Further Results on Exponential Functionals of Brownian Motion

By

Laboratoire de Probabilités
Université Paris VI
4, Place Jussieu
Tour 56 - 3eme Etage
75252 Paris Cedex 05

Technical Report No. 366
August 1992

Department of Statistics
University of California
Berkeley, California 94720

June 1992

Further results on exponential functionals of Brownian motion.

Marc Yor

Laboratoire de Probabilités - Université Paris VI - 4, Place Jussieu - Tour 56 - 3ème Etage - 75252 PARIS CEDEX 05

1. Introduction.

Let $(B_t, t \geq 0)$ denote real-valued Brownian motion starting from 0, and let ν be a real.

The so-called geometric Brownian motion with drift ν, which is defined as :

$$\exp(B_t + \nu t), \ t \geq 0,$$

is often taken as the main stochastic model in Mathematical finance. This process may be represented as :

$$\exp(B_t + \nu t) = R^{(\nu)}_{A^{(\nu)}_t} \ , \quad t \geq 0, \tag{1.a}$$

with :

$$A^{(\nu)}_t = \int_0^t ds \exp 2(B_s + \nu s), \tag{1.b}$$

and $(R^{(\nu)}_u, u \geq 0)$ denotes a Bessel process with index ν, starting from 1. Both for theoretical reasons, and practical purposes, the law of the process $(A^{(\nu)}_t, t \geq 0)$, taken at various (possibly random) times has been of some interest in recent years.

In a previous paper [10], the following result was obtained :

let T_λ be an exponential variable, with parameter λ, which is independent of $(B_t, t \geq 0)$; then, we have :

$$P\left(\exp(B^{(\nu)}_{T_\lambda}) \in d\rho,\ A^{(\nu)}_{T_\lambda} \in du\right) = \frac{\lambda}{2\rho^{2+\mu-\nu}}\, p^{\mu}_{u}(1,\rho)d\rho \tag{1.c}$$

where $\mu = \sqrt{2\lambda+\nu^2}$, and $p^{\mu}_{u}(a,\rho)d\rho$ is the semi-group, taken at time u, of the Bessel process $(R^{(\mu)}_u, u \geq 0)$ starting from a.

A number of results about Bessel processes are presented in [10], among which the following formula for $p^{\mu}_{u}(a,\rho)$:

$$p^{\mu}_{u}(a,\rho) = \left(\frac{\rho}{a}\right)^{\mu} \frac{\rho}{u} \exp\left(-\frac{a^2+\rho^2}{2u}\right) I_{\mu}\left(\frac{a\rho}{u}\right), \tag{1.d}$$

where I_μ denotes the modified Bessel function of index μ.

Using formula *(1.d)* , together with the (implicit) Laplace transform in λ presented in *(1.c)*, it is possible to obtain an expression for the joint law of $(\exp(B^{(\nu)}_t), A^{(\nu)}_t)$, for a fixed time t. Indeed, taking up the notation in Section 6 of [10], we define $a_t(x,u)$ as :

$$P(A^{(\nu)}_t \in du \,|\, B^{(\nu)}_t = x) = a_t(x,u)du \tag{1.e}$$

(as explained in [10], a does not depend on ν). Then, we have :

$$\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) a_t(x,u) = \frac{1}{u} \exp\left(-\frac{1}{2u}(1+e^{2x})\right)\theta_{e^x/u}(t) \tag{1.f}$$

where $\theta_r(u)$ is characterized by the formula :

$$I_{|\nu|}(r) = \int_0^\infty \exp\left(-\frac{\nu^2 u}{2}\right) \theta_r(u)du \qquad (\nu \in \mathbb{R},\ r > 0). \tag{1.g}$$

An integral formula for $\theta_r(u)$ is presented in ([10], formula *(6.b)*), which in turn leads to an integral formula for the joint law of $(\exp(B^{(\nu)}_t), A^{(\nu)}_t)$ (see [10], formula *(6.e)*). This result has just been used by Kawazu-Tanaka [6] to obtain some asymptotics for certain diffusions in random media. The same result may also be used to give an expression of the important quantity :

(1.h) $$E[(A_t^{(\nu)}-k)^+]$$

which governs the so-called financial Asian options.
However, it appears that the expression thus obtained for *(1.h)* is too difficult for computational purposes and, at this point, it seems better to consider again the randomized functional : $A_{T_\lambda}^{(\nu)}$, which has a remarkably simple distribution, presented in Theorem 1 below. Consequently, the expression :

$$E[(A_{T_\lambda}^{(\nu)}-k)^+] = \lambda \int_0^\infty dt\ e^{-\lambda t}\ E[(A_t^{(\nu)}-k)^+]$$

has also a simple form (see Corollary 1.3 below) which should doubtless be possible to use for practical purposes.

Theorem 1 : *Let* $\lambda > 0$ *; define* $\mu = \sqrt{2\lambda+\nu^2}$.

Then, if T_λ *denotes a random time, which is exponentially distributed, with parameter* λ, *and independent of* B, *one has :*

(1.i) $$A_{T_\lambda}^{(\nu)} \overset{(law)}{=} \frac{Z_{1,a}}{2Z_b}\ ,\ \text{where}\quad a = \frac{\mu+\nu}{2}\ ,\quad b = \frac{\mu-\nu}{2}$$

and $Z_{\alpha,\beta}$ *, resp.* Z_γ *, denotes a beta variable with parameters* (α,β)*, resp. with parameter* γ*, that is :*

$$P(Z_{\alpha,\beta} \in du) = \frac{u^{\alpha-1}(1-u)^{\beta-1}du}{B(\alpha,\beta)}\ ;\quad P(Z_\gamma \in du) = \frac{e^{-u}u^{\gamma-1}du}{\Gamma(\gamma)}\ ,$$

and $Z_{1,a}$ *and* Z_b *are further assumed to be independent.*

We now illustrate Theorem 1 with the two following Corollaries.

