Prof. O. Stern
with the kindest regards
of the author

CONGRÈS INTERNATIONAL DES SCIENCES ANTROPOLOGIQUES ET ETHNOLOGIQUES

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(Extrait)

NATURAL PHILOSOPHY AND HUMAN CULTURES

By Prof. *Niels Bohr* (Copenhagen).

IT is only with great hesitation that I have accepted a kind invitation to address this assembly of distinguished representatives of the anthropological and ethnographical sciences of which I, as a physicist, have of course no first-hand knowledge. Still, on this special occasion when even the historical surroundings speak to everyone of us about aspects of life other than those discussed at the regular congress proceedings, it might perhaps be of interest to try with a few words to draw your attention to the epistemological aspect of the latest development of natural philosophy and its bearing on general human problems. Notwithstanding the great separation between our different branches of knowledge, the new lesson which has been impressed upon physicists regarding the caution with which all usual conventions must be applied as soon as we are not concerned with everyday experience, may, indeed, be suited to remind us in a novel way of the dangers, well known to humanists, of judging from our own standpoint cultures developed within other societies.

Of course it is impossible to distinguish sharply between natural philosophy and human culture. The physical sciences are, in fact, an integral part of our civilization, not only because our ever increasing mastery of the forces of nature has so completely changed the material conditions of life, but also because the study of these sciences has contributed so much to clarify the background of our own existence. What has it not meant in this respect that we no more consider ourselves as privileged in living at the centre of the universe, surrounded by less fortunate societies inhabiting the edges of the abyss, but that through the development of astronomy and geography we have realized that we are all sharing a small spherical planet of the solar system which again is only a small part of still larger systems. How forceful an admonition about the relativity of all human judgements have we not also in our days received through the renewed revision of the presuppositions underlying the unambiguous use of even our most elementary concepts such as space and time, which, in disclosing the essential dependence of every physical phenomenon on the standpoint of the observer, has contributed so largely to the unity and beauty of our whole world-picture.

While the importance of these great achievements for our general outlook is commonly realized, it is hardly yet so as regards the unsuspected epistemological lesson which the opening of quite new realms of physical research has given us in the latest years. Our penetration into the world of atoms, hitherto closed to the eyes of man, is indeed an adventure which may be compared with the great journeys of discovery of the circumnavigators and the bold explorations of astronomers into the depths of celestial space. As is well known, the marvellous development of the art of physical experimentation has not only removed the last traces of the old belief that the coarseness of our senses would for ever prevent us from obtaining direct information about individual atoms, but has even shown us that the atoms themselves consist of still smaller corpuscles which can be isolated and the properties of which can be investigated separately. At the same time we have, however, in this fascinating field of experience been taught that the laws of nature hitherto known, which constitute the grand edifice of classical physics, are only valid when dealing with bodies consisting of practically infinite numbers of atoms. The new knowledge concerning the behaviour of single atoms and atomic corpuscles has, in fact, revealed an unexpected limit for the subdivision of all physical actions extending far beyond the old doctrine of the limited divisibility of matter and giving every atomic process a peculiar individual character. This discovery has, in fact, yielded a quite new basis for the understanding of the intrinsic stability of atomic structures, which, in the last resort, conditions the regularities of all ordinary experience.

How radical a change in our attitude towards the description of nature this development of atomic physics has brought about is perhaps most clearly illustrated by the fact that even the principle of

causality, so far considered as the unquestioned foundation for all interpretation of natural phenomena, has proved too narrow a frame to embrace the peculiar regularities governing individual atomic processes. Certainly everyone will understand that physicists have needed very cogent reasons to renounce the ideal of causality itself; but in the study of atomic phenomena we have repeatedly been taught that questions which were believed to have received long ago their final answers had most unexpected surprises in store for us. You will surely all have heard about the riddles regarding the most elementary properties of light and matter which have puzzled physicists so much in recent years. The apparent contradictions which we have met in this respect are, in fact, as acute as those which gave rise to the development of the theory of relativity in the beginning of this century and have, just as the latter, only found their explanation by a closer examination of the limitation imposed by the new experiences themselves on the unambiguous use of the concepts entering into the description of the phenomena. While in relativity theory the decisive point was the recognition of the essentially different ways in which observers moving relatively to each other will describe the behaviour of given objects, the elucidation of the paradoxes of atomic physics has disclosed the fact that the unavoidable interaction between the objects and the measuring instruments sets an absolute limit to the possibility of speaking of a behaviour of atomic objects which is independent of the means of observation.

We are here faced with an epistemological problem quite new in natural philosophy, where all description of experiences has so far been based upon the assumption, already inherent in ordinary conventions of language, that it is possible to distinguish sharply between the behaviour of objects and the means of observation. This assumption is not only fully justified by all everyday experience but even constitutes the whole basis of classical physics, which, just through the theory of relativity, has received such a wonderful completion. As soon as we are dealing, however, with phenomena like individual atomic processes which, due to their very nature, are essentially determined by the interaction between the objects in question and the measuring instruments necessary for the definition of the experimental arrangements, we are, therefore, forced to examine more closely the question of what kind of knowledge can be obtained concerning the objects. In this respect we must, on the one hand, realize that the aim of every physical experiment — to gain knowledge under reproducible and communicable conditions — leaves us no choice but to use everyday concepts, eventually refined by the terminology of classical physics, not only in all accounts of the construction and manipulation of the measuring instruments but also in the description of the actual experimental results. On the other hand, it is equally important to understand that just this circumstance implies that no result of an experiment concerning a phenomenon which, in principle, lies outside the range of classical physics, can be interpreted as giving information about independent properties of the objects, but is inherently connected with a definite situation in the description of which the measuring instruments interacting with the objects also enter essentially. This last fact gives the straightforward explanation of the apparent contradictions which appear when results about atomic objects obtained by different experimental arrangements are tentatively combined into a self-contained picture of the object.

