**Proofs of Jacobi Triple Product from the perspectives of q-series and partition theorems.**

by

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**Abstract**

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These are expository notes on various proofs of Jacobi Triple Product Identity and its partition theoretic interpretation. When we further prove a polynomial analogue of Jacobi Triple Identity, this reveals the deeper interaction between q-hypergeometric series and partition of colored integers. Also, we discuss a simple proof using Frobenius partition and its other implications.

# Chapter 1

# Jacobi Triple Product

Jacobi Triple Product was introduced by Jacobi(1829) to transform his four theta functions into infinite products. It is known as one of the Macdonald identity, especially for the affine root system of $A_1$ and the Weyl denominator formula for the corresponding affine Kac-Moody algebra. For complex numbers $q$ with $|q| < 1$ and nonzero $z$, the identity is :

$$(q/z, z, q; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{\binom{n-1}{2}}, \tag{1.1}$$

where $(a_1, a_2, \cdots, a_r; q)_N = (a_1; q)_N (a_2; q)_N \cdots (a_r; q)_N$, and

$$(a; q)_N := (a)_N = \begin{cases} 0 & \text{if } N > 0 \\ \prod_{i=0}^{N-1}(1 - aq^i) & \text{otherwise} \end{cases}$$

with $(a)_\infty = \lim_{N\to\infty}(a)_N$ for $|q| < 1$. (1.1) can be written as other forms such as

$$(-qz, -qz^{-1}, q; q)_\infty = \sum_{n\geq 0} q^{T_n} \frac{z^{-n} + z^{1+n}}{1 + z}, \tag{1.2}$$

where $T_n = \frac{n(n+1)}{2}$, or if one replace $q$ by $q^2$ and $z$ by $zq$ in (1.2), then

$$(z^{-1}q, zq, q^2; q^2)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n^2}. \tag{1.3}$$

**Remark 1.** *C.G.J.Jacobi, in his Fundamenta Nova(1829), defined one theta function $\vartheta(\zeta, \tau)$ and its three other auxiliary functions which he transformed into infinite products by using the generalized version of Jacobi Triple Pdroduct:*

$$\vartheta(\zeta, \tau) = \sum_{n=-\infty}^{\infty} exp(\pi i n^2 \tau + 2\pi i n \zeta)$$

$$= 1 + 2\sum_{n=1}^{\infty}(e^{\pi i\tau})^{n^2}\cos(2n\pi\zeta),$$

$$\Rightarrow \quad \vartheta(z,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} z^n, \tag{1.4}$$

*where* $q = e^{\pi i\tau}$ *and* $\zeta = e^{2\pi\zeta}$ *in the last equality. The generalized version of Jacobi Triple Product is*

$$1 + \sum_{n=1}^{\infty}(-1)^n x^{an^2}(z^{bn} + z^{-bn}) \\ = \prod_{n=1}^{\infty}(1 - z^b x^{a(2n-1)})(1 - z^{-b}x^{a(2n-1)})(1 - x^{2an}), \tag{1.5}$$

*by which he transformed* (1.4) *into the following infinite products*

$$\vartheta(\zeta,\tau) = \prod_{n=1}^{\infty}(1 - e^{2n\pi i\tau})(1 + e^{(2n-1)\pi i\tau + 2\pi i\zeta})(1 + e^{(2n-1)\pi i\tau - 2\pi i\zeta}),$$

$$\Rightarrow \quad \vartheta(z,q) = \prod_{n=1}^{\infty}(1 - q^{2n})(1 + q^{2n-1}z^2)(1 + q^{2n-1}/z^2). \tag{1.6}$$

*Also, Jacobi Triple Product can be expressed in terms of Ramanujan's two variable theta function.*

$$f(a,b) = f(b,a) := \sum_{n=-infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \\ = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty \tag{1.7}$$

# Chapter 2

# Basic q-hypergeometric series and the proofs of Jacobi Triple Product

There are infinite numbers of different ways to prove Jacobi Triple Product using basic hypergeometric series (${}_r\phi_s$) and bilateral basic hypergeometric series (${}_r\psi_s$). In this chapter, only a few of the most interesting ones will be discussed. First, let us define ${}_r\phi_s$ and ${}_r\psi_s$.

**Def 1** (Basic q-hypergeometric series, and bilateral basic hypergeometric series)**.** *For $q, z$ such that each term of the series well-defined:*

$$\begin{aligned} {}_r\phi_s(a_1, \cdots a_r; b_1, \cdots, b_s; q, z) &\equiv {}_r\phi_s\left(\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, z\right) \\ &:= \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_n}{(q, b_1, \cdots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n, \quad (2.1) \\ {}_r\psi_s(a_1, \cdots a_r; b_1, \cdots, b_s; q, z) &\equiv {}_r\psi_s\left(\begin{matrix} a_1, a_2 \cdots, a_r \\ b_1, b_2 \cdots, b_s \end{matrix}; q, z\right) \\ &:= \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \cdots, a_r; q)_n}{(b_1, b_2, \cdots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{s-r} z^n. \end{aligned} \quad (2.2)$$

(2.2) *becomes*

$$= \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_n}{(b_1, \cdots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{s-r} z^n + \sum_{n=1}^{\infty} \frac{(q/b_1, \cdots, q/b_s; q)_n}{(q/a_1; q \cdots, q/a_r; q)_n} \left(\frac{b_1 \cdots b_s}{a_1 \cdots a_r z}\right)^n$$

*by applying* $(a; q)_{-n} := \frac{1}{(aq^{-n}; q)_n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a; q)_n}$.

## 2.1 q-binomial theorem, Euler's identities and Heine's transformation formula for ${}_2\phi_1$

Jacobi Triple Product can be derived from only using either Euler's identity or Heine's transformation formula, both of which utilize q-binomial theorem.

**Theorem 2** (q-binomial theorem by Cauchy [Andrews(1998)]).

$$ {}_1\phi_0\left(\begin{matrix} a \\ - \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a;q)_n z^n}{(q;q)_n} = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{2.3}$$

*Proof.* [Chan(2011)] For fixed $a$ and $q$ inside $|z| < 1$, let

$$\frac{(az;q)_\infty}{(z;q)_\infty} := F(z) = \sum_{n=0}^{\infty} A_n(a,q) z^n,$$

where $F(z)$ is uniformly convergent in $|z| \le \epsilon$, $\epsilon \in (0,1)$, and analytic inside $|z| < 1$. Now, observe $(1-z)F(z) = (1-az)F(zq)$. Therefore, $A_n$ can be obtained by comparing the coefficients of $z^n$ of both sides such that

$$\begin{aligned} A_n - A_{n-1} &= q^n A_n - aq^{n-1} A_{n-1}, \\ A_n &= \frac{1-aq^{n-1}}{1-q^n} A_{n-1} = \frac{(a;q)_n}{(q;q)_n} A_0, \end{aligned}$$

where $A_0 = F(0) = 1$. □

**Theorem 3** (Euler). *For $|z| < 1$, $|q| < 1$:*

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty} \tag{2.4}$$

$$\sum_{n=0}^{\infty} \frac{z^n q^{n(n-1)/2}}{(q;q)_n} = (-z;q)_\infty \tag{2.5}$$

*Proof.* In [Andrews(1998)], (2.4) follows from (2.3) by setting $a = 0$. To show (2.5), replace $a$ by $a/b$ and $z$ by $bz$ for $|bz| < 1$ in (2.3). Then,

$$\sum_{n=0}^{\infty} \frac{(b-a)(b-aq)\cdots(b-aq^{n-1})}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(bz;q)_\infty}.$$

With $b = 0$ and $a = -1$, derive (2.5). □

Now, let us prove Jacobi Triple Product (1.3) by using Theorem 3.

*Proof.* [Andrews(1998)] For $z \neq 0$, $|q| < 1$ and $|z| > |q|$:

$$\begin{aligned}
(zq;q^2)_\infty &= \sum_{m=0}^{\infty} \frac{z^m q^{m^2}}{(q^2;q^2)_m} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{m=-\infty}^{\infty} z^m q^{m^2} (q^{2m+2};q^2)_\infty, \qquad ((q^{2m+2};q^2)_\infty = 0, \text{ for } m \in \mathbb{Z}_{<0}), \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{m=-\infty}^{\infty} z^m q^{m^2} \sum_{r=0}^{\infty} \frac{(-1)^r q^{r^2+2mr+r}}{(q^2;q^2)_r} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{r=0}^{\infty} \frac{(-1)^r z^{-r} q^r}{(q^2;q^2)_r} \sum_{m=-\infty}^{\infty} q^{(m+r)^2} z^{m+r} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{r=0}^{\infty} \frac{(\frac{-q}{z})^r}{(q^2;q^2)_r} \sum_{m=-\infty}^{\infty} q^{m^2} z^m = \frac{1}{(q^2, \frac{-q}{z}; q^2)_\infty} \sum_{m=-\infty}^{\infty} q^{m^2} z^m.
\end{aligned}$$

□

Next, let us consider a more general setting in q-series, especially using Heine's transformation formula from which Euler's formulas (2.4) and (2.5) can be derived.

**Theorem 4** (Heine)**.**

$$ {}_2\phi_1 \left( \begin{matrix} a, b \\ c \end{matrix} ; q, z \right) = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1 \left( \begin{matrix} c/b, z \\ az \end{matrix} ; q, z \right). \tag{2.6}$$

*Proof.* [Gasper and Rahman(2004)],p13. For $|z| < 1$ and $|b| < 1$,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(a,b;q)_n z^n}{(q,c;q)_n} &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n z^n}{(q;q)_n} \frac{(cq^n;q)_\infty}{(bq^n;q)_\infty} \\
&= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n z^n}{(q;q)_n} \sum_{m=0}^{\infty} \frac{(\frac{c}{b};q)_m (bq^n)^m}{(q;q)_m} \\
&= \frac{(b,az;q)_\infty}{(c,z;q)_\infty} \sum_{m=0}^{\infty} \frac{(\frac{c}{b}, z; q)_m b^m}{(q,az;q)_m} \quad \text{(by changing the order of sum).}
\end{aligned}$$

□

Now, let us prove the Jacobi Triple Product by using Theorem 4.

