

Furthermore, the martingales are orthogonal in the sense that their predictable covariation processes satisfy $< M_{ji}, M_{lk} > (t) = 0$ if either $j \neq l$ or $j \neq l$ and $i \neq k$, $j, l = 1, \ldots, 4$, $k = 1, \ldots, n$.

The estimates $\hat{\Lambda}(t)$ and $\hat{F}(t)$ of Section 1 share many of the properties of the usual Aalen-Nelson (Aalen (1978), Nelson (1972)) and Kaplan-Meier (1958) estimates. Here we shall need consistency properties of these estimates. Set $\overline{H}(t) = \overline{H}_1(t) + \overline{H}_2(t)$ where

$$\overline{H}_1(t) \; = \; P(|Z_i| \geq t \;\; \varepsilon_i = 1) \; = \; \int \overline{G}(u) \{ \int_u^\infty \psi(u-t, y) \, dy \} \, du$$

$$\overline{H}_2(t) \; = \; P(|Z_i| \geq t \;\; \varepsilon_i = -1) \; = \; \int \overline{G}(u) \{ \int_u^\infty \psi(x, u-t) \, du \} \, du .$$

Define

$$\Lambda(t) \; = \; -\int_0^t \overline{H}^{-1} (d\overline{F}_1 \; + \; d\overline{F}_2)$$

and let $\overline{F}(t)$ be the corresponding product integral

$$\overline{F}(t) \; = \; \prod_{s \leq t} \{ 1 - \Lambda(ds) \} .$$

For uncensored data, under the hypothesis of bivariate symmetry $\overline{F}(t)$ is the survival function of $|X_{2i} - X_{1i}|$ and $\Lambda(t)$ is the corresponding cumulative hazard function. If $H(\tau) > 0$ then $|\hat{\Lambda}(t) - \Lambda(t)| \to_P 0$ and $|\hat{F}(t) - \overline{F}(t)| \to_P 0$ uniformly in $t \in [0, \tau]$.

Let us consider now the behaviour of the statistic $T(t)$ under the hypothesis of bivariate symmetry. If the survival times have a symmetric density $\psi(x,y) = \psi(y,x)$ then clearly $\overline{F}_1 = \overline{F}_2$, $\overline{F}_3 = \overline{F}_4$, $\Lambda_1 = \Lambda_2$ and $\Lambda_3 = \Lambda_4$. Define $T' = T'(\infty)$ by

$$T'(t) = \int_0^t K_u (U_1 - U_2)\, d\Lambda_1 + \int_0^t K_c (U_3 - U_4)\, d\Lambda_3.$$

Under the hypothesis of bivariate symmetry T' is equal to zero with probability one because

$$\int_0^t K_u U_1\, d\Lambda_1 + \int_0^t K_c U_3\, d\Lambda_3$$

has the same distribution as

$$\int_0^t K_u U_2\, d\Lambda_2 + \int_0^t K_c U_4\, d\Lambda_4.$$

Further, $n^{-1/2}\{T(t) - T'(t)\}$ is a mean zero square integrable martingale with predictable variation process

$$n^{-1} < T - T' > (t) = n^{-1}\int_0^t K_u^2 (U_1 + U_2)\, d\Lambda_1 + n^{-1}\int_0^t K_c^2 (U_3 + U_4)\, d\Lambda_3$$

In particular, if $K_u(t) = J_u\{1 - \hat{F}_-(t)\}$ and $K_c = J_c\{1 - \hat{F}_-(t)\}$ for some continuous functions J_u and J_c then the consistency of $\hat{F}$ and Rebolledo's Central Limit Theorem (Theorem 3.2 in Andersen and Borgan (1985)) imply that $n^{-1/2}\{T(t) - T'(t)\}$ converges weakly to a mean zero normal variable with variance

$$\sigma^2(t) = 2\int_0^t J_u^2 (1 - \overline{F})\overline{F}_1\, d\Lambda_1 + 2\int_0^t J_c^2 (1 - \overline{F})\overline{F}_3\, d\Lambda_3.$$

Proposition 2.1. Suppose that the score generating functions J_u and J_c are continuous, and

$$|J_u(v)| \leq a(1-v)^{-1/2+\delta} \quad |J_c(v)| \leq a(1-v)^{-1/2+\delta} \tag{2.2}$$

for some constants $a > 0$ and $\delta > 0$. Then under the hypothesis of bivariate symmetry, $n^{-1/2}T$ converges weakly to a mean zero normal distribution with variance

$$\sigma_T^2 = 2\{\int J_u^2(1-\overline{F})\overline{F}_1 d\Lambda_1 + \int J_c^2(1-\overline{F})\overline{F}_3 d\Lambda_3\}$$

Proof. Under the assumed growth rate conditions on the score generating functions $\sigma^2(\infty) = \sigma_T^2 < \infty$, where $\sigma^2(\infty) = \lim_{t\uparrow\infty} \sigma^2(t)$. Therefore by Theorem 4.2 in Billingsley (1968), it is enough to show that

$$\lim_{t\uparrow\infty} \overline{\lim} P(n^{-1}\int_t^\infty J_u^2(1-\hat{F}_-)(U_1 + U_2)d\Lambda_1 > \varepsilon) = 0$$

$$\lim_{t\uparrow\infty} \overline{\lim} P(n^{-1}\int_t^\infty J_c^2(1-\hat{F}_-)(U_3 + U_4)d\Lambda_3 > \varepsilon) = 0$$

for all $\varepsilon > 0$. Let us consider the first of these limits, the proof of the second is analogous. We have $\hat{F}_- \geq U/n$ so that (2.2) implies

$$n^{-1}\int_t^\infty J_u^2(1-\hat{F}_-)(U_1 + U_2)d\Lambda_1 \leq a^2 n^{-1}\int_t^\infty (U_1 + U_2)(U/n)^{-1+2\delta} d\Lambda_1.$$

