NONSMOOTH OPTIMIZATION ALGORITHMS FOR THE DESIGN OF CONTROLLED FLEXIBLE STRUCTURES

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NONSMOOTH OPTIMIZATION ALGORITHMS FOR THE DESIGN OF CONTROLLED FLEXIBLE STRUCTURES[1]

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ABSTRACT

First we show that both open-loop and closed-loop optimal control problems can be expressed in the form of nonsmooth optimization problems. Then we present the basics of a class of nonsmooth optimization algorithms which solve constrained optimization problems involving the maxima of differentiable functions. The described algorithms are shown to be natural extensions of the method of centers.

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1. INTRODUCTION

It is reasonably obvious that the solution of constrained open-loop optimal control problems requires the use of appropriate optimization algorithms. It is less obvious that the design of compensators for closed-loop systems can be cast as an optimization problem which can be treated effectively by optimization algorithms. In this paper, we make a strong case for the use of *nonsmooth* optimization algorithms in the solution of constrained open-loop optimal control problems, as well as of linear feedback system design problems, and we present an introduction to a set of appropriate nonsmooth optimization algorithms.

Feedback is used to achieve various desirable properties in a control system, such as stability, disturbance attenuation, and low sensitivity to changes in the plant. Since these properties depend on the shape of various closed-loop system responses, all control system design techniques are at least partially based on response shaping. In the sixties and seventies, the most popular control system design methods were based on weighted least-squares unconstrained minimization in the form of linear-quadratic regulator (LQR) theory (see e.g. [Kwa.1]). The main drawback of the LQR approach is that it tends to result in poor stability robustness and output-disturbance rejection (see e.g., [Doy.1]). Furthermore, the least-squares approach does not permit imposition of hard bounds on system responses. A more recent approach for shaping a *single* frequency response is to minimize not a weighted quadratic norm, as in the LQR approach, but a weighted sup-norm (H^∞-norm) (for a survey see [Fra.1]). This is usually done in conjunction with a compensator parametrization which makes all transfer functions affine in the design parameter, and hence reduces the response shaping to a *convex, unconstrained optimization problem in* H^∞.

However, most closed-loop system design problems require shaping of *several* frequency and time domain responses, some of which may be subject to *hard* constraints. For example, while minimizing the norm of the sensitivity matrix over the bandwidth of the feedback system, the norm of the transfer matrix from the command input to the plant input has to be upper-bounded. Otherwise the command input can drive the plant input outside the linearization region, which may lead to performance deterioration and even instability. The requirement of simultaneous shaping several frequency responses can be dealt with in various ways. For example, in [Doy.2], loop transformations and weighting functions are used to transform the multiloop shaping problem into a problem of unconstrained minimization of the norm of an affine matrix function in H^∞. This approach can be quite conservative. Furthermore, when there are hard bounds on the norms of some of transfer function matrices, the weighting approach cannot be used, because it is not known how to transform a constrained H^∞ minimization problem into an unconstrained one by using weights.

A second approach, first presented in [Kwa.2], is based on the fact that many essential design objectives can be formulated as bounds on the weighted sensitivity matrix $S(j\omega)$ and/or on the weighted complementary sensitivity matrix $T(j\omega) = I - S(j\omega)$. It is concluded in [Kwa.2] that a balance between conflicting design objectives can be achieved by minimizing a performance criterion of the form $\sup_{\omega} [\ |V(j\omega)S(j\omega)|^2 + |W(j\omega)T(j\omega)|^2\ ]$, where $V(\cdot)$ and $W(\cdot)$ are weighting functions selected by the designer. This approach can also be conservative.

>From a designer's point of view, both LQR and the above approaches to complex design problem solution suffer from the drawback that they use design weights which are very difficult to select. This drawback can be further accentuated by the fact that the solution of a weight-dependent, unconstrained optimization problem, can be very sensitive to the weights, which implies that whenever a constrained problem is somehow converted to an unconstrained one by means of weights, a large amount of time may have to be devoted to weight selection.

In [Pol.7], the reader will find a formulation of finite dimensional, linear, time invariant feedback-system design, subject to various hard constraints, as convex, nonsmooth optimization problems. In [Pol.7], affine compensator parametrizations are used, as in the H^∞ approach. In this paper we use a direct compensator parametrization which enables us to select the degree of the compensator, because affine parametrization would result in an infinite dimensional compensator. As a trade-off, we give up problem convexity. We show the mathematical unity of both open- and closed-loop optimal control problems and we present a sequence of progressively more complex algorithms for their solution.

For further reading on nonsmooth optimization and optimal control algorithms, we refer the reader to [Gon.1, Kiw.1, Kle.1, May.1, May.2, Pir.1, Pir.2, Pol.3, Pol.4, Pol.5, Pol.6, Pol.10].