*Proof.* Begin with replacing $z$ with $c/ab$ in (2.3) for $|c| < |ab|$. Then,

$$ {}_2\phi_1 \left( \begin{matrix} a, & b \\ & c \end{matrix} ; q, c/ab \right) = \frac{(b, c/b; q)_\infty}{(c, c/ab; q)_\infty} \sum_{n=0}^{\infty} \frac{(c/ab;q)_n b^n}{(q;q)_n} $$

$$= \frac{(b, c/b; q)_\infty}{(c, c/ab; q)_\infty} \frac{(c/a; q)_\infty}{(b; q)_\infty} = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}. \tag{2.7}$$

In (2.6), set $c = bzq^{1/2}$ and then let $b \to 0, a \to \infty$. It becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{(q; q)_n} = (zq^{1/2}; q)_\infty. \tag{2.8}$$

Similarly in (2.6), set $c = zq$ and then let $a, b \to \infty$. It becomes

$$\sum_{n=0}^{\infty} \frac{q^{n^2} z^n}{(q, zq; q)_n} = \frac{1}{(zq; q)_\infty}. \tag{2.9}$$

By using (2.8),

$$\begin{aligned}
(zq^{1/2}, z^{-1}q^{1/2}; q)_\infty &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n} q^{(m^2+n^2)/2} z^{m-n}}{(q; q)_m (q; q)_n} \quad (set \quad m - n = k) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k^2/2} z^k}{(q; q)_k} \sum_{n=-k}^{\infty} \frac{q^{n^2+nk}}{(q, q^{k+1}; q)_n} \\
&\quad + \sum_{k=1}^{\infty} \frac{(-1)^k q^{k^2/2} z^k}{(q; q)_k} \sum_{n=-k}^{\infty} \frac{q^{n^2+kn}}{(q, q^{k+1}; q)_n}.
\end{aligned} \tag{2.10}$$

Since $n \geq 0$, the inner sums of (2.10) can be written as

$$\frac{1}{(q; q)_k} \sum_{n=0}^{\infty} \frac{q^{n^2} q^{nk}}{(q, q^{k+1}; q)_n} = \frac{1}{(q; q)_k (q^{k+1}; q)_\infty} = \frac{1}{(q; q)_\infty}. \tag{2.11}$$

Simplify (2.10) by using (2.11) and replace $q$ with $q^2$ to obtain Jacobi Triple Product (1.3). □

Note that (2.4) and (2.9) are equivalent identities as shown below.

$$\begin{aligned}
\frac{1}{(z; q)_\infty} &= \sum_{n=0}^{\infty} \frac{q^{n^2-n} z^n}{(q, z; q)_n} \\
&= \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n}.
\end{aligned} \tag{2.12}$$

**Remark 5.** *Before moving into more delicate analytic q-series approach, it is important to notice there is a simple proof of Jacobi Triple Product only using functional equations of infinite products and a recurrence relation of coefficients of those products.*

*Proof.* [Borwein and Borwein(1987)] From (1.3), let $F(z) := (-z^{-1}q, zq; q^2)_\infty$ and observe

$$F(zq) = qz^2 F(z).$$

Also, let $G(z) := (q^2)_\infty F(z)$ so that $G(zq) = \frac{G(z)}{z^2 q}$. Then, $G(z)$ can be written as an even function in a Laurent series such as $G(z) = \sum_{n=-\infty}^{\infty} a_m z^{2m}$. After plugging this into the above functional equation of $G(z)$, we can obtain

$$\sum_{n=-\infty}^{\infty} a_n z^{2n} = \sum_{n=-\infty}^{\infty} a_n q^{2n-1} z^{2m},$$

which shows the recurrence relation of $a_n$ such that $a_n = a_{n-1}q^{2n-1}$. By letting $n \to \infty$ in this recurrence relation, $a_n$ becomes $a_0 q^{n^2}$. $a_0 = 1$ is derived by calculating $F(1)$ and $G(1)$. □

## 2.2 Ramanujan's ${}_1\psi_1$-summation

In this section let us only consider $r = s$ in ${}_r\psi_s$. ${}_r\psi_r$ converges in $\left|\frac{b_1 \cdots b_r}{a_1 \cdots a_r}\right| < |z| < 1$. In this section, we will see Jacobi Triple Product is a limiting case of Ramanujan's ${}_1\psi_1$-series ([Gasper and Rahman(2004)], [Andrews(1998)]). Let's begin with the bilateral summation formula ${}_1\psi_1$ with the proof given by Andrews and Askey(1978).

**Theorem 6** (Ramanujan's ${}_1\psi_1$-summation)**.** *For $|b/a| < |z| < 1$:*

$${}_1\psi_1(a; b; q, z) = \frac{(\frac{b}{a}, az, \frac{q}{az}, q; q)_\infty}{(\frac{q}{a}, \frac{b}{az}, b, z; q)_\infty}. \tag{2.13}$$

*Proof.* Let $f(b) := {}_1\psi_1(a; b; q, z) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n + \sum_{n=1}^{\infty} \frac{(a/b;q)_n}{(q/a;q)_n} (b/az)^n$, and observe

$${}_1\psi_1(a; b; q, z) - a{}_1\psi_1(a; b; q, qz) = \sum_{n=-\infty}^{\infty} \frac{(a;q)_{n+1}}{(b;q)_n} z^n = \frac{1}{z}\left(1 - \frac{b}{q}\right) a{}_1\psi_1(a; b; q, qz),$$

which can be written as

$$\Rightarrow \quad f(bq) - z^{-1}(1-b)f(b) = a{}_1\psi_1(a; bq; q, qz). \tag{2.14}$$

But since we can write $q^n$ as $\frac{(1-bq^n-1)}{b}$,

$$a{}_1\psi_1(a; bq; q, qz) = -\frac{a}{b}(1-b)f(b) + \frac{a}{b} f(bq). \tag{2.15}$$

By combining (2.14) and (2.15),

$$f(b) = \frac{1 - b/a}{(1-b)(1-b/az)} f(bq) = \cdots = \frac{(b/a;q)_n}{(b;q)_n (b/az;q)_n} f(bq^n). \tag{2.16}$$

Since $f(b)$ is analytic for $|b| < min(1, |az|)$, $f(b) = \frac{(b/a;q)_\infty}{(b;q)_\infty(b/az;q)_\infty} f(0)$ as $n \to \infty$. Replace $b$ by $q$ and use $f(q) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n$. Then,

$$f(0) = \frac{(q, q/az; q)_\infty}{(q/a; q)_\infty} f(q) = \frac{(q, q/az, az; q)_\infty}{(q/a, z; q)_\infty}.$$

□

Now, let us derive Jacobi Triple Product from Theorem 6. Replace $z$ by $z/a$ and let $a \to \infty$ in (2.13).

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{(b;q)_n} = \frac{(z, \frac{q}{z}, q; q)_\infty}{(\frac{b}{q}, b; q)_\infty}. \tag{2.17}$$

By setting $b = 0$ in (2.17), we can obtain Jacobi Triple Product (1.1).

So far, we have proved Jacobi Triple Product by directly using several different transformation formulas of q-hypergeometric series. From now on, we will look into the proofs of Jacobi Triple Product by using polynomial analogues of Jacobi Triple Product (1.1), (1.2) and (1.3), all of which involve the use of Gaussian polynomials.

## 2.3 Gaussian polynomials and q-binomial theorem

In this section, we will derive two different polynomial analogues of Jacobi Triple product and prove Jacobi Triple Product by letting $n \to \infty$ in each identity. To derive a polynomial analogue of (1.1), we will only need the following identity.

**Lemma 7** (a polynomial analogue of Euler's identity (2.5))**.**

$$(z;q)_n = \sum_{k=0}^{n} (-1)^k q^{k(k-1)/2} z^k \begin{bmatrix} n \\ k \end{bmatrix}_q, \tag{2.18}$$

*where Gaussian Polynomial is defined as*

$$\begin{bmatrix} n+k \\ k \end{bmatrix}_k := \begin{cases} \dfrac{(q)_{n+k}}{(q)_n (q)_k} & \textit{if } n, k > 0 \\ 0 & \textit{otherwise.} \end{cases} \tag{2.19}$$

**Remark 8.**

$$\lim_{L\to\infty} \begin{bmatrix} 2L \\ L+k \end{bmatrix}_q = \lim_{L\to\infty} \frac{1}{(q;q)_{L+k}} \lim_{L\to\infty} \frac{(q;q)_{2L}}{(q;q)_{L-i}} = \frac{1}{(q;q)_\infty}. \tag{2.20}$$

*Proof.* The proof of Lemma 7 is quite similar to the one given for Thm.2. Let $(z;q)_n := f(z) = \sum_{k\geq 0} a_k(q)z^k$. By using $(1-z)f(zq) = (1-zq^n)f(z)$ with $f(0) = a_0(q) = 1$, and then equating the coefficients of $z^k$, we can obtain $a_k$ as shown in the right hand side of (2.18). Especially, if $n \to \infty$, then (2.18) becomes (2.5) because of (2.20). □

The proof for $\frac{1}{(z;q)_n} = \sum_{k\geq 0} z^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q$ is quite similar as above, except that now $\frac{1}{(z;q)_n} := g(z)$, and $(1-z)g(zq) = (1-zq)g(z)$.

Next, with the help of (2.18), let us prove Jacobi Triple Product (1.1).

*Proof.* [Chan(2011)]

$$(z;q)_{2n} = (-z)^n q^{\frac{(n-1)n}{2}} \left(1-\frac{1}{z}\right)\left(1-\frac{1}{zq}\right)\cdots\left(1-\frac{1}{zq^{n-1}}\right)(1-zq^n)\cdots(1-zq^{2n-1})$$

(by replacing $z$ by $z/q^n$)

$$\begin{aligned} &= (-z)^n q^{\frac{-n(n+1)}{2}}(1-q^n/z)(1-q^{n-1}/z)\cdots(1-q/z)(1-z)\cdots(1-zq^{n-1}) \\ &= (-z)^n q^{\frac{-n(n+1)}{2}}(q/z;q)_n(z;q)_n. \end{aligned}$$

Now, we can use (2.19) to give the following:

$$(z;q)_{2n} = (-z)^n q^{\frac{-n(n+1)}{2}}(q/z;q)_n(z;q)_n = \sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}}(-z)^k. \qquad (2.21)$$

Finally, in the second equality of (2.21) set $l = k-n$ and simplify. Then,

$$(z;q)_n(q/z;q)_n = \sum_{l=-n}^{n} \begin{bmatrix} 2n \\ n+l \end{bmatrix}_q q^{l(l-1)/2}(-z)^l. \qquad (2.22)$$

As $n \to \infty$, (2.22) becomes Jacobi Triple Product (1.1). □

**Remark 9** (The second definition of Gaussian polynomials).
*In [Prodinger and Knopfmacher(2000)], Gaussian polynomials are defined using non-commuting variables. For $A$ and $B$ such that $BA = qAB$,*

$$(A+B)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} A^{n-k}B^k.$$

*In [Chan(2011)] it is shown that this alternative definition and* (2.19) *are equivalent and this new definition gives another way to prove* (2.19).

A polynomial anlogue of Jacobi Triple Product (1.3) was derived by MacMahon. In [MacMahon(2004)] he showed that a polynomial analogue of Jacobi Triple Product (1.3) is

$$\sum_{k=-\infty}^{\infty} z^k q^{k^2} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_{q^2} = (-qz, q/z, q^2)_n, \tag{2.23}$$

If $n \to \infty$ in (2.23), it becomes Jacobi Triple Product (1.3) by using (2.20).