By Theorem 1.1 in van Zuijlen (1978), for given $\varepsilon > 0$ we have $(U/n)^{-1+2\delta} \leq \beta\overline{H}^{-1+2\delta}$ with probability at least $1-\varepsilon$, where $\beta = \varepsilon^{-1+2\delta}$. On the set where this holds the last integral is less than

$$a^2\beta n^{-1}\int_t^\infty (U_1 + U_2)\,\overline{H}^{-1+2\delta} d\Lambda_1 \to_P 2a^2\beta\int_t^\infty \overline{H}^{-1+2\delta}\overline{F}_1 d\Lambda_1 \leq -2a^2\beta\int_t^\infty \overline{H}^{-1+2\theta} d\overline{H}$$

and the right hand side tends to 0 as $t \to \infty$.

For practical purposes we shall need to estimate σ_T^2 from the data. We mimic procedures for the two sample case (Gill (1980), Andersen et al. (1982) and Harrington and Fleming (1982)) and estimate σ_T^2 by $\hat{\sigma}_T^2$ where

$$\hat{\sigma}_T^2 = \Sigma_{i=1}^{2} \int_0^\infty K_u^2 (1 - \frac{\Delta N_i - 1}{U_i - 1}) \frac{dN_i}{U_i} + \Sigma_{i=3}^{4} \int_0^\infty K_c^2 (1 - \frac{\Delta N_i - 1}{U_i - 1}) \frac{dN_i}{U_i}.$$

Under assumptions of Proposition 2.2, $\hat{\sigma}_T^2$ is a consistent estimate of σ^2.

To conclude this section we give a Corollary which summarizes the weak convergence results for the test statistics of Section 1. We need to verify the growth rate condition (2.2). If the score generating function corresponding to uncensored data is selected as $J_u(v) = -\phi'(z)/\phi(z)$ with $z = \Phi^{-1}(1/2 + v/2)$ for some symmetric density ϕ, then for most choices of ϕ arising in practice $J_u(v)$ satisfies the condition (2.2). In particular this holds for normal, logistic and double exponential ϕ. Further, if the score generating function $J_c(v)$ corresponding to censored data is chosen so that the relation (1.1) holds then

$$|J_c(v)| = (1 - v)^{-1} | \int_v^1 J_u(w)\, dw | \leq$$

$$a_1 (1 - v)^{-1} \int_u^1 (1 - w)^{-1/2+\delta} dw = a_2 (1 - v)^{-1/2+\delta}$$

where $a_2 = a_1 / (1/2 + \delta)$. Hence the score functions J_u and J_c satisfy (2.2) with $a = \max(a_1, a_2)$.

Corollary 2.1. Under the null hypothesis the sign , signed Wilcoxon and signed normal scores test statistics are asymptotically mean zero normal with asymptotic variances given by

$$\sigma_S^2 = 2\{\int \overline{F}_1 \, d\Lambda_1 + \int \overline{F}_3 \, d\Lambda_3\} = P(\varepsilon_i = 1) + P(\varepsilon_i = -1)$$
$$\sigma_W^2 = 2\{\int (1-\overline{F})^2 \overline{F}_1 d\Lambda_1 + \int (1-\overline{F}/2)^2 \overline{F}_3 \, d\Lambda_3\}$$
$$\sigma_N^2 = 2\{\int w_1^2(\overline{F}) \overline{F}_1 d\Lambda_1 + \int w_2^2(\overline{F}) \overline{F}_3 \, d\Lambda_3\}$$

where $w_1(s) = \Phi^{-1}(1 - s/2)$, $w_2(s) = 2s^{-1}\phi\{\Phi^{-1}(1 - s/2)\}$, and ϕ and Φ denote the density and the distribution function of the standard normal distribution.

3. Asymptotic distributions under contiguous alternatives.

3.1. The log-likelihood expansion for $(|Z_i|, \varepsilon_i, \delta_{1i}, \delta_{2i})$

In this section we assume that under the null hypothesis (X_{1i}, X_{2i}) have symmetric density ψ, $\psi(x,y) = \psi(y,x)$ and C_i have distribution function G. We consider alternatives of the form $\psi_n(x,y) = \psi(x,y)\ \{1 + n^{-1/2}\gamma_n(x,y)\}$ where γ_n is a sequence of functions such that $\gamma_n(x,y) \rightarrow \gamma(x,y)$ for almost all (x,y) and

$$\int \gamma_n(x,y)\psi(x,y)\,dxdy = \int \gamma(x,y)\psi(x,y)\,dxdy = 0.$$

This condition ensures that ψ_n is a density. In the case of parametric models, if $\psi_{\theta_0}(x,y)$ is a symmetric density and the alternatives are $\psi_{\theta_n}(x,y)$ with $\theta_n = \theta_0 + cn^{-1/2}$, the function γ reduces to c times the derivative of $\log \psi_\theta(x,y)$ at $\theta = \theta_0$.