Corollary 1.1 : *Let* T *be an exponentially distributed random variable, with parameter* 1, *i.e.* E(T) = 1, *which is assumed to be independent of the Brownian motion* $(B_t, t \geq 0)$.

Then, considering successively, in the statement of Theorem 1, the values $\nu = 0$, $\nu = 1/2$ *and* $\nu = -1/2$, *we obtain :*

$$\int_0^{T/2} ds \ \exp(B_s) \overset{(law)}{=} \frac{U}{4T} \quad ; \quad \int_0^T ds \ \exp(2B_s + s) \overset{(law)}{=} U\sigma \ ;$$

(1.j)

$$\int_0^T ds \ \exp(2B_s - s) \overset{(law)}{=} \frac{1-U^2}{2T} \ ,$$

where U *is a uniform random variable valued in* [0,1], $\sigma = \inf\{t : B_t = 1\}$, *and the random variables which are featured on the right-hand sides in each of the three identities in law are assumed to be independent.*

Corollary 1.2 : *Let* $\nu > 0$. *Then, one has :*

$$\text{(1.k)} \qquad \int_0^\infty ds \ \exp 2(B_s - \nu s) \overset{(law)}{=} \frac{1}{2Z_\nu} \ .$$

The identity in law *(1.k)* is easily deduced from *(1.i)*, in which we let λ converge to 0 ; *(1.k)* was obtained first by D. Dufresne [3], and then reproved in [9], where the connection with last passage times for Bessel processes is established.

As announced before stating Theorem 1, we now present an expression of the Laplace transform (with respect to the variable t) of the quantity *(1.h)*.

Corollary 1.3 : *For every* $\nu \geq 0$, $\lambda > 2(1+\nu)$, *and* $k \geq 0$, *one has :*

$$\text{(1.}\ell\text{)} \qquad \lambda \int_0^\infty dt \ e^{-\lambda t} \ E[(A_t^{(\nu)} - k)^+] = \frac{\displaystyle\int_0^{1/2k} dt \ e^{-t} \ t^{\frac{\mu-\nu}{2} - 2} (1-2kt)^{\frac{\mu+\nu}{2} + 1}}{(\lambda - 2(1+\nu)) \ \Gamma(\frac{\mu-\nu}{2} - 1)}$$

It has been pointed out to the author that the actual quantity of interest for financial Asian options is not so much *(1.h)*, but :

$$E[(\frac{1}{t} A_t^{(\nu)} - k)^+] \equiv \frac{1}{t} E[(A_t^{(\nu)}-kt)^+]. \qquad (1.h')$$

Of course, if an explicit, simple expression for *(1.h)* were available, then, it would suffice to replace in such an expression the constant k by (kt), and then divide the obtained quantity by t. Since, despite *(1.f)* and *(1.g)*, no such simple quantity has been obtained, we shall show, in section 7, that, at the cost of a further Laplace transform (with respect to the variable k) an expression for *(1.h')* may be obtained.

(1.3) We now give the details of the organization of the present paper :

- in section 2, some prerequisites about Bessel processes, which complete those presented in [10], are given ;

- in section 3, two proofs of Theorem 1 are given, together with a partial explanation of the identity in law *(1.i)* ; furthermore, the arguments of the proofs of Theorem 1 lead to some other identities in law, presented at the end of section 3 ;

- in section 4, we obtain more identities in law, after first reproving Bougerol's identity in law :

$$\text{for fixed } t \geq 0, \quad \sinh(B_t) \stackrel{(law)}{=} \gamma_{A_t}, \qquad (1.m)$$

where $(\gamma_u, u \geq 0)$ is a Brownian motion starting from 0, which is assumed to be independent of B, and $A_t \equiv A_t^{(0)}$.

We already gave a (computational) proof of *(1.m)* in [10] , using some classical integral formulae for Bessel functions ; here, the proof of *(1.m)* relies upon our previous identity in law *(1.i)*.

Then, with the help of *(1.m)*, we are able to obtain the laws of variables of the form : A_S , where S varies amongst a family of random variables which are independent of B, and have some particular distributions ;

- in section 5, we use the conformal invariance of planar Brownian motion, and, more generally, the skew-product representation of Brownian motion in $\mathbb{R}^n$ $(n \geq 2)$ in order to obtain some more identities in law for variables of the form $A_S^{(\nu)}$, as we just described.

- in section 6, we present some generalizations of the results obtained in the previous sections ; indeed, both the Brownian motion with drift $(B_t+\nu t, t \geq 0)$ and the exponential function may be replaced by, respectively, some adequate diffusion, resp : function ; again, such generalizations may have some important applications in Mathematical finance, when the archetype model of geometric Brownian motion is replaced by some other models ;

- in section 7, finally, we look at some computational issues which arise in the study of Asian options.

(1.4) Some of the results discussed in this paper have been presented, without proof, in [11], whereas a detailed discussion of the implications in Mathematical finance is being made in Geman-Yor [4].

2. Some complements on Bessel processes.

(2.1) For the clarity of the exposition below, we need to take up the main part of the discussion on Bessel processes presented in Section 2 of [10], and to add some complements which will be used below.