## 2.4 A polynomial analogue of Jacobi Triple Product (1.2) by Alladi and Berkovich

In [Alladi and Berkovich(2004)], they derived a polynomial analogue of Jacobi Triple Product (1.2) as a continuation of their research about a bounded version Göllnitz's partition theorem.

$$\sum_{n=0}^{L} q^{T_n} \frac{z^{-n} + z^{1+n}}{1+z} = \sum_{i,j,k \geq 0} q^{T_i+T_j+T_k} z^{i-j} (-1)^k \begin{bmatrix} L-i \\ j \end{bmatrix}_q \begin{bmatrix} L-j \\ k \end{bmatrix}_q \begin{bmatrix} L-k \\ i \end{bmatrix}_q, \tag{2.24}$$

where $T_n = \binom{n}{2}$. By letting $n \to \infty$, the right hand side of (2.24) becomes

$$\sum_{i,j,k \geq 0} \frac{q^{T_i+T_j+T_k}}{(q)_i (q)_j (q)_k} z^{i-j} (-1)^k = (-qz, -qz^{-1}, q; q)_\infty \tag{2.25}$$

due to (2.18) i.e. $\sum_{i \geq 0} \frac{q^{T_i}}{(q)_i} z^i = (-zq)_\infty$.

In this section, we will see how (2.24) can be derived by using Heine's ${}_2\phi_1$ transformation, q-Chu-Vandermonde sum and the Sears-Carlitz transformation between ${}_3\phi_2$ and ${}_5\phi_3$. In chapter 3, we will go over its combinatorial implication, i.e.(2.24) as a special case of bounded version of Gollnitz's (big) partition theorem with a brief introduction of partitions of colored integers. First, we need to consider the following q-hypergeometric identities.

**Lemma 10.** *(2nd q-Chu-Vandermonde)*

$${}_2\phi_1 \left( \begin{matrix} a, q^{-n} \\ c \end{matrix} ; q, q \right) = \frac{a^n (\frac{c}{a})_n}{(c)_n}, \tag{2.26}$$

*(q-Kummer sum)*

$${}_2\phi_1 \left( \begin{matrix} a, b \\ \frac{aq}{b} \end{matrix} ; q, -\frac{q}{b} \right) = \frac{(aq, \frac{aq^2}{b^2}; q^2)_\infty (-q; q)_\infty}{(\frac{-q}{b}, \frac{aq}{b}; q)_\infty}, \tag{2.27}$$

*(Heine's transformation for ${}_2\phi_1$)*

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) = \frac{(\frac{abz}{c})_\infty}{(z)_\infty} {}_2\phi_1\left(\begin{matrix} \frac{c}{a}, \frac{c}{b} \\ c \end{matrix}; q, \frac{abz}{c}\right), \tag{2.28} $$

*(Sears-Carlitz transformation of a terminating ${}_3\phi_2$ for $a = q^{-n}$)*

$$ {}_3\phi_2\left(\begin{matrix} a, b, c \\ \frac{aq}{b}, \frac{aq}{c} \end{matrix}; q, \frac{aqz}{bc}\right) = \frac{(az)_\infty}{(z)_\infty} {}_5\phi_4\left(\begin{matrix} \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, \frac{aq}{bc} \\ \frac{aq}{b}, \frac{aq}{c}, az, \frac{q}{z} \end{matrix}; q, q\right). \tag{2.29} $$

*Proof.* In [Wenchang et al.(2004)Wenchang, Leontina, et al.], the 1st q-Chu-Vandermonde identity is the terminating case with $a = q^{-n}$ in (2.7) and (2.26) is derived from the 1st q-Chu-Vandermonde by changing the order of summation as follows:

$$ \begin{aligned} \frac{(c/b)_n}{(c)_n} &= {}_2\phi_1\left(\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, \frac{cq^n}{b}\right) = \sum_{k=0}^{n} \frac{(q^{-n}, b; q)_{n-k}}{(q, c; q)_{n-k}} \left(\frac{q^n c}{b}\right)^{n-k} \\ &= \frac{(q^{-n}, b; q)_n}{(q, c; q)_n} \left(\frac{q^n c}{b}\right)^n \sum_{k=0}^{n} \frac{(q^{-n}, \frac{q^{1-n}}{c}; q)_k}{(q, \frac{q^{1-n}}{b}; q)_k} q^k, \end{aligned} \tag{2.30} $$

where $(a;q)_{n-k} = (-1)^k a^{-k} q^{\binom{k+1}{2} - nk} \frac{(a)_n}{(q^{1-n}/a)_k}$. The last equality in (2.30) can be further simplified by utilizing $(q^{-n};q)_n = (-1)^n q^{-\binom{n+1}{2}} (q)_n$:

$$ {}_2\phi_1\left(\begin{matrix} q^{-n}, \frac{q^{1-n}}{c} \\ \frac{q^{1-n}}{b} \end{matrix}; q, q\right) = (-1)^n q^{-\binom{n}{2}} \left(\frac{b}{c}\right)^n \frac{(c/b)_n}{(b)_n}. $$

Next, we may replace $\frac{q^{1-n}}{c} \to B$, $\frac{q^{1-n}}{b} \to C$. Since $(q^{1-n}/C;q)_n = (-1)^n q^{-\binom{n}{2}} C^{-n} (C;q)_n$,

$$ {}_2\phi_1\left(\begin{matrix} q^{-n}, \quad B \\ C \end{matrix}; q, q\right) = B^n \frac{(C/B)_n}{(C)_n}. $$

In [Andrews(1998)], q-Kummer sum is proven by using Theoremm 4. For $|q| <$min$(1, |b|)$:

$$ \begin{aligned} \sum_{n=0}^{\infty} \frac{(b, a; q)_n (\frac{-q}{b})^n}{(q, \frac{aq}{b})_n} &= \frac{(a, -q)_\infty}{(\frac{aq}{b}, \frac{-q}{b})_\infty} \sum_{m=0}^{\infty} \frac{(\frac{q}{b}, \frac{-q}{b})_m a^m}{(q, -q)_m} \\ &= \frac{(a, -q)_\infty (\frac{aq^2}{b^2}; q^2)_\infty}{(\frac{aq}{b}, \frac{-q}{b})_\infty (a; q^2)_\infty} = \frac{(-q)_\infty (aq, \frac{aq^2}{b^2}; q^2)_\infty}{(\frac{aq}{b}, \frac{-q}{b})_\infty}. \end{aligned} $$

To show (2.28), iterate the transformation formula (2.6) several times.

Finally, in [Carlitz(1969)], (2.29) is proven as follows. Assume $|\frac{aqz}{bc}| < 1$ for the convergence of the q-series and $z \neq q^j, j \in \mathbb{Z}$. We use the identity shown below:

$$\sum_{j=0}^{k} \frac{(q^{-k}, q^k a, \frac{qa}{bc})_j}{(q, \frac{qa}{b}, \frac{qa}{c})_j} q^j = \frac{(c, q^{1-k}b)_k}{(\frac{qa}{b}, \frac{q^{-k}c}{a})_k} = \frac{(b,c)_k}{(\frac{qa}{b}, \frac{qa}{c})_k} \left(\frac{qa}{bc}\right)^k.$$

Now, ${}_3\phi_2(a, b, c; qa/b, qa/c; q, qax/(bc))$ becomes

$$\begin{aligned}
{}_3\phi_2(\cdots) &= \sum_{k=0}^{\infty} \frac{(a,b,c)_k}{(q, \frac{qa}{b}, \frac{qa}{c})_k} \left(\frac{qax}{bc}\right)^k = \sum_{k=0}^{\infty} \frac{x^k}{(q)_k} \sum_{j=0}^{k} \frac{(q^{-k}, \frac{qa}{bc})_j (a)_{j+k}}{(q, \frac{qa}{b}, \frac{qa}{c})_j} q^j \\
&= \sum_{k=0}^{\infty} \frac{x^k}{(q)_k} \sum_{j=0}^{k} (-1)^j q^{\binom{j}{2} - kj} \frac{(q)_k (\frac{qa}{bc})_j (a)_{j+k}}{(a, \frac{qa}{b}, \frac{qa}{c})_j (q)_{k-j}} \\
&= \sum_{j=0}^{\infty} (-1)^j q^{-\binom{j-1}{2}} x^j \frac{(a)_{2j} (\frac{qa}{bc})_j}{(q, \frac{qa}{b}, \frac{qa}{c})_j} \sum_{k=0}^{\infty} \frac{(q^{2j}a)_k}{(q)_k} q^{-jk} x^k,
\end{aligned}$$

where the inner sum is equal to $\frac{(q^j ax)_\infty}{(q^{-j}x)_\infty}$. Compare the last identity with the right hand side of (2.29) . □

At this point, we can look at Alladi and Berkovich's proof of a polynomial analogue of Jacobi Triple Product (1.2).

*Proof.* Let us begin with rewriting the right hand side of (2.24) as follows.

$$\begin{aligned}
&\sum_{i,j,k \geq 0} q^{T_i + T_j + T_k} z^{i-j} (-1)^k \begin{bmatrix} L-i \\ j \end{bmatrix}_q \begin{bmatrix} L-j \\ k \end{bmatrix}_q \begin{bmatrix} L-k \\ i \end{bmatrix}_q \\
&= \sum_{\substack{i,j \geq 0 \\ i+j \leq 0}} q^{T_i + T_j} z^{i-j} \frac{(q)_L}{(q)_i (q)_j (q)_{L-i-j}} {}_2\phi_1 \left( \begin{matrix} q^{i-L}, q^{j-L} \\ q^{-L} \end{matrix} ; q, q^{1+L-i-j} \right),
\end{aligned} \tag{2.31}$$

where for $0 \leq i, j \leq L, i + j \leq L$. By (2.28) and then (2.26),

$$\begin{aligned}
{}_2\phi_1 \left( \begin{matrix} q^{i-L}, q^{j-L} \\ q^{-L} \end{matrix} ; q, q^{1+L-i-j} \right) &= (q)_{L-i-j} {}_2\phi_1 \left( \begin{matrix} q^{i-L}, q^{j-L} \\ q^{-L} \end{matrix} ; q, q \right) \\
&= (q)_{L-i-j} \frac{(q^{i-L})_j}{(q^{-L})_j}.
\end{aligned}$$

Next, we use the relation $T_i + T_j - ij = T_{i-j} + j$ and make a change of variable so that (2.31) becomes

$$\sum_{\substack{-L \leq i \leq L \\ 0 \leq j \leq L}} q^{T_i + j} z^i \frac{(q)_L (q^{i+j-L})_j}{(q)_{i+j} (q)_j (q^{-L})_j} = \sum_{\substack{0 \leq i \leq L \\ 0 \geq j \geq L}} + \sum_{\substack{-L \leq i \leq 0 \\ 0 \leq j \leq L}} - \sum_{\substack{i=0 \\ 0 \leq j \leq L}}, \tag{2.32}$$

where the second sum is $\sum_{\substack{0\le i\le L\\ 0\le j\le L}} q^{T_i+j} z^{-i} \dfrac{(q)_L (q^{i+j-L})_j}{(q)_{i+j}(q)_j(q^{-L})_j}$ by changing $i \to -i$ .