Let P and P_n denote the joint distributions of $(|Z_i|, \varepsilon_i, \delta_{1i}, \delta_{2i})$ under the null hypothesis and under the alternative, respectively. Set $\eta_i = (1-\delta_{1i})(1-\delta_{2i})$. Then

$$\log dP_n/dP = \Sigma_{j=1}^{4} \int \log (f_{jn}/f_j)\, dN_j + \Sigma V_i$$
$$V_i = \eta_i \log (p_{0n}/p_0) + (1-\eta_i) \log \{(1-p_{0n})/(1-p_0)\},$$

where p_0 and p_{0n} represent the probability of a doubly censored observation under the null hypothesis and under the alternative, respectively. We have

$$p_0 = \int \{ \int_u^{\infty} \int_u^{\infty} \psi(x,y)\, dxdy \}\, dG(u)$$

and similarly

$$p_{0n} = p_0 + n^{-1/2} \int \{ \int_u^{\infty} \int_u^{\infty} \gamma_n(x,y)\, \psi(x,y)\, dxdy \}\, dG(u) = p_0 + n^{-1/2} h_n.$$

Assuming that limits can be taken under the integral signs, it can be easily verified that as $n \to \infty$

$$A_{jn}(t) = 2n^{1/2} [\{f_{jn}(t)/f_j(t)\}^{1/2} - 1] \to A_j(t) \quad j = 1, \ldots, 4 \tag{3.1}$$

where

$$A_1(t) = f_1(t)^{-1} \int \overline{G}(u)\, \gamma(u-t,u)\, \psi(u-t,u) du$$
$$A_3(t) = f_3(t)^{-1} \int \{ \int_u^{\infty} \gamma(u-t,y)\, \psi(u-t,y)\, dy \}\, dG(u).$$

The functions A_2 and A_4 are defined similarly to A_1 and A_3 except that $\gamma(u-t,u)$ and $\gamma(u-t,y)$ should be replaced by $\gamma(u,u-t)$ and $\gamma(y,u-t)$, respectively. Moreover,

$$A_{0n}(d) = 2n^{1/2} [(p_{0n}/p_0)^{1/2} - 1] d + 2n^{1/2} [\{(1-p_{0n})/(1-p_0)\}^{1/2} - 1](1-d) \to$$
$$p_0^{-1} hd - (1-p_0)^{-1} h (1-d) = A_0(d) \tag{3.2}$$

for d = 0 or 1 and

$$h = \int\{\int_u^\infty\int_u^\infty \gamma(x,y)\,\psi(x,y)\,dxdy\}\,dG(u)$$

By Le Cam's Second Lemma (Hájek and Šidák (1968)) it is enough to consider the asymptotic distribution of $L_n = L_{1n} + L_{2n}$ where

$$L_{1n} = n^{-1/2}\Sigma_{j=1}^4 \int A_{jn}\,dN_j \quad L_{2n} = n^{-1/2}\Sigma_{i=1}^n A_{0n}(\eta_i).$$

We consider L_{1n} first. Assume

$$\lim_{n\to\infty} \int \{A_{jn}(t) - A_j(t)\}^2 f_j(t)\,dt = 0 \tag{3.3}$$

for j = 1, . . . , 4. Set

$$S = n^{-1/2}\Sigma_{j=1}^4 \int A_j\,dN_j. \tag{3.4}$$

Under the null hypothesis S is asymptotically mean zero normal with variance

$$\sigma^2 = \int (A_1^2 + A_2^2)\bar{F}_1\,d\Lambda_1 + \int (A_3^2 + A_4^2)\bar{F}_3\,d\Lambda_3. \tag{3.5}$$

By (3.3)

$$E\,L_{1n} = n^{1/2}\Sigma_{j=1}^4 \int A_{jn}(t)\,f_j(t)\,dt = -1/4\,\Sigma_{j=1}^4 \int A_{jn}^2(t)\,f_j(t)\,dt \to -1/4\,\sigma^2,$$

$$\mathrm{Var}(L_{1n} - S) \le \Sigma_{j=1}^4 \int (A_{jn}(t) - A_j(t))^2 f_j(t)\,dt \to 0.$$

It follows that under the null hypothesis L_{1n} is asymptotically normal with mean $-\sigma^2/4$ and variance σ^2.

As for the term L_{2n}, we have

$$\begin{aligned} EL_{2n} &= -n\,(p_{0n}^{1/2} - p_0^{1/2})^2 - n\,\{(1 - p_{0n})^{1/2} - (1 - p_0)^{1/2}\} \\ &= -1/4\,\mathrm{Var}\,A_{0n}(\eta_i) \to -1/4\,h^2 p_0^{-1}(1 - p_0)^{-1}. \end{aligned}$$

Berry-Esseen's Central Limit Theorem implies that L_{2n} converges weakly to a normal distribution with mean $-1/4\, h^2 p_0^{-1}(1-p_0)^{-1}$ and variance $h^2 p_0^{-1}(1-p_0)^{-1}$. Furthermore, it is easy to see that L_{1n} and L_{2n} are uncorrelated. Therefore, it follows that L_n is asymptotically normal with mean $-1/4\,\sigma_0^2$ and variance σ_0^2 where

$$\sigma_0^2 = \int (A_1^2 + A_2^2)\overline{F}_1 d\Lambda_1 + \int (A_3^2 + A_4^2)\overline{F}_3 d\Lambda_3 + h^2 p_0^{-1}(1-p_0)^{-1}. \qquad (3.6)$$

Le Cam's First and Second Lemmas (Hájek and Šidak (1968)) imply now the following result.

Proposition 3.1. Under the null hypothesis if (3.1) (3.2) and (3.3) hold, $\log dP_n / dP$ converges weakly to a normal distribution with mean $-1/2\,\sigma_0^2$ and variance σ_0^2 and the family P_n is contiguous to P.