Let Q_x^δ denote the law, on $C(\mathbb{R}_+,\mathbb{R}_+)$, of the square, starting from x, of a Bessel process with dimension δ ; one of the main properties of the family $(Q_x^\delta\ ;\ \delta > 0, x \geq 0)$ is the additivity property :

$$(2.a) \qquad Q_{x+x'}^{\delta+\delta'} = Q_x^\delta * Q_{x'}^{\delta'} \qquad (\delta,\delta',x,x' \geq 0).$$

From this additivity property, we deduce the following important consequence : fix $t \geq 0$, and $u \geq 0$; then, the function of (x,δ) :

$$q(x,\delta) \overset{\text{def}}{=} Q_x^\delta(\exp - tX_u)$$

is multiplicative in both arguments, i.e. :

$$q(x+x',\delta+\delta') = q(x,\delta)\, q(x',\delta') \qquad (x,x' \geq 0 \; ; \; \delta,\delta' \geq 0).$$

It is then easy to obtain :

$$Q_x^{\delta}(\exp(-tX_u)) = (1+2tu)^{-\delta/2} \exp\left(-x \frac{t}{1+2tu}\right) \tag{2.b}$$

by computing first this quantity for $\delta = 1$, say.

(2.2) The following remarks about the absolute continuity relationship :

$$P_a^{\nu}|_{\mathcal{R}_t \cap (t<T_o)} = \left(\frac{R_t}{a}\right)^{\nu} \exp\left(-\frac{\nu^2}{2}\int_0^t \frac{ds}{R_s^2}\right) \cdot P_a^{o}|_{\mathcal{R}_t} \tag{2.c}$$

which is valid for $a > 0$, and $\nu \in \mathbb{R}$, will also play an important role in the sequel. Here, in agreement with the notation in [10], section 2, P_a^{ν} denotes the law on $C(\mathbb{R}_+,\mathbb{R}_+)$ of the Bessel process with index ν, starting from a, $(R_t)_{t\geq 0}$ is the process of coordinates, $\mathcal{R}_t = \sigma(R_s, s \leq t)$, and $T_o = \inf\{t : R_t = 0\}$ $(\leq \infty)$.

As a consequence of *(2.c)*, we remark that, for $\nu > 0$, we have :

$$P_a^{-\nu}|_{\mathcal{R}_t \cap (t<T_o)} = \left(\frac{a}{R_t}\right)^{2\nu} \cdot P_a^{\nu}|_{\mathcal{R}_t}\ , \tag{2.d}$$

and, as an application, we derive the identity :

$$P_a^{-\nu}(T_o > t) = E_a^{\nu}\left(\left(\frac{a}{R_t}\right)^{2\nu}\right) \tag{2.e}$$

which demonstrates, if need be, that the $(P_a^{(\nu)},(\mathcal{R}_t)_{t\geq 0})$ local martingale : $\left(\left(\frac{1}{R_t}\right)^{2\nu}, t \geq 0\right)$ is not a martingale.

(2.3) It will also be interesting to consider the following time reversal result :

(2.f) *Under* $P_a^{-\nu}$ *, the process* $(R_{T_o - t}\ ;\ t \le T_o)$ *is distributed as* $(R_t\ ,\ t \le L_a)$ *under* P_a^{ν} *, where* $L_a = \sup\{t \ge 0 : R_t = a\}$.

Putting *(2.e)* and *(2.f)* together, we obtain the following

Proposition 1 : *The common distribution of* T_o *, under* $P_a^{-\nu}$ *, and of* L_a *, under* P_o^{ν} *, is that of :* $a^2/_{2Z_\nu}$.

Proof : i) The fact that T_o , under P_a^{ν} , and L_a , under P_o^{ν} , have the same law, follows from *(2.f)*.

ii) In order to show that this common distribution is that of $a^2/_{2Z_\nu}$, we now use the identity *(2.e)* and formula *(2.b)*.

Using the elementary formula :

$$\frac{1}{x^{\nu}} = \frac{1}{\Gamma(\nu)} \int_0^{\infty} ds\ \exp(-sx)\ s^{\nu-1},$$

we deduce from *(2.e)* that :

$$P_a^{\nu}(T_o > t) = a^{2\nu} \frac{1}{\Gamma(\nu)} \int_0^{\infty} ds\ s^{\nu-1}\ E_a^{\nu}(\exp - sR_t^2).$$

Then, applying formula *(2.b)* in the equivalent form :

$$E_a^{\nu}(\exp - sR_t^2) = \frac{1}{(1+2st)^{1+\nu}} \exp\left(- \frac{a^2 s}{1+2st}\right),$$

one obtains the result stated in the Proposition. □

Remark : Another proof of the statement in Proposition 1 is given in [9], together with references to previous papers by Getoor and Pitman-Yor.

3. Two proofs of Theorem 1, and a partial explanation.

(3.1) In order to obtain the law of $A^{(\nu)}_{T_\lambda}$, we start with the following chain of equalities, which is a consequence of the time change relation *(1.a)*:

$$P(A^{(\nu)}_{T_\lambda} \geq u) = \lambda \int_0^\infty dt\ e^{-\lambda t}\ P(A^{(\nu)}_t \geq u)$$

$$= \lambda \int_0^\infty dt\ e^{-\lambda t}\ P(H^{(\nu)}_u \leq t) = E[\exp(-\lambda\ H^{(\nu)}_u)],$$

where $H^{(\nu)}_u \overset{def}{=} \int_0^u \frac{ds}{(R^{(\nu)}_s)^2}$.