The first sum in (2.32) can be rewritten as

$$(q)_L \sum_{i=0}^{L} \frac{z^i q^{T_i}}{(q)_i} \lim_{\substack{z\to q^{-i}\\ b\to q}} {}_4\phi_3 \left( \begin{matrix} q^{\frac{i-L}{2}}, -q^{\frac{i-L}{2}}, q^{\frac{1+i-L}{2}}, -q^{\frac{1+i-L}{2}} \\ q^{i-L}\frac{q}{b}, q^{i-L}z, \frac{q}{z} \end{matrix} ; q, q \right), \tag{2.33}$$

where $\lim_4 \phi_3 = \lim_{\substack{c\to\infty\\ z\to q^{-i}\\ b\to q}} {}_5\phi_4 \left( \begin{matrix} q^{\frac{i-L}{2}}, -q^{\frac{i-L}{2}}, q^{\frac{1+i-L}{2}}, -q^{\frac{1+i-L}{2}}, \frac{q^{1+i-L}}{bc} \\ q^{i-L}\frac{q}{b}, q^{i-L}\frac{q}{c}, q^{i-L}z, \frac{q}{z} \end{matrix} ; q, q \right)$.
After applying (2.29), we can simplify (2.33) so that it becomes

$$\begin{aligned} \sum_{i\ge 0} z^i q^{T_L} (-1)^{L-i} \sum_{n=0}^{L-i} (-1)^n q^{T_{n-1}-Ln} &= \sum_{n=0}^{L} (-1)^{L-n} q^{T_{L-n}} \sum_{i=0}^{L-n} (-z)^i \\ &= \sum_{n=0}^{L} (-1)^n q^{T_n} \frac{1+(-1)^n z^{n+1}}{1+z}. \end{aligned}$$

As similar to the case of the first sum, the second sum becomes

$$\sum_{n=0}^{L} (-1)^n q^{T_n} \frac{z+(-1)^n z^{-n}}{1+z},$$

by replacing $z$ by $1/z$. To simplify the last sum, use the following identity:

$$(q^{j-L})_j = \frac{(q^{-L})_{2j}}{(q^{-L})_j} = \frac{(q^{\frac{-L}{2}}, -q^{\frac{-L}{2}}, q^{\frac{1-L}{2}}, -q^{\frac{1-L}{2}})_j}{(q^{-L})_j}$$

for $j \le L$. Let us briefly sketch how to rearrange the last sum as follows:

$$\begin{aligned} \sum_{j=0}^{L} q^j \frac{(q)_L (q^{j-L})_j}{{(q)_j}^2 (q^{-L})_j} &= (q)_L \lim_{\substack{z\to 1\\ b\to q}} {}_4\phi_3 \left( \begin{matrix} q^{\frac{-L}{2}}, -q^{\frac{-L}{2}}, q^{\frac{1-L}{2}}, -q^{\frac{1-L}{2}} \\ q^{-L}\frac{q}{b}, q^{-L}z, \frac{q}{z} \end{matrix} ; q, q \right) \\ &= (q)_L \lim_{\substack{c\to\infty\\ z\to 1\\ b\to q}} {}_5\phi_4 \left( \begin{matrix} q^{-L/2}, -q^{-L/2}, q^{\frac{1-L}{2}}, -q^{\frac{1-L}{2}}, \frac{q^{1-L}}{bc} \\ q^{-L}\frac{q}{b}, q^{-L}\frac{q}{b}, q^{-L}z, \frac{q}{z} \end{matrix} ; q, q \right) \end{aligned}$$

(we now apply (2.29) and use $(q^{-L})_L = (q)_L (-1)^L q^{-T_L}$)

$$= (q)_L \lim_{\substack{c\to\infty\\ z\to 1\\ b\to q}} \frac{(z)_\infty}{(zq^{-L})_\infty} {}_3\phi_2 \left( \begin{matrix} q^{-L}, b, c \\ -q^{-L}\frac{q}{b}, q^{-L}\frac{q}{c} \end{matrix} ; q, \frac{zq^{1-L}}{bc} \right)$$

$$= \frac{(q)_L}{(q^{-L})_L} \sum_{n=0}^{L} (-1)^n q^{T_{n-1}-Ln} = \sum_{n=0}^{L} (-1)^{L-n} q^{T_L+T_{n-1}-Ln}$$
$$= \sum_{n=0}^{L} (-1)^n q^{T_n}.$$

Finally, we can combine the first, second and last sum.

$$\sum_{n=0}^{L} (-1)^n q^{T_n} \left\{ \frac{1+(-1)^n z^{n+1}}{1+z} + \frac{z+(-1)^n z^{-n}}{1+z} - \frac{1+z}{1+z} \right\}$$
$$= \sum_{n=0}^{L} q^{T_n} \frac{z^{-n}+z^{n+1}}{1+z}.$$

□

# Chapter 3

# Partition Theorems and its relation to the proof of Jacobi Triple Product identity

## 3.1 Euler's identity and Rogers-Ramanujan identities

According to [Alder(1969)], Euler's famous identity states that the number of partitions of $n$ into distinct parts $p_D(n)$ is equal to the number of partitions of $n$ into odd parts $p_O(n)$. Its proof uses the equivalence of the generating functions of $p_D(n)$ and $p_O(n)$, i.e., $(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}$. While this identity involves partitions with each parts differing by at least 1, Rogers-Ramanujan identities deal with partitions with each parts differing at least 2. First Rogers-Ramanujan identity shows that the number of partitions of $n$ into parts differing by at least 2 is equal to the number of partitions of $n$ into parts $\equiv$ 1 or 4 (mod 5), i.e.,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} &= \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \\ &= 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + \cdots . \end{aligned} \tag{3.1}$$

Similarly, second Rogers-Ramanujan identity shows the number of partitions of $n$ into parts differing by at least 2 with each part at least 2, is equal to the number of partitions of $n$ into parts which are $\equiv$ to 2 or 3 (mod 5), i.e.,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} &= \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty} \\ &= 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + \cdots . \end{aligned} \tag{3.2}$$

Rogers proved (3.1) and (3.2) by converting the right hand sides to the left hand sides with help of Jacobi Triple Product and utilized recurrence relations of auxiliary functions [Andrews(1986)].

More generally, if we denote by $q_{d,m}(n)$ the number of partitions of $n$ into parts differing by $\geq d$ with each part $\geq m$, and denote by $p_{d,m}(n)$ the number of partitions of $n$ into parts taken from a fixed set $S_{d,m}$, then Euler's identity and Rogers-Ramanujan identities are all of the type shown below:

$$q_{d,m}(n) = p_{d,m}(n). \tag{3.3}$$

In case of the first Rogers Ramanujan identity, $S_{2,1}$ represents the set of numbers $\equiv 1$ or 4 (mod 5). But according to Lehmer(1946) and Alder(1948), (3.3) is true only if $d = 1$ or $d = 2$, $m = 1, 2$. Although (3.3) was not as universal as we might expect, there have been many successful papers which generalize Rogers-Ramanujan identities in terms of q-series. For example, Alder, Singh and Carlitz proved the following identities:

$$\frac{(q^k, q^{k+1}; q^{2k+1})_\infty}{(q, q^2, \cdots, q^{2k}; q^{2k+1})_\infty} = \sum_{n=0}^{\infty} \frac{G_{k,n}(x)}{(q)_n}, \tag{3.4}$$

$$\frac{1}{(q^2, q^3, \cdots, q^{2k-1}; q^{2k+1})_\infty} = \sum_{n=0}^{\infty} \frac{G_{k,n}(x)x^n}{(q)_n}, \tag{3.5}$$

where the left side of (3.4) (respectively (3.5)) is the generating function for the number of partitions into parts $\not\equiv 0, \pm k$ (mod 2k+1) (respectively $\not\equiv 0, \pm 1$ (mod 2k+1) ) and the $G_{k,n}(x)$ are polymials in $x$ and equal to $x^{n^2}$ for $k = 2$. Also, Watson([Gasper and Rahman(2004)], pg 45) showed the following general formula which generalized Rogers-Ramanujan identities:

$$\sum_{n=0}^{\infty} (-1)^n a^{2n} q^{\frac{n(5n-1)}{2}} \frac{(1 - aq^{2n})(a)_n}{(1 - a)(q)_n} = (a)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q)_n}. \tag{3.6}$$

He proved (3.6) with his transformation formula for ${}_8\phi_7$ of a terminating balanced ${}_4\phi_3$ series,

$$\begin{aligned} &{}_8\phi_7 \left( \begin{matrix} a, qa^{1/2}, -qa^{1/2}, & b, & c, & d, & e, & q^{-n} \\ a^{1/2}, -a^{1/2}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1} \end{matrix} ; q, \frac{a^2 q^{2+n}}{bcde} \right) \\ &= \frac{(aq, aq/de; q)_n}{(aq/d, aq/e; q)_n} {}_4\phi_3 \left( \begin{matrix} q^{-n}, & d, & e, & aq/bc \\ aq/b, & aq/c, & deq^{-n}/a \end{matrix} ; q, q \right). \end{aligned} \tag{3.7}$$

Let $b, c, d, e \to \infty$ in (3.7). Then, it becomes

$$\begin{aligned} &\sum_{r=0}^{n} (1 - aq^{2r}) a^{2r} q^{2r^2} q^{nr} \frac{(aq)_{r-1}(q^{-n})_r}{(q, aq^{n+1}; q)_r} \\ &\qquad = (aq)_n \sum_{r=1}^{n} (-1)^r q^{\frac{r(r+1)}{2}} \frac{(q^{-n})_r a^r q^{nr}}{(q)_r}. \end{aligned} \tag{3.8}$$

If we take the limit $n \to \infty$ of (3.8), then it is simplified to (3.6). If we let $a = 1$ in (3.6),

$$(q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = 1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(5n-1)}{2}} = 1 + \sum_{n=1}^{\infty} (-1)^n \{q^{n(5n-1)/2} + q^{n(5n+1)/2}\}$$

$$= \sum_{n=-\infty}^{\infty} q^{n/2} q^{5n^2/2} = (q^3, q^2, q^5; q^5)_\infty.$$

By dividing both sides of the above identity by $(q)_\infty$, we can recover the first Rogers-Ramanujan identity. To obtain second Rogers-Ramanuhan identity, we may put $a = q$ in (3.6).