3.2. Efficacies of tests.

To obtain efficacies of tests it is enough to find the joint distribution of $n^{-1/2}T$ and $\log(dP_n / dP)$ under the null hypothesis. Let $S' = S'(\infty)$ be given by

$$S'(t) = n^{-1/2} \Sigma_{j=1}^{4} \int_0^t A_j (dN_j - U_j d\Lambda_j)$$

where A_j, $j = 1, \ldots, 4$ is as in Section 3.1. Under the null hypothesis $S'(t)$ is a mean zero square integrable local martingale with predictable variation process

$$< S' > (t) = n^{-1} \Sigma_{j=1}^{4} \int_0^t A_j^2 U_j d\Lambda_j$$

Moreover, the predictable covariation process of $S'(t)$ and $n^{-1/2}\{T(t) - T'(t)\}$ is given by

$$n^{-1/2} < S', T - T' > (t) = n^{-1} \int_0^t K_u (A_1 U_1 - A_2 U_2)\, d\Lambda_1$$

$$+ \ n^{-1} \int_0^t K_c (A_3 U_3 - A_4 U_4)\, d\Lambda_3.$$

In particular, if the processes K_u and K_c are given by $K_u = J_u(1 - \hat{F}_-)$ and $K_c = J_c(1 - \hat{F}_-)$ for some continuous score generating functions J_u and J_c, then the right-hand side converges in probability to

$$c_T(t) = \int_0^t J_u(1 - \overline{F})(A_1 - A_2)\overline{F}_1\, d\Lambda_1 + \int_0^t J_c(1 - \overline{F})(A_3 - A_4)\overline{F}_3\, d\Lambda_3.$$

Under conditions of Propositions 2.1 and 3.1, Rebolledo's Central Limit Theorem and Cramer-Wold device applied in a fashion analogous to Hájek and Šidak (1968) and Gill (1980), entail that $S'(\infty)$ and $n^{-1/2}\{T(\infty) - T'(\infty)\}$ are jointly asymptotically normal with mean zero and covariance matrix

$$\begin{bmatrix} \sigma^2 & c_T \\ c_T & \sigma_T^2 \end{bmatrix} \tag{3.7}$$

where $c_T = c_T(\infty)$ and σ^2 is given by (3.5). Since $n^{-1/2}T'(\infty) = o_P(1)$ and S' has the same asymptotic distribution as the statistic (3.4), it follows that L_{1n} and $n^{-1/2}T$ are jointly asymptotically normal with mean $(-\sigma^2/4, 0)$ and the above covariance matrix. It remains to show that L_{2n}, the term corresponding to doubly censored observations is uncorrelated with $T(\infty) - T'(\infty)$. Fix $t < \infty$ and let us approximate the martingale $T(t) - T'(t)$ by the Lebesgue-Stieltjes integral

$$\Sigma_{i=1}^{n} \sum_{s_v} K_u(s_v)\{dM_{1i}(s_v) \; - \; dM_{2i}(s_v)\} \; +$$

$$\Sigma_{i=1}^{n} \sum_{s_v} K_c(s_v)\{dM_{3i}(s_v) \; - \; dM_{4i}(s_v)\}$$

where $0 = s_1 < \cdots < s_m = t$ is a fine grid. For $i = l$ we have $A_{0n}(\eta_l)\, dM_{ji}(s_v) = 0$, $j = 1, \ldots, 4$. Further, for $i \neq l$ $A_{0n}(\eta_l)$ and $dM_{ji}(s_v)$ are conditionally independent given the history $F_{s_v -}$ so that using the predictability of the processes K_u and K_c we get

$$\begin{aligned}
&E[\sum_{s_v} A_{0n}(\eta_l) K_u(s_v)\{dM_{1i}(s_v) - dM_{2i}(s_v)\} \\
&+ \sum_{s_v} A_{0n}(\eta_l) K_c(s_v)\{dM_{3i}(s_v) \; - \; dM_{4i}(s_v)\}] \\
&= E[\sum_{s_v} K_u(s_v) E\{A_{0n}(\eta_l) | F_{s_v -}\} E[\{dM_{1i}(s_v) - dM_{2i}(s_v)\} | F_{s_v -}]] \\
&+ E[\sum_{s_v} K_c(s_v) E\{A_{0n}(\eta_l) | F_{s_v -}\} E[\{dM_{3i}(s_v) - dM_{4i}(s_v)\} | F_{s_v -}]] \\
&= 0.
\end{aligned}$$

An appropriate limiting argument shows now that L_{2n} is uncorrelated with $T(\infty) - T'(\infty)$. This can be proved more rigorously by first considering the case when K_u and K_c are predictable step functions and then for general predictable processes (see Meyer (1971)).

Proposition 3.2. Suppose that the assumptions of Propositions 2.1 and 3.1 are satisfied. Then under the null hypothesis $(\log dP_n / dP, n^{-1/2} T)$ converge weakly to a normal distribution with mean $(-1/2\,\sigma_0^2, 0)$ and covariance matrix (3.7) with σ^2 replaced by σ_0^2 given by (3.6).

Le Cam's Third Lemma implies now that the efficacy of the test statistic T is given by c_T^2 / σ_T^2. In particular, the following corollary gives explicitly the efficacies of the sign, signed Wilcoxon and signed normal scores tests.