Using now the absolute continuity relationship *(2.c)*, we obtain :

$$E[\exp(-\lambda H^{(\nu)}_u)] = E^0_1[(R_u)^\nu \exp(-\frac{\mu^2}{2} H_u)] = E^\mu_1\left[\frac{1}{(R_u)^{\mu-\nu}}\right],$$

so that we have obtained the equality :

$$P(A^{(\nu)}_{T_\lambda} \geq u) = E^\mu_1\left[\frac{1}{(R_u)^{\mu-\nu}}\right]. \tag{3.a}$$

We now obtain an integral representation of the right-hand side of formula *(3.a)*.

Proposition 2 : *Let* $\mu \geq b \geq 0$. *Then, for every* $r \geq 0$, *one has :*

$$E^\mu_r\left(\frac{1}{(R_u)^{2b}}\right) = \frac{1}{\Gamma(b)} \int_0^{1/2u} dv\ \exp(-r^2 v)v^{b-1}(1-2uv)^{\mu-b}. \tag{3.b}$$

Proof : It uses the same arguments as those found in the proof of Proposition 1. Precisely, from the elementary formula :

$$\frac{1}{x^b} = \frac{1}{\Gamma(b)} \int_0^\infty dt\ \exp(-tx) t^{b-1},$$

one deduces :

$$E_r^\mu\left[\frac{1}{(R_u)^{2b}}\right] = \frac{1}{\Gamma(b)} \int_0^\infty dt\ t^{b-1}\ E_r^\mu(\exp -tR_u^2).$$

We now write formula *(2.b)* in the equivalent form :

$$E_r^\mu[\exp(-tR_u^2)] = \frac{1}{(1+2tu)^{\mu+1}} \exp\left(-\ r^2\ \frac{t}{1+2tu}\right),$$

and we finally obtain formula *(3.b)* by changing the variable t into :
$v = \dfrac{t}{1+2tu}$. □

Using formula *(3.b)*, we now remark that formula *(3.a)* may be written in the following form : using the notation $\hat{Z}_b$ for $1/_{2Z_b}$, one gets :

(3.c) $$P(A_{T_\lambda}^{(\nu)} \geq u) = E[\hat{Z}_b \geq u\ ;\ (1 - \frac{u}{\hat{Z}_b})^a].$$

Now, if we use the notation : X for $Z_{1,a}$, we have :

$$P(X \geq x) = (1-x)^a$$

and we obtain, assuming that X and $\hat{Z}_b$ are independent :

$$P(X\hat{Z}_b \geq u) = P(\hat{Z}_b \geq u\ ;\ X \geq \frac{u}{\hat{Z}_b}) = E[\hat{Z}_b \geq u\ ;\ (1 - \frac{u}{\hat{Z}_b})^a].$$

Hence, we deduce from *(3.c)* that :

(3.d) $$P(A_{T_\lambda}^{(\nu)} \geq u) = P(X\hat{Z}_b \geq u),$$

which proves the identity in law *(1.h)*.

(3.2) It is, in fact, possible to give another proof of the identity in law *(1.h)*, without using the explicit Laplace transform formula *(2.b)*. First, we remark that, from formula *(2.d)*, and formula *(3.a)*, we have :

$$P\left(A^{(\nu)}_{T_\lambda} \geq u\right) = E_1^{-\mu}\left(T_o \geq u \; ; \; \frac{1}{(R_u)^{2(b-\mu)}}\right). \tag{3.e}$$

On the other hand, thanks to Proposition 1, the right-hand side of formula *(3.c)* is equal to :

$$E_1^{-\mu}\left((T_o \geq u) \; ; \; (2(T_o-u))^{\mu-b} \frac{\Gamma(\mu)}{\Gamma(b)}\right). \tag{3.f}$$

Hence, since formula *(3.c)* is equivalent to the identity in law *(1.i)*, we must be able, in order to give a second proof of this identity in law, to pass directly from *(3.e)* to *(3.f)*, that is : to prove the following identity :

$$E_1^{-\mu}\left((T_o \geq u) \; ; \; (R_u)^{2(\mu-b)}\right) = E_1^{-\mu}\left(T_o \geq u) \; ; \; (2(T_o-u))^{\mu-b} \frac{\Gamma(\mu)}{\Gamma(b)}\right). \tag{3.g}$$

Indeed, the right-hand side of *(3.g)* is, thanks to the strong Markov property, equal to :

$$E_1^{-\mu}\left((T_o \geq u) \; ; \; E_{R_u}^{-\mu}\left((2T_o)^{\mu-b}\right) \frac{\Gamma(\mu)}{\Gamma(b)}\right)$$

and this latter expression is equal to the left-hand side of *(3.g)* since, from Proposition 1, we deduce :

$$E_r^{-\mu}\left((2T_o)^{\mu-b}\right) \frac{\Gamma(\mu)}{\Gamma(b)} = r^{2(\mu-b)}. \qquad \square$$

(3.3) In order to illustrate the method of time-change and change of probability which we used in subsection **(3.1)**, we now prove two other identities in law

Proposition 3 : *We keep the notation :* $a = \frac{\mu+\nu}{2}$ *and* $b = \frac{\mu-\nu}{2}$. *Then, we have :*

1) for $r \geq 1$, $P\left(\sup_{u \leq T_\lambda} \exp(B_u+\nu u) \geq r\right) = \frac{1}{r^{2b}}$, *i.e.* $\sup_{u \leq T_\lambda} \exp(B_u+\nu u) \overset{(law)}{=} \frac{1}{Z_{2b,1}}$

2) *for* $\rho \le 1$, $P\left(\inf_{u\le T_\lambda} \exp(B_u+\nu u) \le \rho\right) = \rho^{2a}$, *i.e.* $\inf_{u\le T_\lambda} \exp(B_u+\nu u) \overset{(law)}{=} Z_{2a,1}$.