2. FORMULATION OF OPTIMAL DESIGN PROBLEMS

We propose to consider both open-loop optimal control and closed-loop optimal control of flexible structures. By *open-loop optimal control* we mean the computation of optimal open-loop controls which take a structure from an initial state to a desired state subject to various constraints on the control and state, while by *closed-loop optimal control* we mean the computation of optimal, finitely parametrized, finite-dimensional closed-loop compensators, subject to constraints on various time- and frequency-domain constraints.

2.1. CANONICAL FORMS

Our first task is to show that these problems can be cast in the form of the two canonical problems, below. Note that the two problems differ only in the space on which they are defined. For the design of finite-dimensional closed-loop compensators, which is a problem with a finite dimensional design vector, we adopt the canonical form

$$\min \{ \psi^0(x) \mid \psi^j(x) \leq 0 \,,\ j \in \mathbf{m} \,,\ \ x \in \mathbf{X} \} \,, \tag{2.1a}$$

where $\mathbf{m} \triangleq \{ 1, 2, \ldots, m \}$, $\mathbf{X} \subset \mathbf{R}^n$ is a set with a very tractable description, e.g., $\mathbf{X} = \mathbf{R}^n$ or

$$\mathbf{X} \triangleq \{ x \in \mathbf{R}^n \mid |x^k| \leq b^k \,,\ k = 1, 2, 3, \ldots, n \} \,, \tag{2.1b}$$

and, with $\mathbf{M} \triangleq \{ 0, 1, 2, \ldots, m \}$,

$$\psi^j(x) = \max_{y_j \in \mathbf{Y}_j} \phi^j(x, y_j) \,,\ \ \forall j \in \mathbf{M} \,, \tag{2.1c}$$

where $\phi^j : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$. We will assume that the functions $\phi^j(\cdot,\cdot)$ and their gradients $\nabla_x \phi^j(\cdot,\cdot)$ are Lipschitz continuous on bounded sets. In addition, we will assume that the intervals $\mathbf{Y}_j = [a_j, b_j] \subset \mathbf{R}$ are compact. We note that when $\mathbf{Y}_j$ contains only one point, i.e., $a_j = b_j$, the function $\psi^j(x) = \phi^j(x, a_j) \triangleq f^j(x)$ is differentiable (otherwise it need not be); thus we see that the formulation (2.1a-c) allows that some of the $\psi^j(x)$ are ordinary differentiable functions.

Similarly, for open-loop optimal control, we adopt the canonical form

$$\min \{ \psi^0(u\,,T) \mid \psi^j(u\,,T) \le 0\,,\ j \in \mathbf{m}\,,\ (u\,,T) \in \mathbf{U} \times \mathbf{T} \}\,, \tag{2.2a}$$

where $\mathbf{U} \triangleq \{ u \in L_2^p[0\,,1] \mid u(t) \in U\ \forall t \in [0\,,1] \}$, $U \subset \mathbf{R}^p$ either is compact or else equal to $\mathbf{R}^p$, $\mathbf{T} \triangleq [T_0\,,T_f]$, and for $j \in \mathbf{M}$,

$$\psi^j(u\,,T) = \max_{t \in \mathbf{Y}_j} \phi^j(u\,,T\,,t)\,, \tag{2.2b}$$

with $\phi^j : L_2^p[0\,,1] \times \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$. We will assume that the functions $\phi^j(\cdot\,,\cdot\,,\cdot)$ and their gradients[2] $\nabla_u \phi^j(\cdot\,,\cdot\,,\cdot)$ are Lipschitz continuous on bounded sets. In addition, we will assume that the intervals $\mathbf{Y}_j \subset \mathbf{R}$ are compact.

2.2. TRANSCRIPTION OF OPEN-LOOP OPTIMAL CONTROL PROBLEMS INTO CANONICAL FORM

Since the transcription of open-loop optimal problems into the form (2.2a) is simpler than the transcription of closed-loop optimal control problems into the form (2.1a), we will do it first. Although in practice one usually computes with second order dynamics, the explanation becomes simpler if we adopt as our model for the dynamics the first order differential equation

$$\dot{z}(t) = TAz(t) + Th(z(t)\,,u(t))\,,\quad t \in [0\,,1]\,,\quad z(0) = z_0\,,\quad T > 0\,,\quad u \in \mathbf{U}\,, \tag{2.3}$$

where the *state vector* $z(t)$ is an element of a Hilbert space, H, so that (2.3) can, in fact, be a *partial* differential equation, and the control $u(t) \in \mathbf{R}^p$ is finite dimensional. We will assume that A is an infinitesimal generator of a C_0 semigroup[3] and that the operator $h(\cdot\,,\cdot)$ is bounded and continuously differentiable. The parameter T is a time-scaling parameter which enables us to convert free time problems into fixed time form, as well as to avoid some well known pathological behavior of the discretizations that are needed in solving optimal control problems. When (2.3) represents an ODE, $A = 0$ holds. We note that (2.3) can be used to represent a broad class of dynamical systems described either by ODEs or PDEs, including PDEs derived using Lagrangian dynamics.