Corollary 3.1. The efficacies of the sign, signed Wilcoxon and signed normal scores tests are given by

$$
\begin{aligned}
e_S &= \{\int (A_1 - A_2)\overline{F}_1 \, d\Lambda_1 \; + \; \int (A_3 - A_4)\overline{F}_3 \, d\Lambda_3\} / \sigma_S^2 \\
e_W &= \{\int (A_1 - A_2)(1 - \overline{F})\overline{F}_1 \, d\Lambda_1 \; + \; \int (A_3 - A_4)(1 - \overline{F}/2)\overline{F}_3 \, d\Lambda_3\} / \sigma_W^2 \\
e_N &= [\int (A_1 - A_2) w_1(\overline{F})\overline{F}_1 \, d\Lambda_1 \; + \; \int (A_3 - A_4) w_2(\overline{F})\overline{F}_3 \, d\Lambda_3\} / \sigma_N^2
\end{aligned}
$$

where w_1, w_2, σ_S^2, σ_W^2 and σ_N^2 are as in Corollary 2.1.

From Proposition 3.2 it follows that the signed rank tests are in general inefficient. This is in contrast with tests for two-sample comparisons under the equal censoring model (Gill (1980), Harrington and Fleming (1982)). In the present setting the scores of asymptotically optimal tests depend on the censoring distribution, while censored data signed rank tests do not take this distribution into the account. Moreover, information carried by doubly censored observations is omitted. In the next Section we shall examine the efficiency loss due to censoring in more detail.

If the density of the paired survival times (X_1, X_2) belongs to a parametric family $\psi_\theta(x, y)$ and the censoring distribution is known (e.g. in the case of fixed censoring), tests for symmetry can be based for instance on the likelihood ratio statistic. The efficacy of this test is σ_0^2 given by (3.6). In general however, the form of the censoring distribution is unknown, tests have to be constructed adaptively using methods appropriate for semiparametric models. This problem will be considered elsewhere.

4. Some comparisons.

For comparison purposes we consider the Farlie-Gumbel-Morgenstern (FGM) family and assume that the joint density of the survival times (X_1, X_2) has form

$$\Delta e^{-s} e^{-\Delta t} \{1 + \alpha (1 - 2e^{-s})(1 - 2e^{-\Delta t})\}. \tag{4.1}$$

Here the marginals are standard exponential for X_1 and exponential with scale parameter Δ, $\Delta > 0$ for X_2. The parameter α ranges between -1 and 1 and accounts for the degree of dependence between X_1 and X_2. For $\alpha = 0$ the two survival times are independent. In general the correlation between X_1 and X_2 is equal to $\alpha/4$ and thus it ranges between -.25 and .25.

The efficiencies of the tests will be considered for uncensored data and for exponential and fixed censoring. In the exponential case, we assume that C has exponential distribution with scale parameter λ and choose $\lambda = .05, .2, .5, 1.4$ and 3. For this choice of λ and $\alpha = 0$ the probabilities of uncensored observations are .93, .76, .53, .25 and .1, respectively. For $\alpha \neq 0$ the probabilities of uncensored observations do not differ much from these values. In the case of fixed censoring C assumes value c with probability 1. We choose $c = 3, 2, 1.25, .7$ and .4. When $\alpha = 0$ the probabilities of uncensored observations are approximately .90, .75, .51, .25 and .11, respectively.

To discuss the signed rank tests we apply the log transformation to X_1, X_2 and C. This transformation turns the exponential FGM model (4.1) into a FGM family with extreme value marginals. Further, set

$$x_1 = e^{-t} + \Delta \quad x_2 = 2e^{-t} + \Delta \quad x_3 = e^{-t} + 2\Delta \quad x_4 = 2e^{-t} + 2\Delta$$
$$y_1 = 1 + \Delta e^{-t} \quad y_2 = 2 + \Delta e^{-t} \quad y_3 = 1 + 2\Delta e^{-t} \quad y_4 = 2 + 2\Delta e^{-t}.$$

In the case of exponential censoring the densities $f_{i\Delta}(t)$, $i = 1, \ldots, 4$ of (2.1) are given by

$$\begin{aligned}
f_{1\Delta}(t) &= \Delta e^{-t}\{(1+\alpha)/(x_1+\lambda)^2 - 2\alpha/(x_2+\lambda)^2 - 2\alpha/(x_3+\lambda)^2 + 4\alpha/(x_4+\lambda)^2\} \\
f_{2\Delta}(t) &= \Delta e^{-t}\{(1+\alpha)/(y_1+\lambda)^2 - 2\alpha/(y_2+\lambda)^2 - 2\alpha/(y_3+\lambda)^2 + 4\alpha/(y_4+\lambda)^2\} \\
f_{3\Delta}(t) &= \lambda e^{-t}\{(1+\alpha)/(x_1+\lambda)^2 - 2\alpha/(x_2+\lambda)^2 - \alpha/(x_3+\lambda)^2 + 2\alpha/(x_4+\lambda)^2\} \\
f_{4\Delta}(t) &= \lambda\Delta e^{-t}\{(1+\alpha)/(y_1+\lambda)^2 - \alpha/(y_2+\lambda)^2 - 2\alpha/(y_3+\lambda)^2 + 2\alpha/(y_4+\lambda)^2\}.
\end{aligned}$$