Information regarding the behaviour of an atomic object obtained under definite experimental conditions, may, however, according to a terminology often used in atomic physics, be adequately characterized as *complementary* to any information about the same object obtained by some other experimental arrangement excluding the fulfilment of the first conditions. Although such kinds of information cannot be combined into a single picture by means of ordinary concepts, they represent indeed equally essential aspects of any knowledge of the object in question which can be obtained in this domain. Just the recognition of such a complementary character of the mechanical analogies by which one has tried to visualize the individual radiative effects has, in fact, led to an entirely satisfactory solution of the riddles of the properties of light alluded to above. In the same way it is only by taking into consideration the complementary relationship between the different experiences concerning the behaviour of atomic corpuscles that it has been possible to obtain a clue to the understanding of the striking contrast between the properties of ordinary mechanical models and the peculiar laws of stability governing atomic structures which form the basis for every closer explanation of the specific physical and chemical properties of matter.

Of course I have no intention, on this occasion, of entering more closely into such details, but I hope that I have been able to give you a sufficiently clear impression of the fact that we are here in no way

concerned with an arbitrary renunciation as regards the detailed analysis of the almost overwhelming richness of our rapidly increasing experience in the realm of atoms. On the contrary, we have to do with a rational development of our means of classifying and comprehending new experience which, due to its very character, finds no place within the frame of causal description that is only suited to account for the behaviour of objects as long as this behaviour is independent of the means of observation. Far from containing any mysticism contrary to the spirit of science, the view-point of "complementarity" forms indeed a consistent generalization of the ideal of causality.

However unexpected this development may appear in the domain of physics, I am sure that many of you will have recognized the close analogy between the situation as regards the analysis of atomic phenomena, which I have described, and characteristic features of the problem of observation in human psychology. Indeed, we may say that the trend of modern psychology can be characterized as a reaction against the attempt at analyzing psychical experience into elements which can be associated in the same way as are the results of measurements in classical physics. In introspection it is clearly impossible to distinguish sharply between the phenomena themselves and their conscious perception, and although we may often speak of lending our attention to some particular aspect of a psychical experience, it will appear on closer examination that we really have to do, in such cases, with mutually exclusive situations. We all know the old saying that, if we try to analyze our own emotions, we hardly possess them any longer, and in that sense we recognize between psychical experiences, for the description of which words such as "thoughts" and "feelings" are adequately used, a complementary relationship similar to that between the experiences regarding the behaviour of atoms obtained under different experimental arrangements and described by means of different analogies taken from our usual ideas. By such a comparison it is, of course, in no way intended to suggest any closer relation between atomic physics and psychology, but merely to stress an epistemological argument common to both fields, and thus to encourage us to see how far the solution of the relatively simple physical problems may be helpful in clarifying the more intricate psychological questions with which human life confronts us, and which anthropologists and ethnologists so often meet in their investigations.

Coming now closer to our subject of the bearing of such view-points on the comparison of different human cultures, we shall first stress the typical complementary relationship between the modes of behaviour of living beings characterized by the words "instinct" and "reason". It is true that any such words are used in very different senses; thus instinct may mean motive power or inherited behaviour, and reason may denote deeper sense as well as conscious argumentation. What we are concerned with is, however, only the practical way in which these words are used to discriminate between the different situations in which animals and men find themselves. Of course, nobody will deny our belonging to the animal world; and it would even be very difficult to find an exhaustive definition characterizing man among the other animals. Indeed, the latent possibilities in any living organism are not easily estimated, and I think that there is none of us who has not sometimes been deeply impressed by the extent to which circus animals can be drilled. Not even with respect to the conveyance of information from one individual to another would it be possible to draw a sharp separation between animals and man; but of course our power of speech places us in this respect in an essentially different situation, not only as regards the exchange of practical experience, but above all as regards the possibility of transmitting through education to children the traditions concerning behaviour and reasoning which form the basis of any human culture.

As regards reason compared with instinct, it is, above all, essential to realize that no proper human thinking is imaginable without the use of concepts framed in some language which every generation has to learn anew. This use of concepts is, in fact, not only to a large extent suppressing instinctive life, but stands even largely in an exclusive relationship of complementarity to the display of inherited instincts. The astonishing superiority of lower animals compared with man in utilizing the possibilities of nature for the maintenance and propagation of life has certainly often its true explanation in the fact that, for such animals, we cannot speak af any conscious thinking in our sense of the word. At the same time the amazing capacity of so-called primitive people to orientate themselves in forests or deserts, which, though apparently lost in more civilized societies, may on occasion be revived in any of us, might justify the conclusion that such feats are only possible when no recourse is taken to conceptual thinking, which on its side is adapted to far more varied purposes of primary importance

for the development of civilization. Just because it is not yet awake to the use of concepts, a new-born child can hardly be reckoned as a human being; but belonging to the species of man, it has, of course, though more helpless a creature than most young animals, the organic possibilities of receiving through education a culture which enables it to take its place in some human society.

Such considerations confront us at once with the question whether the widespread belief that every child is born with a predisposition for the adoption of a specific human culture is really well-founded, or whether one has not rather to assume that any culture can be implanted and thrive on quite different physical backgrounds. Here we are of course touching a subject of still unsettled controversies between geneticists, who pursue most interesting studies on the inheritance of physical characters. In connection with such discussions, however, we must above all bear in mind that the distinction between the concepts genotype and phaenotype, so fruitful for the clarification of heredity in plants and animals, essentially presupposes the subordinate influence of the external conditions of life on the characteristic properties of the species. In the case of the specific cultural characters of human societies the problem is, however, reversed in the sense that the basis for the classification is here the traditional habits shaped by the histories of the societies and their natural environments. These habits, as well as their inherent presuppositions, must therefore be analyzed in detail before any possible influence of inherited biological differences on the development and maintenance of the cultures concerned can be estimated. Indeed, in characterizing different nations and even different families within a nation, we may to a large extent consider anthropological traits and spiritual traditions as independent of each other, and it would even be tempting to reserve by definition the adjective "human" for just those characters which are not directly bound to bodily inheritance.