**Remark 11.** *By using Watson's transformation formula for* ${}_8\phi_7$*, we can find other expressions for* $(aq)_\infty$ *and* $\frac{1}{(q)_\infty}$ *shown in Theorem 3 [Slater(1966)]. Set* $bc = aq$ *in* (3.7)*:*

$$1 + \sum_{n=1}^{\infty} (-1)^n a^n q^{n(3n-1)/2} (1 - aq^{2n}) \frac{(a;q)_n}{(q;q)_n} = (aq;q)_\infty. \tag{3.9}$$

*With* $a = 1$ *in* (3.9)*, we obtain Euler's pentagonal number theorem, which is a special case of Jacobi Triple Product:*

$$1 + \sum_{n=1}^{\infty} (-1)^n \{q^{n(3n-1)/2} + q^{n(3n+1)/2}\} = (q;q)_\infty.$$

*Likewise, if we set* $b = a^{1/2}, c = -a^{1/2}$ *and let* $d, e, q^{-N} \to \infty$ *in* (3.7)*, then*

$$\sum_{n=0}^{\infty} a^n q^{n(3n+1)/2} \frac{(a;q)_n}{(q;q)_n} = (aq;q)_\infty \sum_{n=0}^{\infty} a^n q^{n^2} \frac{(-q;q)_n}{(q;q)_n (aq;q)_n}. \tag{3.10}$$

*By plugging* $a = 1$ *in* (3.10)*, we can obtain* $\sum_{n=0}^{\infty} \frac{q^{n^2}}{{(q;q)_n}^2} = \frac{1}{(q;q)_n}$.

## 3.2 The partition theorems by Schur and Göllnitz.

In Euler's identity, $p_O(n)$ can be interpreted as 'parts $\equiv \pm 1$ (mod 4).' Also, in the first Rogers-Ramanujan identity, replace 'parts $\equiv$ 1 or 4 (mod 5)' by 'parts $\equiv \pm 1$ (mod 5).' We may generalize this argument further. Define $p_d(n)$ as the number of partitions of $n$ into parts $\equiv \pm 1$ (mod $d + 3$). Then (3.3) can be written as either $q_{1,1}(n) = p_1(n)$ for Euler's identity or $q_{2,1}(n) = p_2(n)$ for first Rogers-Ramanujan identity. Thanks to Alder, (3.3) is not true for $d > 3$, i.e., $q_{3,1}(n) \neq p_3(n)$. But Schur found a new relation between the two such that $q_{3,1}(n) - p_3(n)$ is the number of partitions of $n$ into parts differing by $\geq 3$ and containing at least 2 consecutive multiples of 3.

**Theorem 12** (partition theorem of Schur, 1926)**.** *Let* $P_1(n)$ *denote the number of partitions of* $n$ *into distinct parts* $\equiv -2^1, -2^0$ $(mod\, 3)$*, let* $p_3(n)$ *denote the number of partitions of* $n$ *into parts* $\equiv \pm 1$ $(mod\, 6)$*, and let* $G_1(n)$ *denote the number of partitions of* $n$ *with each part* $\geq 3$ *with equlity only if a part is* $\equiv -2^1, -2^0$ $(mod 3)$*. Then,* $G_1(n) = P_1(n) = p_3(n)$

*Proof.* [Andrews(1986)] The equivalence of $P_1(n)$ and $p_3(n)$ can be easily achieved by the generating function argument.

$$\sum_{n=0}^{\infty} P_1(n)q^n = (-q,-q^2;q^3)_\infty = \frac{(q^2,q^4;q^6)_\infty}{(q,q^2;q^3)_\infty} = \frac{(q^2,q^4;q^6)_\infty}{(q,q^4;q^6)_\infty(q^2,q^5;q^6)_\infty} = \sum_{n=0}^{\infty} p_3(n)$$

Bressoud(1980) proved the equivalence of $P_1(n)$ and $G_1(n)$ by showing the existence of a simple bijection of the two numbers. □

**Remark 13.** *The above notations $P_i(n)$ and $G_i(n)$, for $i \geq 1$ follows the ones in Alladi, Andrews, and Berkovich's paper on a four parater q-series identity and their implication will become clear in the next theorems.*

However, Alder(1948) proved the generalization of combinatorial interpretation for $q_{d,1}(n) - p_d(n)$ wasn't available for $d \geq 4$. In particular, he proved the number of partitions of $n$ into parts differing by at least $d$ among which no two consecutive multiples of $d$ appear, is not equal to the number of partitions of n into parts taken from any set of integers whatsoever for $d \geq 4$. Instead of trying to find relations between $q_{d,1}(n)$ and $p_d(n)$, Göllnitz and Gordon discovered theorems concerning $q_{d,m}$ with $d = 2$.

**Theorem 14** (The first Göllnitz(1960)-Gordon(1965) identity)**.** *The number of partitions of n into parts differing by at least 2 among which no two consecutive even numbers appear is equal to the number of partitions of n into parts $\equiv 1, 4, 7 \, (mod\, 8)$.*

For example, for $n = 11$, $\{11, 10+1, 9+2, 8+3, 7+4, 7+3+1\} \Leftrightarrow \{9+1+1, 7+4, 7+1+1+1+1, 4+4+1+1+1, 4+1+\cdots+1, 1+\cdots+1\}$.

**Theorem 15** (The second Göllnitz(1960)-Gordon(1965) identity)**.** *The number of partitions of n into parts differing by at least 2 among which no two consecutive even numbers appear and with each part $\geq 3$ is equal to the number of partitions of n into parts $\equiv$ 3,4,5 (mod 8).*

For example, for $n = 11$, $\{11, 8+3, 7+4\} \Leftrightarrow \{11, 5+3+3, 4+4+3\}$. The q-series identities of Thm.14 and Thm.15 are as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} &= \frac{1}{(q,q^4,q^7;q^8)_\infty} \\ &= 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 2q^6 + 3q^7 + 4q^8 + \cdots, \end{aligned} \tag{3.11}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(q^2;q^2)_n} &= \frac{1}{(q^3,q^4,q^5;q^8)_\infty} \\ &= 1 + q^3 + q^4 + q^5 + q^6 + q^7 + 2q^8 + 2q^9 + 2q^{10} \cdots, \end{aligned} \tag{3.12}$$

both of which can be found in Ramanujan's lost notebook.

**Theorem 16** (Göllnitz's partition theorem [Göllnitz(1967)]). *Let $P_2(n)$ denote the number of partitions of $n$ into distinct parts $\equiv -2^2, -2^1, -2^0 \ (mod\, 6)$, let $P_2'(n)$ denote the number of partitions of $n$ into parts $\equiv 2, 5, 11 \ (mod\, 6)$ and let $G_2(n)$ denote the number of partitions of $n$ into parts $\neq 1$ or $3$ such that the difference between the parts $\geq 6$, with equality only if a part is $\equiv -2^2, -2^1, -2^0 \ (mod\, 6)$. Then, $G_2(n) = P_2(n) = P_2'(n)$*

## 3.3 Refinement and Generalization of the theorems of Schur and Göllnitz.

According to Alladi, Andrews and Gordon (respectively Alladi and Berkovich) reinterpreted Schur's identity (respectively Göllnitz's theorem) by using a specific type of partitions of colored integers. This process has revealed a much deeper result, a 4 parameter q-series identity [Alladi et al.(2003)Alladi, Andrews, and Berkovich]. To begin with, let us introduce colored integers and their corresponding weighted partitions. Assume all positive integers $n$ occur in two primary colors $\mathbb{A}, \mathbb{B}$, and for $n \geq 2$ in secondary color $\mathbb{AB}$ as well. Given interger $n$, the ordering of colors is $\mathbb{AB}_n < \mathbb{A}_n < \mathbb{B}_n$, i.e.,

$$\mathbb{A}_1 < \mathbb{B}_1 < \mathbb{AB}_2 < \mathbb{A}_2 < \mathbb{B}_2 < \mathbb{AB}_3 < \cdots . \tag{3.13}$$

Consider the following substitutions:

$$\begin{cases} \mathbb{A}_n \to 3n-2, \quad \mathbb{B}_n \to 3n-1 & n \geq 1 \\ \text{and consequently} \quad \mathbb{AB}_n \to 3n-3 & n \geq 2, \end{cases} \tag{3.14}$$

then we can find the ordering in (3.13) becomes $1 < 2 < 3 < \cdots$.

In case the colored integers occur in 3 primary colors $\mathbb{A}, \mathbb{B}, \mathbb{C}$ and for $n \geq 2$ also in secondary colors $\mathbb{AB}, \mathbb{AC}, \mathbb{BC}$, the ordering of colors for a given $n$ is

$$\mathbb{AB}_n < \mathbb{AC}_n < \mathbb{A}_n < \mathbb{BC}_n < \mathbb{B}_n < \mathbb{C}_n. \tag{3.15}$$

As similar to (3.14), we also need the below substitutions which will play a role to confirm that Theorem 12 and 16 are special cases of Theorem 17 and 18.

$$\begin{cases} \mathbb{A}_n \to 6n-4, \quad \mathbb{B}_n \to 6n-2, \quad \mathbb{C}_n \to 6n-1, & n \geq 1 \\ \text{consequently} \quad \mathbb{AB}_n \to 6n-6, \quad \mathbb{AC}_n \to 6n-5, \quad \mathbb{BC}_n \to 6n-3. & n \geq 2 \end{cases} \tag{3.16}$$

Next, let us consider a partition $\pi$ as a word of colored integers in decreasing order. Let $\nu_{\mathbb{A}}(\pi), \nu_{\mathbb{B}}(\pi), \cdots, \nu_{\mathbb{BC}}(\pi)$ denote the number of parts of $\pi$ in colors $\mathbb{A}, \mathbb{B}, \cdots, \mathbb{BC}$. Also, let $\sigma(\pi)$ denote the sum of its parts and $\lambda(\pi)$ its largest part. The weight of each $\mathbb{A}_i \cdots \mathbb{BC}_k$ is defined to be its subscripts $i, \cdots, k$ and the gap between two ordered colors, e.g., $\mathbb{A}_i$ and $\mathbb{AB}_j$ is $|i-j|$. Especially, we will focus our attention on Type-I partitions, $\pi = m_1 + m_2 + \cdots + m_\nu$

where $m_i$s denote colored integers with a predetermined ordering (3.13) (respectively (3.15)) such that the gap between $m_i$ and $m_{i+1}$ is $\geq 1$ with equality only if either

$$\begin{cases} m_i \text{ and } m_{i+1} \text{ are of the same primary colors, or} \\ m_i \text{ is of a higher order color in (3.13) (respectively (3.15)).} \end{cases} \tag{3.17}$$

The generating function of all Type-I partitions with given $\nu_{\mathbb{A}}(\pi), \cdots, \nu_{\mathbb{BC}}(\pi)$ where $\pi$ is minimal (i.e., $\pi$ with minimal $\sigma(\pi)$) is given by

$$\frac{q^{\sigma(\pi)}}{(q)_{\nu(\pi)}}. \tag{3.18}$$

For example, if the colors occur in the order $\{\mathbb{A}, \mathbb{AB}, \mathbb{AB}, \mathbb{AC}, \mathbb{C}, \mathbb{B}, \mathbb{F}\}$, then its minimal $\pi$ is $\pi = \mathbb{A}_{11} + \mathbb{AB}_{10} + \mathbb{AB}_8 + \mathbb{AC}_6 + \mathbb{C}_4 + \mathbb{B}_3 + \mathbb{BC}_2$ with $\sigma(\pi) = 44$, and $\nu(\pi) = 7$. Its corresponding generating function is $\frac{q^{44}}{(q)_7}$. Under specific transformations which will be discussed shortly, the number of Type-I partitions appears to be the same as $G_1$ (respectively $G_2$) in Theorem 12 (respectively in Theorem 16). Now let us explain a two parameter refinement of Schur's theorem due to Alladi-Gordon, and a three parameter refinement of Göllnitz's theorem due to Alladi, Andrews and Gordon by using the partitions of colored integers.