In the case of fixed censoring these densities are given by

$$\begin{aligned}
f_{1\Delta}(t) &= \Delta e^{-t}\{(1+\alpha)I_c(x_1) - 2\alpha I_c(x_2) - 2\alpha I_c(x_3) + 4\alpha I_c(x_4)\} \\
f_{2\Delta}(t) &= \Delta e^{-t}\{(1+\alpha)I_c(y_1) - 2\alpha I_c(y_2) - 2\alpha I_c(y_3) + 4\alpha I_c(y_4)\} \\
f_{3\Delta}(t) &= ce^{-t}\{(1+\alpha)e^{-cx_1} - 2\alpha e^{-cx_2} - \alpha e^{-cx_3} + 2\alpha e^{-cx_4}\} \\
f_{4\Delta}(t) &= \Delta ce^{-t}\{(1+\alpha)e^{-cy_1} - \alpha e^{-cy_2} - 2\alpha e^{-cy_3} + 2\alpha e^{-cy_4}\}
\end{aligned}$$

where

$$I_c(x) = \int_0^c te^{-tx}\,dt.$$

The score functions $A_i(t)$, $i = 1, \ldots, 4$ of Section 3 can be next obtained by differentiating $\log f_{\Delta i}(t)$ with respect to Δ at $\Delta = 1$. Furthermore, the probabilities $p_{0\Delta}$ of doubly censored observations are given by

$$p_{0\Delta} = \lambda[(1+\alpha)/(1+\Delta+\lambda) + \alpha\{1/(2+2\Delta+\lambda) - 1/(2+\Delta+\lambda) - 1/(1+2\Delta+\lambda)\}]$$

in the case of exponential censoring, and

$$p_{0\Delta} = (1+\alpha)\exp\{-c(1+\Delta)\} + \alpha\exp\{-c(2+2\Delta)\}$$
$$-\alpha\exp\{-c(2+\Delta)\} - \alpha\exp\{-c(1+2\Delta)\}$$

in the case of fixed censoring. The contribution of doubly censored observations to the asymptotic lower bound (3.6) is then given by $h^2 p_0^{-1}(1-p_0)^{-1}$ where p_0 is given by $p_{0\Delta}$ evaluated at $\Delta = 1$ and h is the derivative of $p_{0\Delta}$ with respect to Δ at $\Delta = 1$.

The asymptotic lower bound of Proposition 3.1 and the efficacies of the signed rank test do not have closed form expressions, therefore we computed them numerically using Romberg integration algorithm (e.g. Gerald and Wheatley (1985, p. 281)).

When $\alpha = 0$ and the data are uncensored the signed Wilcoxon test is locally most powerful within the class of signed rank tests and efficient within the class of tests based on $|X_{2i} - X_{1i}|$ and the signs ε_i of $X_{2i} - X_{1i}$ (Doksum (1980)). For $\alpha \neq 0$ the efficient signed rank test within this class of tests depends in a complicated way on α. However, it follows from Table 1 below that the asymptotic relative efficiency (ARE) of the signed Wilcoxon test with respect to the asymptotic lower bound is at least .971 when $\alpha \neq 0$ and thus not much efficiency is lost when applying this test in the case of dependent data as well. For censored data results of Woolson and Lachenbruch (1980) imply that when $\alpha = 0$ the censored data signed Wilcoxon test is locally most powerful within the class of signed rank tests derived from marginal likelihood considerations. Table 1 gives the ARE of the censored data signed Wilcoxon test with respect to the optimal parametric test. For both types of censoring and each of the α values considered, the ARE decreases as the censoring gets heavier. When $\alpha = 0$ the ARE drops down from 1 for uncensored data to .503 for exponential censoring with $\lambda = 3$ and to .418 for fixed censoring at $c = .4$. Thus in this case the efficiency loss is approximately 49.7% and 58.2% for exponential and fixed censoring, respectively. More generally, for λ between 0 and 3 the efficiency loss due to censoring ranges between 30.6% for $\alpha = 1$ and 65.3% for $\alpha = -1$. For fixed censoring and c values considered, the efficiency loss due to censoring ranges between 48.7% for $\alpha = 1$ and 65.3% for $\alpha = -1$

Table 1 about here

In Table 2 we give the ARE of the sign and signed normal scores tests with respect to the signed Wilcoxon test. In the case of exponential censoring the ARE of both tests increases a little as the censoring gets heavier. In the case of the sign test and $\alpha = 0$ the ARE increases from .75 for uncensored data to .841 for $\lambda = 3.0$. Thus in this case there is a 9.1% gain in the ARE. For the remaining α values this gain ranges between 7% for $\alpha = 1$ and 18.8% for $\alpha = -1$. In the case of the normal scores test and $\alpha = 0$ the ARE increases from .955 for uncensored data to 1.005 for $\lambda = 3$ so that there is a 5% increase in ARE. For other α values this gain ranges between 6.2% when $\alpha = 1$ and 2.3% when $\alpha = -1$. As α approaches -1 or $\lambda \geq 3$ the normal scores test performs slightly better than the signed Wilcoxon test. Further, let us consider the fixed censorship model. In the case of sign test we observe a slight decrease in ARE when $c \leq 1.25$, i.e. when the probability of uncensored observations exceeds .5, and a slight increase when $c \geq 1.25$. For $c \leq 1.25$ and $\alpha = 0$ the loss in efficiency is approximately 4.4% while for $c \geq 1.25$ the gain in efficiency is about 9.9%. For the remaining α values and $c \geq 1.25$ the gain in ARE ranges between 5.7% when $\alpha = 1$ and 22.6% when $\alpha = -1$. As for the normal scores test, the ARE is somewhat higher under fixed censoring than for uncensored data. The gain in ARE is 7.5% when $\alpha = 0$ and ranges between 7.3% when $\alpha = 1$ and 8.1% when $\alpha = -1$. For most of the α and c values considered the efficiency of the signed normal scores tests is higher than the signed Wilcoxon test.

Table 2 about here

Rather than base tests for symmetry on the ranks of differences within the pairs, we can consider tests based on differences of ranks of each pair in the pooled sample.