At first sight, it might perhaps appear that such an attitude would mean unduly stressing merely dialectic points. But the lesson which we have received from the whole growth of the physical sciences is that the germ of fruitful development often lies just in the proper choice of definitions. When we think for instance of the clarification brought about in various branches of science by the argumentation of relativity theory we see indeed what advance may lie in such formal refinements. As I have already hinted at earlier in this address, relativistic view-points are certainly also helpful in promoting a more objective attitude as to relationships between human cultures, the traditional differences of which in many ways resemble the different equivalent manners in which physical experience can be described. Still, this analogy between physical and humanistic problems is of limited scope and its exaggeration has even led to misunderstandings of the essence of the theory of relativity itself. The unity of the relativistic world picture, in fact, just implies the possibility for any one observer to predict within his own conceptual frame how any other observer will coordinate experience within the frame natural to him. The main obstacle to an unprejudiced attitude towards the relation between various human cultures is, however, the deep-rooted differences of the traditional backgrounds on which the cultural harmony in different human societies is based and which exclude any simple comparison between such cultures.

It is above all in this connection that the view-point of complementarity offers itself as a means of coping with the situation. In fact, when studying human cultures different from our own, we have to deal with a particular problem of observation which on closer consideration shows many features in common with atomic or psychological problems, where the interaction between objects and measuring tools, or the inseparability of objective content and observing subject, prevents an immediate application of the conventions suited to accounting for experiences of daily life. Especially in the study of cultures of primitive peoples, ethnologists are, indeed, not only aware of the risk of corrupting such cultures by the necessary contact, but are even confronted with the problem of the reaction of such studies on their own human attitude. What I here allude to is the experience, well known to explorers, of the shaking of their hitherto unrealized prejudices through the experience of the unsuspected inner harmony human life can present even under conventions and traditions most radically different from their own. As a specially drastic example I may perhaps here remind you of the extent to which in certain societies the roles of men and women are reversed, not only regarding domestic and social duties but also regarding behaviour and mentality. Even if many of us, in such a situation, might perhaps at first shrink from admitting the possibility that it is entirely a caprice of fate that the people concerned

have their specific culture and not ours, and we not theirs instead of our own, it is clear that even the slightest suspicion in this respect implies a betrayal of the national complacency inherent in any human culture resting in itself.

Using the word much as it is used, in atomic physics, to characterize the relationship between experiences obtained by different experimental arrangements and visualizable only by mutually exclusive ideas, we may truly say that different human cultures are complementary to each other. Indeed, each such culture represents a harmonious balance of traditional conventions by means of which latent potentialities of human life can unfold themselves in a way which reveals to us new aspects of its unlimited richness and variety. Of course, there cannot, in this domain, be any question of such absolutely exclusive relationships as those between complementary experiences about the behaviour of well-defined atomic objects, since hardly any culture exists which could be said to be fully self-contained. On the contrary, we all know from numerous examples how a more or less intimate contact between different human societies can lead to a gradual fusion of traditions, giving birth to a quite new culture. The importance in this respect of the mixing of populations through emigration or conquest for the advancement of human civilization needs hardly be recalled. It is, indeed, perhaps the greatest prospect of humanistic studies to contribute through an increasing knowledge of the history of cultural development to that gradual removal of prejudices which is the common aim of all science.

As I stressed in the beginning of this address, it is, of course, far beyond my capacities to contribute in any direct way to the solution of the problems discussed among the experts at this congress. My only purpose has been to give an impression of a general epistemological attitude which we have been forced to adopt in a field as far from human passions as the analysis of simple physical experiments. I do not know, however, whether I have found the right words to convey to you this impression, and before I conclude, I may perhaps be allowed to relate an experience which once most vividly reminded me of my deficiencies in this respect. In order to explain to an audience that I did not use the word prejudice to imply any condemnation of other cultures, but merely to characterize our necessarily prejudiced conceptual frame, I referred jokingly to the traditional prejudices which the Danes cherish with regard to their Swedish brothers on the other side of the beautiful Sound outside these windows, with whom we have fought through centuries even within the walls of this castle, and from contact with whom we have, through the ages, received so much fruitful inspiration. Now you will realize what a shock I got when, after my address, a member of the audience came up to me and said that he could not understand why I hated the Swedes. Obviously I must have expressed myself most confusingly on that occasion, and I am afraid that also to-day I have talked in a very obscure way. Still, I hope that I have not spoken so unclearly as to give rise to any such misunderstandings of the trend of my argument.

On the Bose-Einstein Condensation

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On the Bose-Einstein Condensation

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A proof is given of the condensation phenomenon of a Bose-Einstein gas. A preliminary discussion of its transport properties is outlined with a view to its possible bearing on the problem of liquid helium.

Introduction

In his well-known papers[1] on the degeneracy of an ideal gas, Einstein mentioned a peculiar condensation phenomenon of the ideal "Bose-Einstein" gas. This very interesting discovery, however, has not appeared in the textbooks, probably because Uhlenbeck in his thesis[2] questioned the correctness of Einstein's argument. Since, from the very first, the mechanism appeared to be devoid of any practical significance, all real gases being condensed at the temperature in question, the matter has never been examined in detail; and it has been generally supposed that there is no such condensation phenomenon.