**Theorem 17.** *[Alladi and Gordon(1995)] Let $P_4(n; i, j)$ denote the number of partitions of $n$ into $i$ distinct parts in color $\mathbb{A}$, $j$ distinct parts in color $\mathbb{B}$ for positive integers $i, j$. Also, let $G_4(n; a, b, ab)$ denote the number of partitions of Type-I with $\nu_{\mathbb{A}}(\pi) = a, \cdots, \nu_{\mathbb{AB}}(\pi) = ab$. Then,*

$$\sum_{\substack{i,j \\ constraints}} G_4(n; a, b, ab) = P_4(n; i, j), \tag{3.19}$$

where '$i, j$ constraints' are

$$i = a + ab \quad j = b + ab. \tag{3.20}$$

To prove (3.19) we may use the equivalence relation of the generating functions of each side:

$$\sum_{\substack{i,j \\ \text{constraints}}} \frac{q^{T_{a+b+ab} + T_{ab}}}{(q)_a (q)_b (q)_{ab}} = \frac{q^{T_i + T_j}}{(q)_i (q)_j}. \tag{3.21}$$

*Proof.* (Step1) It's easy to show the generating function of $P_4$ is $\frac{q^{T_i+T_j}}{(q)_i(q)_j}$. The generating function for all Type-I partitions with i,j constraints, $\sum G_4$ (call it G) is

$$G = \frac{H}{(q)_{i+j-ab}}, \quad \text{where } H = H(i, j, ab) = \sum_{\mu \in M} q^{\sigma(\mu)}. \tag{3.22}$$

$M$ is the set of minimal Type-I partitions with constraints and $\sigma(\mu)$ is sum of parts of a give partition $\mu$. Showing G is the generating function of the left side of (3.21) is equivalent to showing

$$H(i, j, ab) = q^{T_{i+j-ab} + T_{ab}} \begin{bmatrix} i + j - ab \\ i - ab, j - ab, ab \end{bmatrix}_q. \tag{3.23}$$

Since $M$ is a disjoint union of all partitions ending in $a, b, ab$ (call it $A, B, AB$ respectively), $H = H_a + H_b + H_{ab} := \{\sum_{\mu\in A} + \sum_{\mu\in B} + \sum_{\mu\in AB}\}q^{\sigma(\mu)}$. We can compute each $H_a(i,j,ab), H_b(i,j,ab)$, and $H_{ab}(i,j,ab)$ by using the recurrence relations derived from the gap conditions of Type I-partitions (3.17). The detailed calculations are omitted here.
(Step2) Once we obtain the generating functions of $\sum G_4$ and $P_4$, we can move on to prove they are equal. We may assume $j \le i$ and rewrite (3.21) by using $T_i + T_j = T_{i+j} - ij$:

$$\sum_{ab=0}^{j} \begin{bmatrix} i \\ ab \end{bmatrix}_q \begin{bmatrix} j \\ ab \end{bmatrix}_q (q)_{ab} q^{(i-ab)(j-ab)} = 1. \tag{3.24}$$

(3.24) can be written as

$$\sum_{ab=0}^{j} q^{ij-(i+j)ab+(ab)^2} \begin{bmatrix} j \\ ab \end{bmatrix}_q (q^{i-ab+1})_{ab} = 1, \tag{3.25}$$

which becomes

$$\sum_{ab=0}^{j} q^{ij-(i+j)ab+(ab)^2} \begin{bmatrix} j \\ ab \end{bmatrix}_q \sum_{k=0}^{ab} (-1)^k q^{(i-ab)k+T_{ab}} \begin{bmatrix} ab \\ k \end{bmatrix}_q = 1, \tag{3.26}$$

where the inner sum of (3.26) is due to q-binomial theorem (2.18). By making change of variables such as $ab = k+l$ in (3.26) and using $\begin{bmatrix} j \\ k+l \end{bmatrix}_q \begin{bmatrix} k+l \\ k \end{bmatrix}_q = \begin{bmatrix} j \\ j-l \end{bmatrix}_q \begin{bmatrix} j-l \\ k \end{bmatrix}_q$, it is equivalent to the following:

$$\sum_{l=0}^{j} q^{ij-(i+j)l+l^2} \begin{bmatrix} j \\ j-l \end{bmatrix}_q \sum_{k=0}^{j-l} (-1)^k q^{-(j-l)k+T_k} \begin{bmatrix} j-l \\ k \end{bmatrix}_q = 1 \tag{3.27}$$

By applying (2.18) again, we can find the inner sum in (3.27) is equal to

$$(q^{-(j-l)+1})_{j-l} = \begin{cases} 0 & \text{if } j-l > 0 \\ 1 & \text{if } j-l = 0 \end{cases}$$

which proves the left side of (3.27) is equal to 1. As a result, this is equivalent to (3.21). □

Now, we can continute our discussion about how Theorem 17 generalizes and refines of Theorem 12. Let us multiply both sides of (3.21) by $A^i B^j$ and sum over $i, j$ such that

$$\sum_{a,b,ab} A^{a+ab} B^{b+ab} \frac{q^{T_{a+b+ab}+T_{ab}}}{(q)_a (q)_b (q)_{ab}} = (-Aq)_\infty (-Bq)_\infty. \tag{3.28}$$

If we make the following transformations:

$$\begin{cases} q \to q^3 & \text{dilation} \\ A \to q^{-2}, \quad B \to q^{-1} & \text{translation,} \end{cases} \tag{3.29}$$

we can observe the gap condition of Type-I partition (3.17) translates into the gap conditions of $G_1(n)$ in Therem 12. In other words, due to (3.29), (3.28) turns into an analytic q-series version of Schur's partition theorem. In the same way, it is possible to generalize Theorem 16 in the context of Type-I partitions of colored integers.

**Theorem 18.** *[Alladi et al.(1995)Alladi, Andrews, and Gordon] Let $P_5(n;i,j,k)$ denote the number of partitions of $n$ into $i$ distinct parts in color $\mathbb{A}$, $j$ distinct parts in color $\mathbb{B}$, $k$ distinct parts in color $\mathbb{C}$ and let $G_5(n;i,j,k)$ denote the number of Type-I partitions $\pi$ of $n$ with $\nu_{\mathbb{A}}(\pi) = a, \cdots, \nu_{\mathbb{BC}}(\pi) = bc$. Then,*

$$\sum_{\substack{i,j,k \\ constraints}} G_5(n;i,j,k) = P_5(n;i,j,k), \tag{3.30}$$

*where 'i,j,k constraints' are*

$$i = a + ab + ac, \quad j = b + ab + bc, \quad k = c + ac + bc. \tag{3.31}$$

**Remark 19.** *The proof of Theorem 18 depends on the following identity, which implies the equivalence of the generating functions of each side in* (3.30)*:*

$$\sum_{\substack{i,j,k \\ constraints}} \frac{q^{T_t + T_{ab} + T_{ac} + T_{bc-1}}(1 - q^a + q^{a+bc})}{(q)_a (q)_b (q)_c (q)_{ab} (q)_{ac} (q)_{bc}} = \frac{q^{T_i + T_j + T_k}}{(q)_i (q)_j (q)_k} \tag{3.32}$$

*where $t := a + b + c + ab + ac + bc$. Let us multiply* (3.32) *by $A^i B^j C^k$ and sum over $i, j, k$. Then it becomes*

$$\sum_{\substack{a,b,c \\ ab,ac,bc}} A^{a+ab+ac} B^{b+ab+bc} C^{c+ac+bc} \frac{q^{T_t + T_{ab} + T_{ac} + T_{bc-1}}(1 - q^a + q^{a+bc})}{(q)_a (q)_b (q)_c (q)_{ab} (q)_{ac} (q)_{bc}} = (-Aq, -Bq, -Cq)_\infty. \tag{3.33}$$

*With the substitutions similar to* (3.29)

$$\begin{cases} q \to q^6 & \textit{dilation} \\ A \to q^{-4}, B \to q^{-2}, C \to q^{-1} & \textit{translation}, \end{cases} \tag{3.34}$$

*the gap conditions on the colored parts of Type-I partitions become identical with the gap conditions on $G_2(n)$ in Göllnitz's partition theorem.*

The proof of (3.32) uses Whipple's q-analogue of Watson's transformation ${}_8\phi_7 \to {}_4\phi_3$, the ${}_6\psi_6$ summation of Bailey and a transformation formula ${}_3\phi_3 \to {}_3\phi_2$. Later, the proof was simplified further by using Jackson's q-analogue of Dougall's summation for ${}_6\phi_5$ in [Alladi and Andrews(1999)].

## 3.4 A bounded key identity of Göllnitz theorem and Jacobi Triple Product.

With the background of the above partition theorems and their reformulation using colored intergers, now we are ready to complete our original quest: giving combinatorial interpretion of (2.24) shown by Alladi and Berkovich. This begins with deriving weighted partition identity from a bounded version of Göllnitz partition theorem.