For uncensored data this approach was adopted by Snijders (1981), Doksum (1980) and Lam and Longnecker (1983) among others. Formally the procedures take the same form as the usual tests for the two-sample comparisons except that the variance of the test statistics is modified so as to take into the account the intrapair dependence. When conditioned on the observed configuration of ranks in the pooled sample, the tests are distribution free. See Snijders (1981) for further discussion.

Censored data analogues of these procedures were developed by O'Brien and Fleming (1987), Dabrowska (1988) and Albers (1988) among others. Here we consider a Prentice type method of ranking of the observations, that is the paired data are pooled, uncensored observations are ranked among themselves and each censored observation is assigned the same rank as the nearest uncensored observation on the left. For suitably chosen score functions $J(u,d)$, $u \in (0,1)$, $d = 1,0$, the test statistics reject the hypothesis of symmetry for large values of $n^{1/2} W / \hat{\sigma}$ where

$$W = n^{-1} \{\Sigma_{i=1}^{n} J(\hat{S}(Y_{1i}), \delta_{1i}) - \Sigma_{i=1}^{n} J(\hat{S}(Y_{2i}), \delta_{2i})\} \tag{4.2}$$

and $\hat{\sigma}^2$ is an estimator of the asymptotic variance of W of the form

$$\hat{\sigma}^2 = n^{-1} \{\Sigma_{j=1}^{2} \Sigma_{i=1}^{n} J^2(\hat{S}(Y_{ji}), \delta_{ji}) - \Sigma_{i=1}^{n} J(\hat{S}(Y_{1i}), \delta_{1i}) J(\hat{S}(Y_{2i}), \delta_{2i})\}.$$

Here $\hat{S}(t)$ is the Kaplan-Meier estimate from the pooled sample or an estimator asymptotically equivalent to it, e.g. $\hat{S}(t) = 1 - \exp\{-\hat{\Lambda}(t)\}$ where $\hat{\Lambda}(t)$ is the Aalen-Nelson estimate based on the pooled sample. The choice $J(u,1) = 2u - 1$ and $J(u,0) = u$ yields the Prenctice-Wilcoxon test statistic (O'Brien and Fleming (1987)) whereas the choice $J(u,1) = -1 - \log(1-u)$ and $J(u,0) = -\log(1-u)$ leads to the paired data log-rank test.

Asymptotic properties of tests based on $n^{1/2} W / \hat{\sigma}$ were studied in Dabrowska (1988). Therein it was assumed that the censoring is bivariate, i.e. each pair member has its own censoring time and the two censoring times are possibly dependent. The univariate censoring mechanism discussed in the previous sections can be accommodated into this framework by considering pairs of censoring times (C_1, C_2) such that $P(C_1 = C_2) = 1$. Thus the marginal survival functions of C_1 and C_2 are common, say $\overline{G}(t)$ whereas their joint survival function is $\min\{\overline{G}(s), \overline{G}(t)\}$.

Let us consider the model described at the beginning of Section 3.1. Let S be the common marginal distribution function of X_{1i} and X_{2i} under the null hypothesis. Further, set

$$\gamma_0(s) = \int \{\gamma(s,t) - \gamma(t,s)\} \psi(s,t)\, dt$$
$$\Gamma_0(s) = \int_s^{\infty} \gamma_0(x)\, dx.$$

The efficacy of the tests based on the statistic (4.2) is then given by c_W^2 / σ_W^2 where

$$c_W = \int J(S(s),1)\overline{G}(s)\gamma_0(s)\, ds + \int J(S(s),0)\Gamma_0(s)\, dG(s)$$
$$\sigma_W^2 = 2\{\int J^2(S(s),1)\overline{G}(s)\, dS(s) + \int J^2(S(s),0)\overline{S}(s)\, dG(s)$$
$$-\Sigma_{p=0}^{1}\Sigma_{q=0}^{1}\int\int J(S(s),p)\, J(S(t),q)\, dH_{pq}(s,t)\}$$

where $H_{pq}(s,t)$ is the bivariate subdistribution function corresponding to those observations for which $\delta_{1i} = p$ and $\delta_{2i} = q$, $p, q = 0, 1$, $i = 1, \ldots, n$. In particular in the case of the exponential FGM model (4.1), S is the standard exponential distribution function, $\gamma_0(s) = -(1-s)e^{-s}$ and $\Gamma_0(s) = se^{-s}$.