Proof : 1) From *(1.a)*, we have :

$$\sup_{u\le T_\lambda} (\exp(B_u+\nu u)) = \sup_{s\le A^{(\nu)}_{T_\lambda}} (R^{(\nu)}_s),$$

hence :

$$P\left\{\sup_{u\le T_\lambda} \exp(B_u+\nu u) \ge r\right\} = \lambda \int_0^\infty dt\ e^{-\lambda t}\ P\left\{\sup_{s\le A^{(\nu)}_t} (R^{(\nu)}_s) \ge r\right\}$$

$$= \lambda \int_0^\infty dt\ e^{-\lambda t}\ P\{A^{(\nu)}_t \ge T^{(\nu)}_r\},\ \text{where}\ \ T^{(\nu)}_r = \inf\{u : R^{(\nu)}_u = r\}$$

$$= \lambda \int_0^\infty dt\ e^{-\lambda t}\ P\left\{t \ge H^{(\nu)}_{T^{(\nu)}_r}\right\} = E\left[\exp(-\lambda\ H^{(\nu)}_{T^{(\nu)}_r})\right] = \frac{1}{r^{\mu-\nu}} = \frac{1}{r^{2b}}\ ,$$

from the absolute continuity relation *(2.c)*.

2) Similarly, we have :

$$P\left\{\inf_{u\le T_\lambda} \exp(B_u+\nu u) \le \rho\right\} = \lambda \int_0^\infty dt\ e^{-\lambda t}\ P\left\{\inf_{s\le A^{(\nu)}_t} (R^{(\nu)}_s) \ge \rho\right\}$$

$$= \lambda \int_0^\infty dt\ e^{-\lambda t}\ P\{A^{(\nu)}_t \ge T^{(\nu)}_\rho\} = \lambda \int_0^\infty dt\ e^{-\lambda t}\ P\{t \le H^{(\nu)}_{T_\rho}\ ;\ T_\rho < \infty\}$$

$$= E\left[\exp(-\lambda H^{(\nu)}_{T_\rho})\ ;\ T_\rho < \infty\right] = \frac{1}{\rho^{2b}}\ P^{\mu}_1(T_\rho < \infty),$$

from the absolute continuity relationship *(2.c)*.

Now, it is well-known that, since $\left(\frac{1}{(R_t)^{2\mu}}, t \geq 0\right)$ is a local martingale under P_1^{μ}, which converges to 0, as $t \longrightarrow \infty$, then :

$$\sup_{t\geq 0} \frac{1}{(R_t)^{2\mu}} \overset{(law)}{=} \frac{1}{U},$$

with U a uniformly distributed random variables on the interval [0,1] ; therefore, one has :

$$P_1^{\mu}(T_\rho < \infty) = P_1^{\mu}\left(\sup_{t\geq 0} \frac{1}{R_t^{2\mu}} \geq \frac{1}{\rho^{2\mu}}\right) = \rho^{2\mu},$$

and finally :

$$P\left\{\inf_{u\leq T_\lambda} \exp(B_u+\nu u) \leq \rho\right\} = \frac{1}{\rho^{2b}} \rho^{2\mu} = \rho^{2a}. \qquad \square$$

(3.4) In order to obtain a better understanting of the "factorization identity" *(1.i)*, we now relate the law of $A_{T_\lambda}^{(\nu)}$ to that of the future supremum of a certain Bessel process with negative index.

Proposition 4 : *Let* $\mu = (2\lambda+\nu^2)^{1/2}$, *and* $a = \frac{\mu+\nu}{2}$, $b = \frac{\mu-\nu}{2}$. *Then, if we denote :* $M = \sup_{v\leq T_o} R_v$, *we have :*

1) $$P_r^{-\mu}(M \in dx) = (2\mu) \frac{r^{2\mu}\, dx}{x^{2\mu+1}} 1_{(x\geq r)},$$

and, consequently :

$$E_r^{-\mu}[M^{2a}] = \frac{\mu}{b} r^{2a} ;$$

2) if we denote : $M_u = \sup_{u\leq v\leq T_o} R_v$, *for* $u \geq 0$, *then, we have :*

(3.h) $$P(A_{T_\lambda}^{(\nu)} \geq \mu) = \frac{E_1^{-\mu}[(M_u)^{2a}]}{E_1^{-\mu}[M_u^{2a}]}$$

3) if, for $x > 0$*, we define :* $L_x = \sup\{u : R_u = x\}$*, then, we have, for* $u \geq 0$ *:*

$$\text{(3.i)} \qquad P(A^{(\nu)}_{T_\lambda} \geq \mu) = \frac{E_1^{-\mu}[M_u^{2a} \; ; \; L_{ZM} \geq u]}{E_1^{-\mu}[M_u^{2a}]} ,$$

where Z *is independent of* $(R_u \geq 0)$*, and is a beta* $(2a,1)$ *random variable, i.e.* $P(Z \in dx) = (2a)x^{2a-1}\,dx \qquad (0 < x < 1)$.

(3.5) Although the proof given in **(3.2)** is certainly much more illuminating than the proof given in **(3.1)**, we have not succeeded to obtain a more satisfactory explanation of the "factorization identity" *(1.i)*.