**Theorem 20.** *[Alladi and Berkovich(2004)] Let $G_L(n; a, b, c, ab, ac, bc)$ be defined as $G(n; a, b, c, ab, ac, bc)$ with the condition that no part exceeds $\mathbb{C}_L$. Let $P_L(n; i, j, k)$ be defined as $P(n; i, j, k)$ such that $\lambda(\mathbb{A}) \leq \mathbb{A}_{L-k}$, $\lambda(\mathbb{B}) \leq \mathbb{B}_{L-i}$ and $\lambda(\mathbb{C}) \leq \mathbb{C}_{L-j}$. Then,*

$$\sum_{\substack{i,j,k \\ constraints}} G_L(n; i, j, k) = P_L(n; i, j, k). \tag{3.35}$$

We may multiply each side of (3.35) by $A^i B^j C^k q^n$ to count a part occuring in color $\mathbb{A}, \cdots$, or $\mathbb{BC}$ with weight $A, \cdots$ or $BC$. By summing over $i, j, k, n$, (3.35) changes into:

$$\sum_{\substack{a,\cdots,bc \geq 0 \\ n \geq 0}} A^{a+ab+ac} B^{b+ab+bc} C^{c+ac+bc} q^n G_L(n; a, \cdots, bc)$$
$$= \sum_{i,j,k \geq 0} q^{T_i+T_j+T_k} A^i B^j C^k \begin{bmatrix} L-i \\ j \end{bmatrix}_q \begin{bmatrix} L-j \\ k \end{bmatrix}_q \begin{bmatrix} L-k \\ i \end{bmatrix}_q. \tag{3.36}$$

Note that for a color $\mathbb{AB}$, its weight is formulated to be $A \cdot B$, i.e., $A$ multiplied by $B$, similarly for $\mathbb{AC}, \mathbb{BC}$ as well. Also note that for Type-I partions, if two parts differ at least by 2, then there is total independence in assigning the colors.

In order to make connection between (3.36) and a polynomial analogue of Jacobi Triple Product (2.24), Alladi and Berkovich used the weight function $\omega$, which is discussed below. Let us consider a partition $\bar{\pi}$ into distinct uncolored parts, and decompose it into chains $\chi$, i.e., a maximal run of consecutive integers. Because of the independence of color assignment, the weight $\omega(\bar{\pi})$ can be defined multiplicatively as $\prod_{\chi \in \bar{\pi}} \omega(\chi)$. The weight of these chains is to be constructed as polynomials in $A, B, C$ such that it will eventually be the same as the one we obtain after assigning colors into integers and summing over all possible such assignments. These assignments are required to follow the gap condition of Type-I partitions. Therefore,

$$\omega(\bar{\pi}) = \sum \prod_{i=1}^{l(\bar{\pi})} X_i, \tag{3.37}$$

where $X_i$ is the weight of i-th part in color $\mathbb{X}$ in $\bar{\pi}$ in some color assignment. For example, from the ordering of 6 colors $A_1 < B_1 < C_1 < AB_2 < AC_2 < A_2 < BC_2 < B_2 < C_2 < AB_3 < AC_3 \cdots$, if $\bar{\pi}_1$ represents $1+2$, then $\omega(\bar{\pi}) = A(BC+B+C)+B(B+C)+C^2$. Also, if $\bar{\pi}_2$ represents $2+4$, then $\omega(\bar{\pi}) = (A+B+C+AB+AC+BC)^2$. Now, the following weighted partition theorem can be established.

**Theorem 21.** *[Alladi and Berkovich(2004)] Let $D_L$ denote the set of all partitions $\bar{\pi}$ into distinct parts $\leq L$ with weights $\omega(\bar{\pi})$. Then,*

$$\sum_{\bar{\pi}\in D_L} \omega(\bar{\pi})q^{|\pi|} = \sum_{i,j,k\geq 0} q^{T_i+T_j+T_k} A^i B^j C^k \begin{bmatrix} L-i \\ j \end{bmatrix}_q \begin{bmatrix} L-j \\ k \end{bmatrix}_q \begin{bmatrix} L-k \\ i \end{bmatrix}_q, \tag{3.38}$$

*where $|\pi|$ is the sum of parts of $\bar{\pi}$ and the formula for $\omega(\chi)$ is given below.*

**Remark 22.** *By using*

$$\lim_{L\to\infty} \begin{bmatrix} L \\ i \end{bmatrix}_q = \frac{1}{(q)_i},$$

*The identity (3.38) becomes the following.*

$$\sum_{\bar{\pi}\in D} \omega(\bar{\pi})q^{|\pi|} = (-Aq, -Bq, -Cq; q)_\infty \tag{3.39}$$

Let us derive the formula for $\omega(\chi)$. Let $\omega_l(\mathbb{X})$ denote weight of the chain $\chi$ of legnth $l$ with smallest part $s(\chi)$ in color $\mathbb{X} \in \{\mathbb{A}, \cdots, \mathbb{BC}\}$. Then, by using the color-gap conditions (3.15) and (3.17), we can find 6 recurrence relations of $\omega_l(\mathbb{X})$, starting from $\omega_l(\mathbb{C}) = C\omega_{l-1}(\mathbb{C})$. With initial condition $\omega(\mathbb{X}) = X$, each $\omega_l(\mathbb{X})$ can be solved as polynomials in $A, B, C$. Therefore, if $s(\chi)$ denote the least part of $\chi$,

$$\omega(\chi) = \begin{cases} \omega_l(\mathbb{A}) + \omega_l(\mathbb{B}) + \omega_l(\mathbb{C}) = \sum\limits_{\substack{i+j+k\\=l}} A^i B^j C^k + ABC \sum\limits_{\substack{i+j+k\\=l-2}} A^i B^j C^k & \text{if } s(\chi) = 1, \\ \omega_l(\mathbb{A}) + \cdots + \omega_l(\mathbb{BC}) \\ = F_l(A,B,C) + A(B-C)F_{l-1}(A,B,C) + A^2BCF_{l-2}(A,B,C) & \text{if } s(\chi) > 1, \end{cases} \tag{3.40}$$

where $F_l(A,B,C) = \sum\limits_{\substack{i+j+k\\=l}} A^i B^j C^k + ABC \sum\limits_{\substack{i+j+k\\=l-2}} A^i B^j C^k + BC \sum\limits_{\substack{j+k\\=l-1}} B^j C^k$.

With $C = -1$ in (3.40),

$$\omega(\chi) = \begin{cases} (-1)^{l(\chi)}(1-AB) & \text{if } s(\chi) > 1, \\ (-1)^{l(\chi)}1 + \sum_{i=1}^{l(\chi)}((-A)^i + (-B)^i) & \text{if } s(\chi) = 1. \end{cases} \tag{3.41}$$

With $A = 1/B = z, C = -1$ in (3.40),

$$\omega(\chi) = \begin{cases} \frac{z^{-l(\chi)}+z^{1+l(\chi)}}{1+z} & \text{if } s(\chi) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.42}$$

Since $|\chi| = 1 + 2 + \cdots + l(\chi) = T_{l(\chi)}$, if $s(\chi) = 1$, (3.38) becomes

$$\sum_{l=0}^{L} q^{T_l} \frac{z^{-l} + z^{1+l}}{1+z} = \sum_{i,j,k\geq 0} q^{T_i+T_j+T_k} z^{i-j} (-1)^k \begin{bmatrix} L-i \\ j \end{bmatrix}_q \begin{bmatrix} L-j \\ k \end{bmatrix}_q \begin{bmatrix} L-k \\ i \end{bmatrix}_q, \tag{3.43}$$

which is a polynomial analogue of Jacobi Triple Product (2.24).

**Remark 23.** *The above discussion about colored integers using 3 primary colors A,B and C can be extended into formulating a new theorem of type-I partitions with 4 primary colors A,B,C and D. In this case, the colored integers occur in 4 primary colors* $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$*, for* $n \geq 2$ *also in secondary colors* $\mathbb{AB}, \mathbb{AC}, \cdots, \mathbb{CD}$*, and for* $n \geq 4$ *also in quaternary color* $\mathbb{ABCD}$ *with the ordering*

$$\begin{cases} \text{if } m < n \text{ as uncolored positive integers, } m \text{ in any color} < n \text{ in any color,} \\ \text{if the same } n \text{ occur in different colors,} \\ \quad \mathbb{ABCD}_n < \mathbb{AB}_n < \mathbb{AC}_n < \mathbb{AD}_n < \mathbb{A}_n < \mathbb{BC}_n < \mathbb{BD}_n < \mathbb{B}_n < \mathbb{CD}_n < \mathbb{C}_n < \mathbb{D}_n. \end{cases} \tag{3.44}$$

*If we do not consider parts in color* $\mathbb{ABCD}_n$*, the gap condition of Type-I partition is the same as in Therem 18 and the ternary colors* $\mathbb{ABC}, \mathbb{ABD}, \mathbb{BCD}$ *are discarded as well. The gap between parts in color* $\mathbb{ABCD}$ *is* $\geq 4$ *such that the least quarternary part is*

$$\begin{cases} \geq 3 + 2\tau & \text{if } \mathbb{A}_1 \text{ is a part} \\ \geq 4 + 2\tau & \text{otherwise,} \end{cases}$$

*where* $\tau$ *is the number of non-quaternary parts in* $\pi$*.*

**Theorem 24.** *[Alladi et al.(2003)Alladi, Andrews, and Berkovich] For nonnegative integers* $i, j, k,$ *and* $l$*, let* $P_6(n; i, j, k, l)$ *denote the number of partitions of* $n$ *into i(respectively j,k,l) distint parts in color* $\mathbb{A}$*(respectively*$\mathbb{B}, \mathbb{C}, \mathbb{D}$*). Let* $G_6(n; a, b, c, d, ab \cdots, cd, Q)$ *denote the number of Type-I partitions* $\pi$ *in colors* $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}, \cdots, \mathbb{CD}, \mathbb{ABCD}$ *such that* $\nu_{\mathbb{A}}(\pi) = a, \cdots, \nu_{\mathbb{CD}}(\pi) = cd$*, and* $\nu_{\mathbb{ABCD}}(\pi) = Q$*. Then,*

$$G_6(n; i, j, k, l) = P_6(n; i, j, k, l), \tag{3.45}$$

*where* $G_6(n; i, j, k, l) = \sum_{\substack{i,j,k,l- \\ constraints}} \bar{G}_6(n; a, b, c, d, ab, \cdots, cd, Q)$*.*

Its proof depends on the following 4-parameter key identity

$$\sum_{i,j,k,l} A^i B^j C^k D^l \sum_{\substack{i,j,k,l- \\ constraints}} \frac{q^{T_\tau + T_{ab} + \cdots + T_{cd} - bc - bd - cd + 4T_{Q-1} + 3Q + 2Q\tau}}{(q)_a (q)_b (q)_c (q)_d (q)_{ab} (q)_{ac} (q)_{ad} (q)_{bc} (q)_{bd} (q)_{cd} (q)_Q}$$
$$\cdot \{(1 - q^q) + q^{a+bc+bd+Q}(1 - q^b) + q^{a+bc+bd+Q+b+cd}\} = (-Aq, -Bq, -Cq, -Dq)_\infty, \tag{3.46}$$

where $ab \neq a$ times $b$, $\tau := a + b + c + d + ab + ac + ad + bc + bd + cd$ and $i, j, k, l$-constraints are

$$i = a + ab + ac + ad + Q, \quad j = b + ab + bc + bd + Q,$$
$$k = c + ac + bc + cd + Q, \quad l = d + ad + bd + cd + Q. \tag{3.47}$$

**Remark 25.** *If we put* $D = -1, C = B^{-1}$*, and* $A = 0$ *in* (3.46)*, then it becomes*

$$(q, Bq, q/B; q)_\infty = \sum_{n=0}^{\infty} q^{T_n} \left( \frac{B^{n+1} + B^{-n}}{1 + B} \right) \tag{3.48}$$

Alladi, Andrews and Berkovich proved that with a substitution similar to (3.14) or (3.16), Therem 24 can have its partition theoretic counterpart which involves the use of missing notations $G_3, P_3$ from our discussion about partitions(Theorem 12, 16, 17, 18 and 24). We may consider this new partition theorem as an extension of the theorems of Schur and Göllnitz [Alladi et al.(2003)Alladi, Andrews, and Berkovich].