In Table 3 we give the ARE of the paired Prentice-Wilcoxon and logrank tests relative to the signed Wilcoxon test. For uncensored data these tests perform considerably better than the signed Wilcoxon test. For $\alpha = 0$ the ARE is 1.125 for the paired Wilcoxon test and 1.5 for the logrank test. For other choices of α the ARE ranges between 1.114 for $\alpha = .5$ and 1.188 for $\alpha = -1$ in the case of the paired Wilcoxon test and between 1.352 for $\alpha = 1$ and 1.690 for $\alpha = -1$ in the case of the logrank test. Under exponential censoring the ARE of both tests decreases as the censoring gets heavier. When $\alpha = 0$ the ARE of the paired Prentice-Wilcoxon test drops down from 1.125 for uncensored data to .862 when $\lambda = 3$ whereas the ARE of the paired logrank test decreases from 1.5 for uncensored data to .882 when $\lambda = 3$. Thus the efficiency loss is 26% for the Prentice-Wilcoxon test and 61.8% for the logrank test. For the remaining α values considered, in the case of the Prentice-Wilcoxon test the efficiency loss is between 25.1% for $\alpha = \pm.5$ and 30% for $\alpha = 1$. In the case of the logrank test this loss ranges between 51.2% for $\alpha = 1$ and 76.8% for $\alpha = -1$. The Prentice-Wilcoxon test performs better than the signed Wilcoxon test only for $\lambda \leq .2$ and from Table 2 we find that the signed normal scores test has higher efficiency than the Prentice-Wilcoxon test when $\lambda \geq .5$ except for $\alpha = -1$ and $\lambda = .5$. In the case of the logrank test the situation is analogous. The logrank test performs better than the signed Wilcoxon test for $\lambda \leq .5$ and the signed normal scores test has higher efficiency than the logrank test for $\lambda \geq 1.4$ except when $\alpha = -1$ and $\lambda = 1.4$. Finally let us consider the fixed censorship model. In the case of the Prentice-Wilcoxon test the ARE decreases as the censoring gets heavier. For $\alpha = 0$ and $c \geq .4$ the efficiency loss due to censoring is about 12.1%. For other α values the efficiency loss ranges between 9.7% for $\alpha = .5$ and 19.5 % for $\alpha = -1$. The Prentice-Wilcoxon test performs slightly better than the signed Wilcoxon test with the exception of $c = .4$ and $\alpha = -.5$ or -1. As for the logrank test, its ARE decreases as the censoring gets heavier except for $\alpha = 1$. For $\alpha = 0$ the efficiency loss caused by censoring is about 48.3%, for other α values the efficiency loss is between 35.8% when $\alpha = .5$ and 73.5% when $\alpha = -1$. For all values considered the logrank test performs better than the signed Wilcoxon test except when $c = .7$ or .4 and $\alpha = -.5$ or -1.

Table 3 about here

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Table 1. ARE of the signed Wilcoxon test with respect to the optimal parametric test.

α	0.0	1.0	.5	–.5	–1.0
	Uncensored data				
	1.0	.998	.999	.995	.971
	Exponential censoring with scale λ				
$\lambda =$ .05	.940	.962	.952	.919	.879
$\lambda =$ .2	.879	.889	.862	.778	.716
$\lambda =$ .5	.715	.816	.772	.647	.571
$\lambda =$ 1.4	.590	.739	.672	.502	.415
$\lambda =$ 3.0	.503	.692	.606	.402	.315
	Fixed censoring at c				
c = 3.0	.516	.620	.571	.459	.396
c = 2.0	.464	.573	.524	.398	.328
c = 1.25	.456	.562	.517	.387	.322
c = .7	.446	.545	.502	.385	.321
c = .4	.418	.511	.467	.368	.318

Table 2. ARE of the sign (S) and signed normal scores (N) tests with respect to the signed Wilcoxon test.

α			0.0	1.0	.5	-.5	-1.0
			Uncensored data				
		S	.750	.757	.764	.709	.638
		N	.955	.938	.940	.980	1.011
			Exponential censoring with scale λ				
λ =	.05	S	.750	.730	.765	.710	.643
		N	.964	.909	.949	.988	1.018
λ =	.2	S	.752	.763	.767	.715	.654
		N	.981	.961	.966	1.003	1.029
λ =	.5	S	.761	.773	.775	.729	.678
		N	.997	.978	.984	1.014	1.034
λ =	1.4	S	.794	.797	.801	.775	.743
		N	1.005	.994	.998	1.014	1.022
λ =	3.0	S	.841	.827	.838	.838	.826
		N	1.005	1.000	1.001	1.007	1.008
			Fixed censoring at c				
c =	3.0	S	.716	.734	.734	.677	.613
		N	1.013	.981	.991	1.044	1.082
c =	2.0	S	.706	.732	.728	.667	.607
		N	1.030	.999	1.009	1.058	1.092
c =	1.25	S	.726	.748	.745	.692	.643
		N	1.026	1.009	1.014	1.042	1.060
c =	.7	S	.785	.777	.786	.773	.750
		N	1.012	1.011	1.010	1.016	1.020
c =	.4	S	.849	.814	.834	.860	.864
		N	1.004	1.007	1.005	1.004	1.003

Table 3. ARE of the paired Prentice-Wilcoxon (W) and Logrank (L) tests with respect to the signed Wilcoxon test.

α			0.0	1.0	.5	-.5	-1.0
			Uncensored data				
		W	1.125	1.141	1.114	1.154	1.188
		L	1.500	1.352	1.415	1.596	1.690
			Exponential censoring with scale λ				
$\lambda =$	.05	W	1.103	1.107	1.089	1.133	1.168
		L	1.447	1.302	1.365	1.539	1.629
$\lambda =$	.2	W	1.049	1.031	1.028	1.081	1.118
		L	1.320	1.186	1.246	1.401	1.483
$\lambda =$	.5	W	.978	.940	.952	1.011	1.046
		L	1.157	1.040	1.095	1.223	1.289
$\lambda =$	1.4	W	.892	.845	.867	.918	.943
		L	.961	.883	.921	1.001	1.038
$\lambda =$	3.0	W	.862	.827	.844	.880	.895
		L	.882	.840	.861	.903	.922
			Fixed censoring at c				
c =	3.0	W	1.072	1.104	1.069	1.097	1.134
		L	1.365	1.346	1.337	1.414	1.475
c =	2.0	W	1.043	1.093	1.049	1.059	1.088
		L	1.245	1.465	1.309	1.225	1.232
c =	1.25	W	1.024	1.091	1.041	1.025	1.037
		L	1.129	1.555	1.264	1.061	1.025
c =	.7	W	1.011	1.071	1.031	1.003	1.000
		L	1.049	1.319	1.144	.992	.971
c =	.4	W	1.004	1.041	1.017	.997	.993
		L	1.017	1.120	1.057	.990	.955