In discussing some properties of liquid helium, I recently realized that Einstein's statement has been erroneously discredited; moreover, some support could be given to the idea that the peculiar phase transition ("λ-point"), that liquid helium undergoes at 2.19°K, very probably has to be regarded as the condensation phenomenon of the Bose-Einstein statistics, distorted, of course, by the presence of molecular forces and by the fact that it manifests itself in the liquid and not in the gaseous state. In a preliminary note,[3] the course of the specific heat of an ideal Bose-Einstein gas was reproduced, but no proof was communicated. As, since then, I have been asked several times for a proof and as even the correctness of the result has been questioned anew, it might perhaps be justified, on this occasion to publish a quite elementary demonstration of the condensation mechanism, discussing only briefly here the possible connection of the Bose-Einstein degeneracy with the problem of liquid helium.[4]

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[1] A. Einstein, Ber. Berl. Akad. 261 (1924); 3 (1925).
[2] G. E. Uhlenbeck, *Dissertation* (Leiden, 1927).
[3] F. London, Nature **141**, 643 (1938).
[4] In a recent paper Uhlenbeck has withdrawn his former objection. G. E. Uhlenbeck and B. Kahn, Physica **5**, 399 (1938).

§1. The Degeneracy of the Bose-Einstein Gas

We start from the well-known fundamental formula for the most probable distribution in a Bose-Einstein gas:

$$N_i = \frac{g_i}{e^{\beta\epsilon_i+\alpha}-1}.$$

Here $\beta = 1/kT$, g_i = statistical weight of the state of energy ϵ_i; the parameter α has to be determined as a function of T by the condition

$$N = \sum_i N_i = \sum_i \frac{g_i}{e^{\beta\epsilon_i+\alpha}-1}. \qquad (1)$$

N = total number of particles. It may be mentioned that α is proportional to Gibb's potential ζ; it is $\zeta = -\alpha kT$. For free particles of the mass M, without spin, in a given volume V, one usually assumes

$$\epsilon_p = (p^2/2M), \quad g(p)dp = (4\pi V/h^3)p^2 dp$$

and obtains

$$N = \frac{4\pi V}{h^3}\int_0^\infty \frac{p^2 dp}{e^{\alpha+p^2/2MkT}-1} = \frac{V}{h^3}(2\pi MkT)^{3/2}\cdot F(\alpha), \qquad (2)$$

where

$$F(\alpha) = \frac{2}{\pi^{1/2}}\int_0^\infty \frac{z^{1/2}dz}{e^{\alpha+z}-1} = e^{-\alpha} + \frac{e^{-2\alpha}}{2^{3/2}} + \frac{e^{-3\alpha}}{3^{3/2}} + \frac{e^{-4\alpha}}{4^{3/2}} + \cdots.$$

Here α must be positive; otherwise some of the N_i would be negative, which, of course, is not admissible. Now $F(\alpha)$ is a monotonously decreasing function, its maximum value being $F(0) = 2.612$. Therefore, no solution $\alpha(T)$ of Eq. (2) can be found for

$$N > \frac{V(2\pi MkT)^{3/2}}{h^3}2.612,$$

i.e., for $\quad T < T_0,$

where
$$T_0 = \left(\frac{N}{2.612V}\right)^{2/3}\frac{h^2}{2\pi Mk}. \qquad (3)$$

For M = mass of the He-atom and for a molar volume of 27.6 cm³ one obtains $T_0 = 3.13°K$.

Equation (1), however, obviously always has a solution $\alpha(T)$, for each temperature T, with positive N_i. The difference of the behavior of Eqs. (1) and (2) comes from the neglect caused by the substitution of the sum by an integral. This can easily be seen if one takes, e.g., the discontinuous energy values of a cubic volume of linear dimension L (periodic boundary conditions):

$$\epsilon_{k,l,m} = (k^2+l^2+m^2)(h^2/2ML^2)$$
$$k, l, m = 0, \pm 1, \pm 2, \cdots$$

and substitutes them into (1):

$$N = \sum_{k,l,m}\frac{1}{e^{\sigma(k^2+l^2+m^2)+\alpha}-1}, \qquad (1a)$$

where $\quad \sigma = \dfrac{h^2}{2L^2MkT} = \dfrac{1.90\pi}{N^{2/3}}\cdot\dfrac{T_0}{T}.$

Here the right-hand side is again a monotonously decreasing function of α, which, however, is not bounded by a finite limit when $\alpha \to 0$. Consequently Eq. (1′) has, for each temperature T, a solution $\alpha(T)$.[5] The difference between (1a) and (2) is due to the fact that in (2) the lowest state $k = l = m = 0$ (or $p = 0$), incorrectly acquires the statistical weight zero, and therefore it becomes entirely suppressed. It is just this *one* term in the sum (1a) which is decisive for the behavior in the limit $\alpha \to 0$, *viz.*, at lowest temperatures. It will therefore be sufficient to treat this lowest term of the sum separately; the other terms may, without detriment, be replaced by an integral.

We collect the states with $k^2+l^2+m^2 \leqslant \rho^2$ where the value of ρ may be chosen in such a way, that the difference between the integral from ρ to ∞ and the sum from ρ to ∞ is negligible (say $\rho \sim 10$). The mean energy of these states is

$$\bar{\epsilon}_0 = \frac{3}{10}\frac{\rho^2 h^2}{ML^2} = \frac{3}{5}\epsilon \quad \text{where} \quad \epsilon = \frac{h^2\rho^2}{2ML^2}$$

and their number is given by

$$g_0 = (4\pi/3)\rho^3.$$

[5] From this fact Uhlenbeck[2] was led to the conclusion that no condensation phenomenon should exist.

Then we may write, in a better approximation than (2),

$$N = \frac{g_0}{e^{\beta\bar{\epsilon}_0+\alpha}-1} + \frac{4\pi V}{h^3}\int_{\rho h/L}^\infty \frac{p^2dp}{e^{\alpha+p^2/2MkT}-1}$$

or with $\alpha' = \beta\bar{\epsilon}_0 + \alpha = \frac{3}{5}\beta\epsilon + \alpha$:

$$1 = \frac{g_0/N}{e^{\alpha'}-1} + \left(\frac{T}{T_0}\right)^{3/2}\frac{2}{\pi^{1/2}\cdot F(0)} \times \int_{\beta\epsilon}^\infty \frac{z^{1/2}dz}{e^{z+\alpha'-3/5\cdot\beta\epsilon}-1}. \qquad (4)$$

The first term of the right-hand side must not be negative, since it gives the number of atoms in the cell of the g_0 lowest states. Therefore in any case $\alpha' > 0$. It may be remarked that the quantity $\beta\epsilon = (h^2\rho^2/2ML^2kT)$ is always a very small number $[= (5.97/N^{2/3})\rho^2(T_0/T)]$.