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[2] By gradients we mean the kernels of linear functionals which play an analogous role to gradients of functions defined on $\mathbb{R}^n$, i.e., the gradient of $f : L_2^p[0\,,1] \to \mathbb{R}$ is defined by the property that $\lim_{t \to 0} [f(u + t\delta u) - f(u) - t\,\langle \nabla f(u)\,, \delta u \rangle_2]/t = 0$, where $\langle \cdot\,, \cdot \rangle_2$ denotes the $L_2^p[0\,,1]$ scalar product.

[3] For a discussion of semigroup theory see [Bal.1] or [Paz.1].

Obviously, we must assume that (2.3) has a weak solution, which we will denote by $z^{u,T}(t)$. In addition, we will assume that the differential, $\delta z^{u,T}(t\,;\delta u\,,\delta T)$, of this solution, with respect to the control u and the time scaling parameter T, is given by the weak solution of linearized equation

$$\delta \dot z(t) = T[A + \frac{\partial h(z^{u,T}(t)\,,u(t))}{\partial z}]\delta z(t) + T\frac{\partial h(z^{u,T}(t)\,,u(t))}{\partial u}\delta u(t) + [A + h(z^{u,T}(t)\,,u(t))]\delta T\,,$$

$$t \in [0\,,1]\,,\quad \delta z(0) = 0\,. \tag{2.4}$$

Now consider the optimal control problem with control, end point and state space constraints:

$$\min_{(u\,,T)} \{\, g^0(z^{u,T}(1)) \mid g^j(z^{u,T}(1)) \le 0\,,\ j = 1\,,2\,,\dots,m_1\,;\ \ g^j(z^{u,T}(t)) \le 0\,,\ \forall t \in [0\,,1]\,,$$

$$j = m_1+1\,,m_1+2\,,\dots,m\,;\ \ u(t) \in U\ \ \forall t \in [0\,,1]\,,T \in [T_o\,,T_f]\,\}\ ,\tag{2.5}$$

where $T_o > 0$ is assumed to be very small and $T_f < \infty$. We assume that the control $u(\cdot)$ is an $L_2^p[0\,,1]$ function, that the set $U \subset \mathbf{R}^p$ is compact and that all the functions $g^j : H \to \mathbf{R}$ are continuously differentiable. Next, we define $\mathbf{Y}_j = \{\,1\,\}$ for $j = 1\,,2\,,\dots,m_1$ and $\mathbf{Y}_j = [0\,,1]$ for $j = m_1+1\,,\dots,m_1+m_2$ (so that $m = m_1 + m_2$), and we define $\phi^j : L_2^p[0\,,1] \times \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$, $j = 0\,,1\,,\dots,m$, by $\phi^j(u\,,T\,,t) = g^j(z^{u,T}(t))$ for $j = 1\,,2\,,\dots,m_1$, and by $\phi^j(u\,,T\,,t) = g^j(z^{u,T}(t))$ for $j = m_1+1\,,\dots,m$, and, finally, if for $0\,,1\,,2\,,\dots,m$, we define the functions $\psi^j : L_2^p[0\,,1] \times \mathbf{R}_+ \to \mathbf{R}$ by $\psi^j(u\,,T) = \max_{t \in Y_j} \phi^j(u\,,T\,,t)$, then we see that problem (2.5) assumes the form (2.2a).

2.3. TRANSCRIPTION OF FEEDBACK-SYSTEM DESIGN INTO CANONICAL FORM

Next we turn to the more arduous task of transcribing closed-loop optimal control problems into the form (2.1a). Consider the n_i–input - n_o–output feedback system S, shown in Fig. 1. We assume that the plant is described by a linear, time-invariant differential equation in a Hilbert space E:

$$\dot z_p(t) = A_p z_p(t) + B_p e_2(t)\,, \tag{2.6a}$$

$$y_2(t) = C_p z_p(t) + D_p e_2(t)\,, \tag{2.6b}$$

where $z_p(t) \in E$, $e_2(t) \in \mathbf{R}^{n_i}$, $y_2(t) \in \mathbf{R}^{n_o}$, for $t \ge 0$. We will assume that the operators $B_p : \mathbf{R}^{n_i} \to E$, $C_p : E \to \mathbf{R}^{n_o}$ and $D_p : \mathbf{R}^{n_i} \to \mathbf{R}^{n_o}$ are bounded, and that A_p may be an unbounded operator from E to E, with domain dense in E, which generates a C_0 semigroup, $\{e^{A_p t}\}_{t \ge 0}$. We will denote the *spectrum* of A_p by $\sigma(A_p)$, and we will denote the *resolvent set* of A_p by $\rho(A_p)$. Referring to [Paz.1], we find that there exist $M \in (1, \infty)$ and $\gamma \in \mathbf{R}$ such that

$$\| e^{A_p t} \| \leq M e^{\gamma t}, \quad \forall t \geq 0 \; ; \tag{2.7}$$

furthermore $\rho(A_p)$ contains the half-plane, $\{ s \in \mathbb{C} \mid \mathrm{Re}\, s > \gamma \}$. We will denote the domain and the range of A_p by $D(A_p)$ and $R(A_p)$, respectively.