Ideally, one would like to find out, for each λ and ν, two independent random variables N and D, which are measurable with respect to $\sigma\{(B_t, t \geq 0) \; ; \; T_\lambda\}$, and such that :

$$\text{(3.j)} \qquad A^{(\nu)}_{T_\lambda} = \frac{N}{D} , \quad N \overset{(law)}{=} Z_{1,a} \quad , \quad D \overset{(law)}{=} 2Z_b .$$

It is not clear at all that this program may be fulfilled, and we explain why :

4. Some applications of Theorem 1.

(4.1) Theorem 1 allows to obtain a quick proof of Bougerol's identity in law *(1.m)*, which may be more natural than the proof given in [10].

Theorem 2 : (Bougerol [1]) *Let* $(B_t, t \geq 0)$ *be a real-valued Brownian motion, starting from* 0*, and define* $A_t = \int_0^t ds \exp(2B_s)$, $t \geq 0$*. Then, we have :*

$$\text{(4.a)} \qquad \textit{for every fixed } t \geq 0 : \quad \sinh(B_t) \overset{(law)}{=} \gamma_{A_t} .$$

Proof : Let $\theta > 0$, and $\lambda = \frac{\theta^2}{2}$. In order to prove *(4.a)*, it suffices, from the injectivity of Laplace transform, to prove the equality :

$$(4.b) \qquad E\left[\left|\sinh(B_{T_\lambda})\right|^\alpha\right] = E\left[\left|\gamma_{T_\lambda}\right|^\alpha\right],$$

for all sufficiently small α's.

It is well-known that $|B_{T_\lambda}|$ is an exponential random variable, with parameter θ. Hence, the left-hand side of *(4.b)* is equal to :

$$(4.c) \qquad \theta \int_0^\infty dx \, \exp(-\theta x)(\sinh x)^\alpha,$$

whereas the right-hand side of *(4.b)* is equal to :

$$(4.d) \qquad E(|N|^\alpha) \; E\left((A_{T_\lambda})^{\alpha/2}\right)$$

where N denotes a gaussian random variable, which is centered, and has variance 1.

Using jointly the duplication formula for the gamma function and the identity in law *(1.i)* for $\nu = 0$, it is easily shown that both quantities *(4.c)* and *(4.d)* have the common value :

$$\frac{1}{2^\alpha} B\left(\frac{\theta-\alpha}{2}, \alpha+1\right),$$

which proves *(4.b)*. □

We have not been able to find an adequate extension of the identity in law *(4.a)* for $\nu \neq 0$, which would relate, say, the law of $B_t^{(\nu)}$ to that of $A_t^{(\nu)}$, for fixed t.

However, we have the following weaker relation

Corollary 2.1 : *Let* $\nu > 0$. *Using the notation introduced in Theorem 1 and Theorem 2.1, we have the following identity in law :*

$$(4.e) \qquad \gamma_{A^{(\nu)}_{T_\lambda}} \overset{(law)}{=} \frac{1}{\sqrt{Z_{b,\nu}}} \sinh\left(B_{T_{\lambda+\frac{\nu^2}{2}}}\right),$$

where, on the right-hand side, $Z_{b,\nu}$ *denotes a beta variable, with parameters* (b,ν), *which is assumed to be independent of the pair* $\left\{(B_t\ ;\ t \geq 0)\ ;\ T_{\lambda+\frac{\nu^2}{2}}\right\}$

Proof : We remark that, from the algebraic relation between beta and gamma variables, we have :

$$(4.f) \qquad Z_b \overset{(law)}{=} Z_{b,\nu}\, Z_a\ ,$$

where, on the right-hand side of *(4.f)*, the variables $Z_{b,\nu}$ and Z_a are assumed to be independent.

Now, using the identity in law *(1.i)*, we obtain :

$$A^{(\nu)}_{T_\lambda} \overset{(law)}{=} \frac{1}{Z_{b,\nu}} A_{T_{\lambda+\frac{\nu^2}{2}}}\ ,$$

so that, from the scaling property of Brownian motion, we deduce :

$$\gamma_{A^{(\nu)}_{T_\lambda}} \overset{(law)}{=} \frac{1}{\sqrt{Z_{b,\nu}}} \gamma_{A_{T_{\lambda+\frac{\nu^2}{2}}}} \overset{(law)}{=} \frac{1}{\sqrt{Z_{b,\nu}}} \sinh\left(B_{T_{\lambda+\frac{\nu^2}{2}}}\right), \quad \text{from } (4.a) \quad \square$$

(4.2) We shall now exploit Bougerol's identity in law *(4.a)* in order to describe the laws of variables of the form A_T , when T varies amongst a fairly large class of random variables, assumed to be independent of $(B_t, t \geq 0)$. To do this, our main tool will be the following elementary

Lemma : *Let* T *be a strictly positive r.v., and* g *be the density of* $\sinh(B_T) \overset{(law)}{=} \gamma_{A_T}$. *Then, the law of* B_T *is given by :*

$$P(B_T \in dy) = dy(\cosh y)\, g(\sinh y).$$

Now, we write down a Table, in which the main examples of distributions for T which we have found to be tractable are presented.

$P(B_T \in dx)/dx$	$\frac{\theta}{2} \exp(-\theta\|x\|)$	$\frac{c_\alpha}{(\cosh(x))^\alpha}$	$\frac{a \cosh x}{\pi(a^2+\sinh^2 x)}$	$\frac{x \coth x - 1}{(\sinh x)^2}$
A_T	$\frac{Z_{1,\theta/2}}{2Z_{\theta/2}}$	$\frac{1}{2Z_{\alpha/2}}$	σ_a	$\frac{1}{2UZ_1}$
T	$T_{\theta^2/2}$	$T^{(\alpha)}$	S_α	$T^{(3)}_{\pi/2} + \tilde{T}^{(3)}_{\pi/2}$

We now explain the Table, column after column :

- first column : this is simply a translation of formula *(1.i)*, in the particular case $\nu = 0$, and $\lambda = \frac{\theta^2}{2}$;

- second column : here, α denotes any strictly positive real, and c_α is the normalizing constant which makes $\frac{c_\alpha}{(\cosh x)^\alpha}$ a density of probability on $\mathbb{R}$.