# Chapter 4

# The method of Frobenius Partition

## 4.1 Jacobi Triple Product and Frobenius Partition

Andrews proved Jacobi Triple Product using Frobenius Partitions(call F-partition) in two steps [Andrews(1984)].

*Proof.* (Step1) We begin with rewriting Jacobi Triple Product (1.1) as follows:

$$\frac{1}{(q;q)_\infty}\sum_{m=-\infty}^{\infty} z^m q^{\binom{m+1}{2}} = (-zq, -z^{-1}; q)_\infty. \tag{4.1}$$

Let $\phi(z)$ denote the right hand side of (4.1) and observe

$$\phi(zq) = \frac{1}{zq}\phi(z). \tag{4.2}$$

$\phi(z)$ can be expressed as a Laurent series in a deleted neighborhood of $z = 0$, i.e.,

$$\phi(z) = \sum_{n=-\infty}^{\infty} A_n(q) z^n. \tag{4.3}$$

Plug (4.3) into (4.2) and find the coefficients of $z^n$,

$$\phi(z) = \sum_{n=-\infty}^{\infty} A_n(q)z^n = zq \sum_{n=-\infty}^{\infty} A_n(q)z^n q^n = \sum_{n=-\infty}^{\infty} A_n(q)z^{n+1}q^{n+1}. \tag{4.4}$$

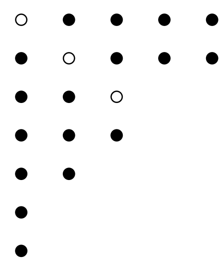
Then, for all $n \in \mathbb{Z}$

$$A_n(q) = q^n A_{n-1}(q) = \cdots = q^{\frac{n(n+1)}{2}} A_0(q) \quad \Rightarrow \phi(z) = \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} A_0(q) z^n.$$

(Step2) What now remains is to check if the constant terms in each side of (4.1) are equal. For this purpose, Andrews used the method of Frobenius partition. First, introduce Frobenius symbol which expresses each partition $\pi$ of $n$ bijectively as

$$\pi = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$$

as follows where $a_1 > a_2 > \cdots > a_r \geq 0,\ b_1 > b_2 \cdots > b_r \geq 0$, such that $n = r + \sum a_i + \sum b_i$. For example, if $\pi = 5+5+3+3+2+1+1$, we can draw its corresponding ferrers diagram like the below and its F-symbol is shown as:



$$\Rightarrow \quad \pi = \begin{pmatrix} 4 & 3 & 0 \\ 6 & 3 & 1 \end{pmatrix}$$

To compute the constant term of $\phi(z)$, multiply r terms in the first product times r terms in the second product from $\prod_{n=1}^{\infty}(1+zq^n)\prod_{m=1}^{\infty}(1+q^{m-1}/z)$. Then, for positive and distinct $a_i$s, together with nonnegative and distinct $b_i$s , the constant term becomes

$$q^{a_1+\cdots+a_r}q^{b_1+\cdots+b_r}.$$

This corresponds to the F-partition with strict decrease in rows such that

$$\begin{pmatrix} a_1-1 & a_2-1 & \cdots & a_r-1 \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

where $n = r + \sum_{i=1}^{r}(a_i-1) + \sum_{i=1}^{r} b_i$ , and $a_1-1 > a_2-1 > \cdots > a_r-1 \geq 0$, and $b_1 > b_2 \cdots > b_r \geq 0$. This implies that the constant term can be interpreted as the generating function for the partition function of $n$, $p(n)$ i.e.,

$$\sum_{n=0}^{\infty} p(n)q^n = \prod \frac{1}{1-q^n} := \frac{1}{(q;q)_\infty}. \tag{4.5}$$

□

## 4.2 Generalized Frobenius Partitions

As shown in the example of the proof of Jacobi Triple Product, in general we can use F-partition to study the generating functions of constant terms of certain infinite products. Let us begin with constructing the generating function of constant terms. Let $f_A(z,q)$ be the generating function for the number of partitions of $n$ into $m$ parts subject to the set of restrictions $A$ such that

$$f_A(z,q) = f_A(z) = \sum P_A(m,n) z_m q^n. \tag{4.6}$$

Then, define $\Phi_{A,B}(q)$ as the generating function of the constant term of $f_A(zq)f_B(z^{-1})$ as follows :

$$\Phi_{A,B}(q) = \sum_{n\geq 0} \phi_{A,B}(n) q^n, \tag{4.7}$$

where $\phi_{A,B}(n)$ is the number of F-partitions,

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$$

with the first row subject to the set of restrictions $A$ and the second row to $B$. If $A_k$ means 'each integer repeated up to $k$ times', then $\phi_{A_1,A_1}(n) = p(n)$ as seen in (4.5). We can consider two gereralized classes of F-partitions. The first of these is F-partition with up to $k$ repetitions of an integer in any row with $a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$ and $b_1 \geq b_2 \geq \cdots \geq b_r \geq 0$. Thus the number of such F-partitions will be

$$\Phi_{A_k,A_k}(q) = \sum_{n\geq 0} \phi_{A_k,A_k}(n) q^n, \tag{4.8}$$

where $A_k$ denotes the condition 'each part repeated at most $k$ times'. For example, the F-partitions for $\phi_{A_2,A_2}(3) := \phi_2(3)$ are

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and $\Phi_2(q) = 1 + q + 3q^2 + 5q^3 + 9q^4 + 14q^5 + 24q^6 + 35q^7 + 55q^8 + \cdots$ .

The second class of generlized F-partitions is F-partition with $A_k$ as the condition of each distinct part taken from $k$ copies of the integers. Thus,

$$C\Phi_{A_k,A_k}(q) = \sum_{n\geq 0} c\phi_{A_k,A_k}(n) q^n \tag{4.9}$$

where we can consider $c\phi_{A_k,A_k}(n)$ as the number of F-partitions of $n$ with $k$ colors. For example, the F-partition enumerated by $c\phi_2(2)$ are

$$\begin{pmatrix} 1_1 \\ 0_1 \end{pmatrix}, \begin{pmatrix} 1_2 \\ 0_1 \end{pmatrix}, \begin{pmatrix} 1_1 \\ 0_2 \end{pmatrix}, \begin{pmatrix} 1_2 \\ 0_2 \end{pmatrix}, \begin{pmatrix} 0_1 \\ 1_1 \end{pmatrix}, \begin{pmatrix} 0_1 \\ 1_2 \end{pmatrix}, \begin{pmatrix} 0_2 \\ 1_1 \end{pmatrix}, \begin{pmatrix} 0_2 \\ 1_2 \end{pmatrix}, \begin{pmatrix} 0_2 & 0_1 \\ 0_2 & 0_1 \end{pmatrix},$$

where the subscripts represent different colors of each integer.
The general formulas for $\Phi_{A_k,A_k}(q)$ and $C\Phi_{A_k,A_k}(q)$ are as follows:

**Theorem 26.** *[Andrews(1984)]*

$$\Phi_{A_k,A_k}(q) = \Phi_k(q) = \frac{\sum\limits_{m_1,\cdots,m_{k-1}=-\infty}^{\infty} \xi^{L(m_1,\cdots,m_{k-1})} q^{Q(m_1,\cdots,m_{k-1})}}{{(q;q)_\infty}^k} \tag{4.10}$$

$$C\Phi_{A_k,A_k}(q) = C\Phi_k(q) = \frac{\sum\limits_{m_1,\cdots,m_{k-1}=-\infty}^{\infty} q^{Q(m_1,\cdots,m_{k-1})}}{{(q;q)_\infty}^k} \tag{4.11}$$

*where* $\xi = e^{\frac{2\pi i}{k+1}}$ *and*

$$\begin{aligned} L(m_1,\cdots,m_{k-1}) &= (k-1)m_1 + (k-2)m_2 + \cdots + m_{k-1}, \\ Q(m_1,\cdots,m_{k-1}) &= \sum_{i=1}^{k-1} {m_i}^2 + \sum_{1\le i<j\le k-1} m_i m_j. \end{aligned} \tag{4.12}$$

*Proof.* $\Phi_k(q)$ is the generating function of the constant term in

$$\prod_{n=1}^{\infty} (1 + zq^n + \cdots + z^k q^{kn})(1 + z^{-1}q^{n-1} + \cdots + z^{-k}q^{k(n-1)}).$$

By using Jacobi Triple Product, we can express (4.12) as

$$\frac{1}{{(q;q)_\infty}^k} \prod_{j=1}^{k} \sum_{m_j=-\infty}^{\infty} (-1)^{m_j} q^{\binom{m_j+1}{2}} z^{m_j} \xi^{j m_j}.$$

Next, since $m_1 + \cdots + m_k = 0$, $\Phi_k(q)$ can be written as

$$\frac{1}{(q;q)_\infty} \sum_{\substack{m_1,\cdots,m_{k-1}\\=-\infty}}^{\infty} \xi^{m_1+\cdots+(k-1)m_{k-1}-k(m_1+\cdots+m_{k-1})} q^{\binom{m_1+1}{2}+\cdots+\binom{m_{k-1}+1}{2}+\binom{-m_1-\cdots-m_{k-1}+1}{2}} \tag{4.13}$$

$C\Phi_k(q)$ is the generating function of the constant term in

$$\prod_{n=1}^{\infty} (1+zq^n)^k (1+z^{-1}q^{n-1})^k = \frac{1}{{(q;q)_\infty}^k} \prod_{j=1}^{k} \sum_{m_j=-\infty}^{\infty} (-1)^{m_j} q^{\binom{m_j+1}{2}} z^{m_j}.$$

The remaining part of the proof is similar to (4.13). □

**Remark 27.** *[Andrews(1998)] Let us denote by* $p(N, M, n)$ *the number of the partitions of* $n$ *into at most* $M$ *parts where each part* $\leq N$*. If* $G(N, M; q)$ *is the generating function of* $p(N, M, n)$*, then*

$$G(N, M; q) = \sum_{n \geq 0} p(N, M, n) q^n = \begin{bmatrix} M + N \\ N \end{bmatrix}. \tag{4.14}$$

*Andrews also pointed out* $p(N, M, n)$ *could be considered as the number of F-partitions of* $n$ *with distinct parts in each row with each top entry is* $< m$ *and each bottom entry is* $< n$*.*

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