Now we have to distinguish two entirely different cases:

(1) $T > T_0$. In this case α' comes out to be a number of the order of magnitude 1. Consequently, the first term of the right-hand side of (4) is of the order $1/N$ and may be neglected compared with the second term. In the latter we may neglect $\beta\epsilon \ll \alpha'$ and obtain

$$(T_0/T)^{3/2} = [F(\alpha')/F(0)] \quad \text{for } T > T_0, \qquad (4a)$$

which is equivalent to (2).

(2) $T < T_0$. The second term of the right-hand side of (4) is in any case smaller than $(T/T_0)^{3/2}$. In particular for $T < T_0$, it is smaller than 1. Thus the first term is required to make up for the rest; consequently, since the numerator of this term is very small, its denominator must be equally small, *viz.*, of the order $1/N$, this means $\alpha' \sim 1/N$; in the second term α' may now be neglected compared with $\beta\epsilon \sim 1/N^{2/3}$ and we can write for (4):

$$1 = \frac{g_0}{N\alpha'} + (T/T_0)^{3/2}\frac{2}{\pi^{1/2}F(0)}\int_{\beta\epsilon}^\infty \frac{z^{1/2}dz}{e^{z-3/5\cdot\beta\epsilon}-1}$$
$$= \frac{g_0}{N\alpha'} + (T/T_0)^{3/2}[1 - 0.173(\beta\epsilon)^{1/2} + \cdots] \qquad (4b)$$

or

$$\alpha' = \frac{g_0}{N}[1 - (T/T_0)^{3/2}(1 - 0.173(\beta\epsilon)^{1/2} + \cdots)]^{-1}, \qquad (5)$$

valid as long as

$$N[1 - (T/T_0)^{3/2}] \gg 1 \quad \text{or for} \quad T_0 - T \gg \frac{2}{3N}T_0,$$

i.e., practically for all temperatures $T < T_0$.

The number of particles in the lowest cell becomes

$$N_0 = N[1 - (T/T_0)^{3/2}(1 - 0.173(\beta\epsilon)^{1/2} + \cdots)]$$
$$= N[1 - (T/T_0)^{3/2}] + 0.422\rho N^{2/3}T/T_0 + \cdots.$$

We see that in first approximation this expression is independent of ρ, i.e., independent of $g_0 = (4\pi/3)\rho^3$, the number of states we have collected in the lowest cell. Therefore, $N(1 - (T/T_0)^{3/2})$ atoms, i.e., *a finite fraction of all atoms* will be assembled in the *one* lowest state. In the spherical shell of radius ρ and thickness $d\rho$ around the lowest state, there will be only 0.422 $N^{2/3}T/T_0 d\rho$ molecules.

Accordingly, the distribution among the lowest quantum states, characterized by the quantum numbers k, l, m (see (1a)), may be written:

$$N_{k,l,m} = \frac{N[1 - (T/T_0)^{3/2}]}{1 + (k^2+l^2+m^2)N^{1/3}29.8T_0/T[1 - (T/T_0)^{3/2}]}.$$

If N is finite, the function $\alpha(T)$ is, of course, an analytic function. However, for increasing values of N, the third derivative of α, near $T = T_0$, becomes greater and greater. In the limit $N \to \infty$, N/V = constant, the function $\alpha(T)$ has a discontinuous second derivative. It consists of two branches which do not cohere analytically. One gets $\alpha = 0$ for $T \leqslant T_0$, while for $T \geqslant T_0$ the function $\alpha(T)$ is given by the inversion of (4a). Taking N = Avogadro's number, and V = molecular volume, we may, for $T < T_0$, write simply:

$$N_0 = N[1 - (T/T_0)^{3/2}]$$
(number of atoms in the lowest state) (6a)

$$N(E)dE = (2\pi V/h^3)(2M)^{3/2}(E^{1/2}dE/(e^{\beta E}-1)).$$
(number of atoms of the interval $(E, E+dE)$ where $E > 0$) (6b)

Therefrom we obtain the energy U per molecular

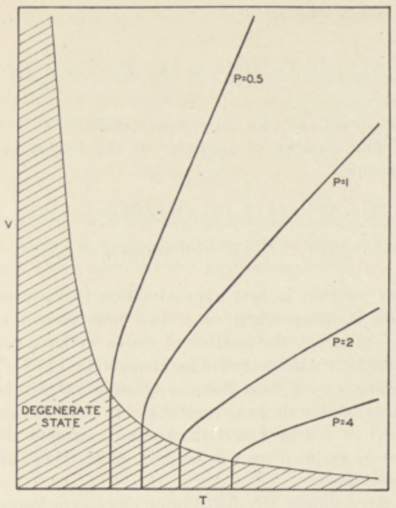

FIG. 1. Isobars of a Bose-Einstein gas.