We find : $c_\alpha = \frac{\Gamma(\frac{\alpha+1}{2})}{\sqrt{\pi}\, \Gamma(\frac{\alpha}{2})}$; the random variable $T^{(\alpha)}$ satisfies :

$$E\left[\exp(-\frac{\lambda^2}{2} T^{(\alpha)})\right] = c_\alpha \int_{-\infty}^{\infty} dx \exp(i\lambda x) \frac{1}{(\cosh)^\alpha} ,$$

and we find :

$$E\left[\exp\left(-\frac{\lambda^2}{2}T^{(\alpha)}\right)\right] = \left|\frac{\Gamma(\frac{\alpha+i\lambda}{2})}{\Gamma(\frac{\alpha}{2})}\right|^2 .$$

The cases $\alpha = 1$ and $\alpha = 2$ are particularly interesting ; for $\alpha = 1$, $T^{(1)}$ may be represented as the first hitting time of $\pi/_2$ by a reflecting Brownian motion starting from 0 ;

for $\alpha = 2$, $T^{(2)}$ may be represented as the first hitting time of $\pi/_2$ by a 3-dimensional Bessel process starting from 0.

- third column : this anticipates upon the discussion in section 5, where the notation for S_α and σ_α are presented ; the case $a = 1$ corresponds to $\alpha = 1$ in the second column.

- fourth column : $T^{(3)}_{\pi/_2}$ and $T^{(3)}_{\pi/_2}$ are two independent copies of the first hitting time of $\pi/_2$ by BES(3), using the notation already introduced in the explanation of the second column.

5. Some applications of the conformal invariance and skew-product representation of Brownian motion in $\mathbb{R}^n$, $n \geq 2$.

(5.1) Let $Z_t = X_t + iY_t$, $t \geq 0$, be a complex valued Brownian motion, i.e. : $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ are two independent real-valued Brownian motions.

P. Lévy [7] remarked that, as a consequence of the conformal invariance of the distribution of Z, if $f : \mathbb{C} \longrightarrow \mathbb{C}$ is an entire fucntion, i.e. f is holomorphic on the whole complex plane $\mathbb{C}$, and f is not constant, then there exists another complex Brownian motion $(\hat{Z}_u, u \geq 0)$ such that :

(5.a) $$f(Z_t) = \hat{Z}\left(\int_0^t ds\ |f'(Z_s)|^2\right), \quad t \geq 0.$$

Several important consequences of this result of Lévy have now been obtained ; see, for example, B. Davis [2], and in another direction, Pitman-Yor [8] who are concerned with the level crossings of a Cauchy process.
In the particular case where $f(z) = \exp(z)$, the equality *(5.a)* becomes :

$$\exp(Z_t) = \hat{Z}\left(\int_0^t ds\, \exp(2X_s)\right), \qquad t \geq 0, \tag{5.b}$$

from which we deduce the following identities in law.

Theorem 3 : *Let* $(B_t, t \geq 0)$ *be a real valued Brownian motion, starting from* 0 *; define, for* $a > 0$, $\sigma_a = inf\{t : B_t = a\}$.

Let $Z_t = X_t + iY_t$, $t \geq 0$, *be a complex Brownian motikon, starting from* 0,

and define $A_t = \int_0^t ds\, \exp(2X_s)$, $t \geq 0$.

1) If $S = inf\{t : |Y_t| = \frac{\pi}{2}\}$, *then, we have :*

$$A_S \overset{(law)}{=} \sigma_1 \; ; \tag{5.c}$$

2) More generally, to any $\alpha \in \mathbb{R}$, *we associate :* $a = (1+\alpha^2)^{-1/2}$ *, and* θ *the unique real in* $]-\frac{\pi}{2},\frac{\pi}{2}]$ *such that :* $tg(\theta) = \frac{1}{\alpha}$.

Then, if $S_\alpha = inf\{t : cos(Y_t) = \alpha\, sin(Y_t)\} \equiv inf\{Y_t = \theta, \text{ or } \theta - \pi\}$, *we have :*

$$A_{S_\alpha} = inf\{t : \hat{X}_t - \alpha\hat{Y}_t = 0\} \overset{(law)}{=} \sigma_a \equiv inf\{t : B_t = a\}. \tag{5.d}$$

The proof follows immediately from the representation *(5.b)*, and the elementary formula : $\exp(z) = \exp(x)(\cos y + i \sin y)$. □

(5.2) We now consider, more generally, $(Z_t, t \geq 0)$ a Brownian motion valued in $\mathbb{R}^n$, $n \geq 2$, and starting from $z_o \neq 0$. For simplicity, we shall assume that $|z_o| = 1$.

As is now well-known (see Itô - Mc Kean [5], p. 270 and sq.), $(Z_t, t \geq 0)$ may be represented in the skew-product form :

(5.e) $$Z_t = |Z_t| \, V(H_t), \quad t \geq 0,$$

where $H_t = \int_0^t \frac{ds}{|Z_s|^2}$, $t \geq 0$, and $(V(u), u \geq 0)$ is a standard Brownian motion on S_{n-1} the unit sphere in $\mathbb{R}^n$, and V is independent of $(|Z_t|, t \geq 0)$.