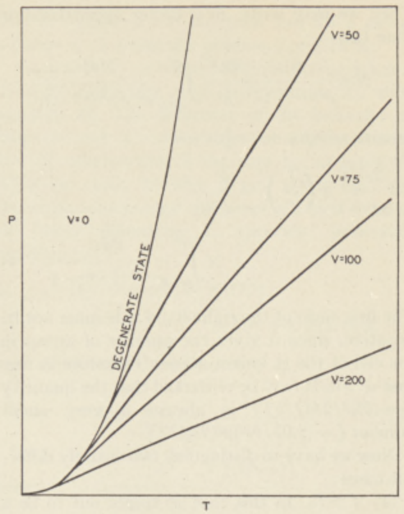

FIG. 2. Isochores of a Bose-Einstein gas.

volume and for $T<T_0$:

$$U_-=\int_0^\infty EN(E)dE=\frac{3}{2}RT\left(\frac{T}{T_0}\right)^{3/2}\frac{\zeta(2.5)}{\zeta(1.5)}$$

$$=\frac{3}{2}0.514RT\left(\frac{T}{T_0}\right)^{3/2}=\frac{3}{2}0.514RCT^{5/2}V, \quad (7a)$$

where $C=\frac{2.612}{N}\left(\frac{2\pi Mk}{h^2}\right)^{3/2}\approx\frac{1}{153}$ for He

and $\zeta(k)=\frac{1}{(k-1)!}\int_0^\infty\frac{x^{k-1}dx}{e^x-1}$.

denotes the Riemann zeta-function ($\zeta(\frac{3}{2})=2.612$; $\zeta(\frac{5}{2})=1.341$).

For $T\geqslant T_0$, Einstein has previously given the semi-convergent expansion:

$$U_+=\frac{3RT}{2}\{1-0.462(T_0/T)^{3/2}$$

$$-0.0225(T_0/T)^3-0.0114(T_0/T)^{9/2}-\cdots\}. \quad (7b)$$

The two branches U_+ and U_- are continuous at $T=T_0$ with a continuous tangent, but the second derivative is discontinuous. Thus the specific heat c_v has a *break* at $T=T_0$ (see Fig. 4):

$$c_{v-}=\frac{15}{4}0.514R(T/T_0)^{3/2} \quad (8a)$$

$$c_{v+}=\tfrac{3}{2}R\{1+0.231(T_0/T)^{3/2}$$

$$+0.045(T_0/T)^3+0.040(T_0/T)^{9/2}+\cdots\}. \quad (8b)$$

The free energy

$$F=-T\int_0^T\frac{U}{T^2}dT$$

is found to be:

$$F_-=-0.514RT(T/T_0)^{3/2}=-0.514CRT^{5/2}V \quad (9a)$$

$$F_+=-\tfrac{3}{2}RT\{\ln(T/T_0)+0.308(T_0/T)^{3/2}$$

$$+0.0075(T_0/T)^3+0.0025(T_0/T)^{9/2}+\cdots\}, \quad (9b)$$

from which one gets the pressure $p=-\partial F/\partial V$

$$p_-=0.514CRT^{5/2} \quad (10a)$$

$$p_+=\frac{RT}{V}\left[1-\frac{0.462}{CVT^{3/2}}-\frac{0.0225}{(CVT^{3/2})^2}\right.$$

$$\left.-\frac{0.0114}{(CVT^{3/2})^3}-\cdots\right]. \quad (10b)$$

This is in agreement with the Virial theorem: $pV=\frac{2}{3}U$.

§2. CONDENSATION IN MOMENTUM SPACE

Attention has already been directed to the remarkable result (10a), that for $T<T_0$ the pressure is independent of density, just as in the case of a van der Waals gas in the transition region below the critical point. However, this independence of density is quite accidental for the Bose-Einstein gas,[6] and it is somewhat misleading to interpret this behavior in analogy with a van der Waals gas by saying that for $T<T_0$ a fraction of the molecules condenses into a state of zero volume and does not contribute to the density.[4] Indeed, in the transition region of the van der Waals gas, a fraction of the molecules is condensed into a state of considerably smaller volume and, since it is separated by gravitation from the other molecules, it does not contribute to the density of the gaseous phase. At this point the analogy fails completely, or is rather to be found in a more remote sense. The supernumerary molecules of a degenerate Bose-Einstein gas which we have denoted by N_0 and which indeed may be considered as belonging to a particular phase, do not, of course, disappear mysteriously from space; they do contribute to the density as any other molecules. They do *not contribute, however, to the pressure,* since their kinetic energy (and momentum), is zero. If one likes analogies, one may say that there is actually a *condensation, but only in momentum space,* and not in ordinary space, i.e., an equilibrium of two phases, *one* containing the molecules N_0 of momentum zero and occupying in the space of momenta, a zero volume; and *another one* showing a distribution over all momenta similar to that which is realized for $T>T_0$. In ordinary space, however, no separation of phases is to be noticed.

In a certain respect the molecules of the condensed phase having the momentum zero, show also a characteristic peculiarity as to their behavior in ordinary space. Since their wave functions are constant over the whole volume, they are particularly inappropriate for forming wave packets of the size of molecular dimensions by superposition of neighboring wave functions (i.e., wave functions of comparable small energy, so

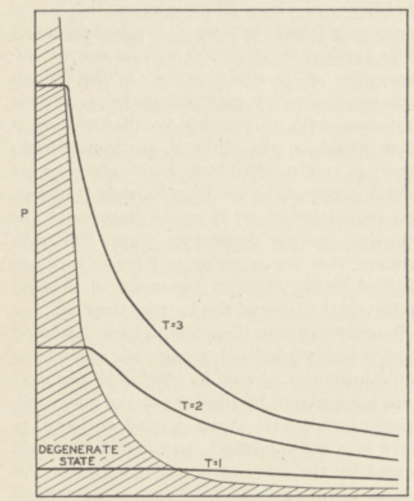

FIG. 3. Isotherms of an ideal Bose-Einstein gas.

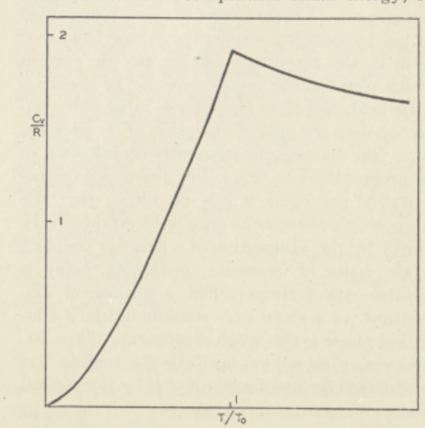

FIG. 4. Specific heat of an ideal Bose-Einstein gas.

[6] This independence of density holds only for an "ideal" Bose-Einstein gas, but is no longer realized as soon as a molecular interaction is taken into account, for instance by adding a term $U_0(V)$ to the energy, representing a mean van der Waals field smeared over the total volume accessible to the molecules. In this case the condensation for constant volume proceeds as before and the specific heat c_v is exactly the same as given by (8). The pressure, however, will no longer be independent of V. It will be given by

$$p_-=0.514\,CRT^{5/2}-\partial U_0/\partial V$$

and for constant pressure the gas will in this case no longer condense to the volume zero.

that the energy distribution of the whole remains unchanged). These molecules therefore will represent a peculiar omnipresence in the total volume at their disposal. Accordingly it will not be allowable to treat their motion in external fields (pressure gradient, etc.), on the lines of Ehrenfest's theorem (approximate validity of classical mechanics for small wave packets), which theorem has been the general basis of the usual corpuscular treatment of the transport phenomena. Here we have before us just the opposite limiting case, namely, wave packets of a very *small extension* in the space of momenta, but which in ordinary space are spread over a region comparable with the extension of the inhomogeneities of macroscopic fields.

The quantum dynamics of this limiting case has been investigated very little. Only one interesting special case has so far been discussed in connection with superconductivity. It has been shown[7] that the macroscopic description given for this phenomenon leads to a relation between electric current and magnetic field, which is identical with that which would be valid in an enormous diamagnetic atom of the dimensions of the metal. Now it is well known that diamagnetism cannot be explained on the basis of classical dynamics of electrons. The currents in a diamagnetic atom are not given by progressive de Broglie waves (or progressive wave packets), as in the usual treatment of the ordinary conductivity phenomena. These currents are represented rather by standing waves; they come from that term in the expression for the electric current which contains the vector potential of the magnetic field, and their structure is intimately connected with the spatial extension *of the quantum states* (the diamagnetic susceptibility is known to be proportional to $\langle r^2\rangle_{Av}$). For the superconductivity in particular it can be shown that the macroscopic phenomena may be interpreted very simply by the assumption of a peculiar coupling in the space of momenta, producing, below a certain critical temperature, a fixation of momentum, as if there were something like a condensed phase in the space of momenta. Thus far, however, it has not yet been possible to base that assumption on a molecular model by the general theory of electrons in metals.

Now the degenerate Bose-Einstein gas provides a good example of a molecular model for such a condensed state in space of momenta, such as seems needed for the superconductive state. Though this fact cannot, of course, be applied to explain superconductivity, as electrons do not obey Bose statistics, it is remarkable on the other hand that the transport properties (viscosity,[8] thermal conductivity[9]), of liquid helium, when passing the λ-point actually change in a very conspicuous manner; thus one speaks of a "superfluidity" and of a super-heat-conductivity."

§3. Superfluidity

We have already emphasized that one might certainly not be justified in formally applying the ordinary corpuscular theory[10] of transport phenomena of a Bose-Einstein gas to this case of degeneracy where the *mean de Broglie wave-length of the particles is comparable with the macroscopic dimensions of the whole system.* On the present occasion, however, it may perhaps be permissible to discuss, with all due reserve, the question of what would be the effect if, in Sommerfeld's theory of conductivity,[11] the Fermi statistics are formally replaced by the Bose-Einstein statistics, and in particular, what would happen at the transition to the degenerate state. One may presume that our attempt will prove as much justified as the classical treatment of diamagnetism on the basis of the Larmor theorem.

In order to fix our ideas and to have a definite, though highly idealized, model, we may depict liquid helium as a metal in which ions and electrons are replaced by particles of the *same* kind, namely both by He atoms. Each He atom may move in a self-consistent (van der Waals) field formed by the other atoms. The states of this system might perhaps, to a certain approximation, be divided into two classes: one, being more of the *Bloch-type* (progressive modulated Schrödinger waves), corresponding to the *electronic states* of the metal and representing a transport of matter; the other one, more of the *Debye-type* (quantized acoustical or elastic waves), corresponding to the *vibrations of the ionic lattice.* In this picture the *fluidity* of the liquid would correspond to the electric *conductivity* of the electrons in a metal; the friction would be due to the dissipation of progressive Bloch waves by inelastic reflection on the Debye waves. Let us finally make the assumption that for these inelastic collisions there exists a mean free path l which is independent of the velocity v for small values of v.

We do not want to insist here on the details of this conception, which is certainly open to much criticism, though perhaps it may prove useful for describing some properties of liquids in general. At any rate it is desirable to know what would be the result if in the theory of metals the Fermi statistics are formally replaced by Bose statistics, even if one does not think of a possible connection with liquid helium, and even if it is more than doubtful whether one is allowed to apply the ordinary collision theory to this limiting case in which the mean value of the de Broglie wave-length is comparable with the macroscopic dimensions of the whole system.

The well-known formula for the electric conductivity $\sigma\sim\langle l/v\rangle_{Av}$ would yield the value $\sigma=\infty$ in the case where a finite fraction of all particles has zero velocity. But actually one has to apply a special consideration for the slowest particles. A particle of mass M and of initial velocity v_0, chosen in the direction of the field F, needs the time τ to traverse the mean free path l:

$$l=v_0\tau+(1/2M)F\tau^2,$$

or $$\tau=(M/F)[(v_0^2+2lF/M)^{\frac{1}{2}}-v_0].$$

The mean change of velocity due to the action of F during this time τ will therefore be given by

$$\langle\delta v_0\rangle_{Av}=\frac{1}{2M}F\tau=\tfrac{1}{2}[(v_0^2+2lF/M)^{\frac{1}{2}}-v_0]. \quad (11)$$

In general (particularly in the case of Fermi statistics, and also in the case $T>T_0$ for the Bose statistics), one is accustomed and entitled to disregard the few particles with $Mv_0^2/2\lesssim lF$ and one may write instead of (11):

$$\langle\delta v_0\rangle_{Av}=\frac{l}{2Mv_0}\cdot F \quad \text{for} \quad Mv_0^2/2\gg lF, \quad (11a)$$

which gives the above-mentioned formula for σ.