We may also represent the radial part of Z, i.e. $R_t = |Z_t|$, as :

(5.f) $$R_t = \exp(B_u + \nu u)\Big|_{u=H_t}, \quad t \geq 0,$$

where $(B_u, u \geq 0)$ is a real-valued Brownian motion, starting from 0.
This latter representation *(5.f)* is nothing else but the representation *(1.a)* we started with, once we have remarked that :

(5.g) $$H_t = \inf\left\{u : A_u^{(\nu)} \equiv \int_0^u ds \exp 2(B_s + \nu s) > t\right\}.$$

We now replace in *(5.e)* the time variable t by $A_u^{(\nu)}$, which gives, thanks to *(5.f)* :

(5.h) $$\exp(B_u + \nu u) \, V(u) = Z_{A_u^{(\nu)}}, \quad u \geq 0.$$

On the left-hand side of *(5.h)*, the processes $(B_u \, ; \, u \geq 0)$ and $(V(u) \, ; \, u \geq 0)$ are independent, whereas, on the right-hand side of *(5.h)*, $A_u^{(\nu)}$ is measurable with respect to $(|Z_u|, u \geq 0)$, as it follows from *(5.g)*.
We now have the following extension of Theorem 3.

Theorem 4 : *Let* $\theta \in \mathbb{R}^n$, $|\theta| = 1$, *and define :*

$$S_\theta = \inf\{u : (\theta, V(u)) = 0\}.$$

Then, we have :

$$A^{(\nu)}_{S_\theta} \overset{(law)}{=} \sigma_{a_\theta} ,$$

where $a_\theta = (\theta, z_o)$ *, and* σ_b *has the same meaning as in Theorem 3.*

The proof is just as immediate as that of Theorem 3, using the fact that $\{(\theta, Z_u), u \geq 0\}$ is a one-dimensional Brownian motion, starting from a_θ. A natural question is now to find out the distribution of S_θ .

6. An extension to some diffusions.

7. Some computational issues.

(7.1) From the applied point of view, formula

$$(1.\ell) \qquad \lambda \int_0^\infty dt\, e^{-\lambda t}\, E[(A^{(\nu)}_t - k)^+] = \frac{\int_0^{1/2k} e^{-t}\, t^{\frac{\mu-\nu}{2} - 2} (1-2kt)^{\frac{\mu-\nu}{2}+1}}{(\lambda - 2(1+\nu))\, \Gamma(\frac{\mu-\nu}{2} - 1)}$$

is not completely satisfactory ; one would like to invent the Laplace transform in λ, which would give on expression of

$$(1.h) \qquad E[(A^{(\nu)}_t - k)^+].$$

In order to do this, we may divide both sides of *(1.ℓ)* by λ, and then inspect ther (new) right-hand side of *(1.ℓ)*, call it : $r(\lambda)$; we can write :

$$(7.a) \qquad r(\lambda) = \frac{1}{\lambda(\lambda - 2(1+\nu))} \int_0^1 \frac{du}{u}\, e^{-u/2k} \left\{ \frac{\left(\frac{u}{2k}\right)^{\frac{\mu-\nu}{2} - 1}}{\Gamma(\frac{\mu-\nu}{2} - 1)} \right\} (1-u)^{\frac{\mu-\nu}{2} + 1}$$

(7.2) Another computational question, which also arises naturally in the applications of the results of the present paper to Mathematical finance is that of finding an explicit expression for the distribution of :

$$H_t^{(\nu)} = \int_0^t \frac{du}{(R_u^{(\nu)})^2} ,$$

where $(R_u^{(\nu)}, u \geq 0)$ is a Bessel process with index ν, starting from 1, say. We shall show below that this problem is closely related to the problem raised in **(7.1)**.

(7.3) Coming back to **(7.1)**, we recall that, as indicated at the beginning of this paper, it is the expression :

$$E[(\frac{1}{t} A_t^{(\nu)} - k)^+] \qquad \textit{(1.h')}$$

which is of interest, rather than *(1.h)*.

REFERENCES

[1] **Ph. Bougerol** : Exemples de théorèmes locaux sur les groupes résolubles. *Ann. I.H.P., vol. 19, n° 4, 1983, p. 369-391.*

[2] **B. Davis** : Brownian motion and analytic functions. *Ann. Prob. 7, 1979, p. 913-932.*

[3] **D. Dufresne** : The distribution of a perpetuity, with applications to risk theory and pension funding. *Scand. Actuarial J., 1990, p. 39-79.*

[4] H. Geman, M. Yor : Bessel processes, Asian options and perpetuities.
Submitted to Mathematical Finance (June 1992).

[5] K. Itô, H.P. Mc Kean : Diffusion processes and their sample paths.
Springer (1965).

[6] K. Kawazu, H. Tanaka : On the maximum of a diffusion process in a drifted Brownian environment.
To appear in Sém. Prob. XXVII, (1993).

[7] P. Lévy : Le mouvement brownien plan. *Amer. J. Math., 62, 1940, p. 487-550.*

[8] J. Pitman, M. Yor : Level crossings of a Cauchy process.
Ann. Prob., 14, n° 3, 1986, p. 780-792.

[9] M. Yor : Sur certaines fonctionnelles exponentielles du mouvement brownien réel.
J. Appl. Proba. 29, 1992, p. 202-208.

[10] M. Yor : On some exponential functionals of Brownian motion.
Adv. Appl. Prob. (September 1992).

[11] M. Yor : Sur les lois des fonctionnelles exponentielles du mouvement brownien, considérées en certains instants aléatoires.
Note aux Comptes Rendus Acad. Sci. Paris, Juin 1992, t. 314, Série I, p. 951-956.