In the case of the Bose degeneracy ($T<T_0$), however, we are not allowed to neglect the *finite fraction* of particles with $v_0=0$. For these we obtain

$$\langle\delta v_0\rangle_{Av}=(lF/2M)^{\frac{1}{2}} \quad \text{for} \quad Mv_0^2/2\ll l\cdot F. \quad (11b)$$

This contribution is, for ordinary values of F, enormously greater than (11a). One may say the "conductivity," *viz.*, the *fluidity* (defined as derivative of the current with respect to F), becomes *infinite* for $F\to 0$, and this abruptly, as soon as $T<T_0$. For a fixed value of F, the current will be proportional to N_0/N, i.e., proportional to the fraction of atoms of velocity zero.

The particles N_0, having the energy zero, will not appreciably contribute to the transport of energy, and therefore, we should not expect a particularly great increase of the *heat conductivity* when passing to the degenerate state.

§4. Thermomechanical Effect

But there is another mechanism which may produce a transfer of heat.[12] The van der Waals forces between the walls and the He atoms are much stronger than those between the He atoms themselves. In a layer, L, of perhaps 10A, or 100A, or even greater thickness along the walls, the van der Waals field will be appreciably stronger than in the interior, I, of the liquid. In this layer the concentration of the degenerate atoms will therefore be much greater than in the interior, the entropy in the layer, S_L, will be much smaller than in the interior, S_I. Thus we have a situation quite analogous to a thermocouple: namely, two different conductors, L and I, in conducting contact.

If q mols of helium pass at a temperature T from I to L they will go into a state of greater order, and the heat

$$Q=T(S_I-S_L)q \quad (12)$$

will be set free. This will occur in a *reversible manner*, quite as in the case of the Peltier effect. Relating all transfer of energy or matter to unit

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of time and to unit of cross section, we may write

$$Q=T(S_I-S_L)J, \qquad (12a)$$

where J is the current density. The "Peltier coefficient" is accordingly given by

$$\Pi_{I,\,L}=T(S_I-S_L). \qquad (13)$$

Assuming for S the expression for an ideal degenerate Bose gas, which is given by (7) (9)

$$S_-=\frac{U_--F_-}{T}=(5/2)0.514CRT^{3/2}V, \qquad (14)$$

we obtain for the Peltier coefficient

$$\Pi_{I,\,L}=(5/2)0.514CRT^{5/2}(V_I-V_L). \qquad (15)$$

Now the arrangement for measuring heat conductivity can simply be considered as a thermocouple consisting of the two "metals" L and I. If we heat at one point of the wall and cool at another one, in such a way that the temperature at the two points may be kept constant at T_1 and T_2 respectively, we may produce a "thermoforce" Φ given, according to W. Thomson's well-known thermodynamic relation, by the formula

$$\Phi=\int_{T_1}^{T_2}\frac{\Pi_{I,\,L}}{T}dT=\int_{T_1}^{T_2}(S_I-S_L)dT$$

or with (14):

$$\Phi=(T_2-T_1)(5/2)0.514CRT_1^{\,3}(V_I-V_L). \qquad (16)$$

This "thermoforce" will produce very great circulation of matter, since the "internal resistance" of the thermoelement is extraordinarily small, and, therefore, the consumption of heat for maintaining even the smallest temperature differences will also be very great. This process, therefore, will appear like an enormous conduction of heat; the "conductivity," however, will depend very strongly on the gradient of temperature. For the lowest temperatures, the helium atoms will everywhere be almost completely degenerate, in the layer as well as in the interior of the liquid. There will be no difference of entropy between the different parts of the liquid, the "thermoforce" will disappear—as it must according to Nernst's Law—and there will no longer be a production of circulation in the liquid. Only ordinary heat conductivity (chiefly by the Debye vibrations), will remain. Nevertheless the great fluidity, since it is proportional to the number N_0 of degenerate atoms, will persist down to the lowest temperatures; only the driving thermoforce and the Peltier heat will disappear.

All this is in qualitative agreement with the experiments, particularly with those of Keesom and Saris,[13] and of Kürti and Simon,[13] on the heat conductivity of He at the lowest temperatures, which below 0.6°K, has been found to become "normal" again, i.e., small and independent of the gradient of temperature. In fact, this transition to normal heat conductivity occurs in just that region where the thermal anomaly of the specific heat, connected with the λ-point, ceases and passes over into an ordinary Debye T^3-law for the specific heat.

This mechanism of reversible transformation of heat into mechanical energy gives a very simple explanation also for the so-called "fountain phenomenon" observed by Allen and Jones,[14] and interprets it as a pump driven by a thermoelement. It may be remarked that according to (12) the flow of matter in the capillary layer has to have the opposite direction to the transfer of heat, if the entropy in the capillary layer is smaller than in the normal liquid. In fact this connection between the directions of the flow of heat and of the flow of matter has actually been observed in the experiment of Allen and Jones. The same has been found in the Knudsen manometer experiment of H. London.[12]

The idea that the transport phenomena of He II might be reversible processes has first been discussed by L. Tisza.[15] The thermodynamic relation (12) has recently been given by H. London.[12] Though it might appear that the logical connection between the facts will not be qualitatively very different from the one we have sketched here, it is obvious that the theoretical basis given thus far is not to be considered more than a quite rough and preliminary approach to the problem of liquid helium, limited chiefly by the lack of a satisfactory molecular theory of liquids.

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