We define the *transfer function* of the plant, $G_p(s)$, by

$$G_p(s) \triangleq C_p(sI - A_p)^{-1}B_p + D_p \, , \; \forall s \in \rho(A_p) \, . \tag{2.8}$$

It follows from [Kat.1, Theorem III 6.7], that $G_p(s)$ is analytic on $\rho(A_p)$. In addition, it is shown in [Jac.1] that for $s \in \{ s \in \mathbb{C} \mid \mathrm{Re}\, s > \gamma \}$, $G_p(s)$ is equal to the Laplace transform of $\{ C_p e^{A_p t} B_p + D_p \delta(t) \}_{t \geq 0}$ and $\lim_{\substack{|s| \to \infty \\ \mathrm{Re}\, s > \gamma}} G_p(s) \to D_p$.

We will assume that the compensator is *finite dimensional, linear, and time-invariant*, with state equations

$$\dot{z}_c(t) = A_c z_c(t) + B_c e_1(t) \, , \tag{2.9a}$$

$$y_1(t) = C_c z_c(t) + D_c e_1(t) \, , \tag{2.9b}$$

where $z_c(t) \in \mathbf{R}^{n_c}$, $e_1(t) \in \mathbf{R}^{n_o}$, $y_1(t) \in \mathbf{R}^{n_i}$ and A_c, B_c, C_c and D_c are matrices of appropriate dimension. We will assume that a number of the elements of the matrices A_c, B_c, C_c and D_c are to be determined by optimization. We group all these elements into a single *design vector* $x \in \mathbf{R}^n$ and will assume that all the compensator matrices are continuously differentiable in this vector. From now on we will show the dependence of the matrices A_c, B_c, C_c and D_c on x explicitly. Similarly, we will refer to the corresponding closed-loop system as $S(x)$.

We note that the compensator transfer function is given by

$$G_c(x, s) = C_c(x)(sI_{n_c} - A_c(x))^{-1}B_c(x) + D_c(x) \, . \tag{2.10}$$

To ensure well-posedness of the feedback system, we assume that $det(I_{n_i} + D_c(x)D_p) \neq 0$. Finally, we define the Hilbert space $H = E \times \mathbf{R}^{n_c}$, on which the inner product is defined as follows:

for $u = (z_p , z_c)$, $v = (z'_p , z'_c)$ in H

$$\langle u , v \rangle_H = \langle z_p , z'_p \rangle_E + \langle z_c , z'_c \rangle_{\mathbb{R}^{n_c}} . \tag{2.11}$$

Since $e_1 = u - y_2 - d_o$ and $e_2 = y_1 + d_i$, where d_o is the plant output disturbance and d_i is the plant input disturbance, the state equations for the feedback system are given by

$$\begin{bmatrix} \dot{z}_p \\ \dot{z}_c \end{bmatrix} = A(x) \begin{bmatrix} z_p \\ z_c \end{bmatrix} + B(x) \begin{bmatrix} u \\ d_o \\ d_i \end{bmatrix} , \tag{2.12a}$$

$$\begin{bmatrix} e_1 \\ e_2 \\ y \end{bmatrix} = C(x) \begin{bmatrix} z_p \\ z_c \end{bmatrix} + D(x) \begin{bmatrix} u \\ d_o \\ d_i \end{bmatrix} , \tag{2.12b}$$

where

$$A(x) = \begin{bmatrix} A_p - B_p D_c(x)(I_{n_o} + D_p D_c(x))^{-1} C_p & B_p (I_{n_i} + D_c(x) D_p)^{-1} C_c(x) \\ -B_c(x)(I_{n_o} + D_p D_c(x))^{-1} C_p & A_c(x) - B_c(x)(I_{n_o} + D_p D_c(x))^{-1} D_p C_c(x) \end{bmatrix} \tag{2.12c}$$

$$B(x) = \begin{bmatrix} B_p D_c(x)(I_{n_o} + D_p D_c(x))^{-1} & -B_p D_c(x)(I_{n_o} + D_p D_c(x))^{-1} & B_p (I_{n_i} + D_c(x) D_p)^{-1} \\ B_c(x)(I_{n_o} + D_p D_c(x))^{-1} & -B_c(x)(I_{n_o} + D_p D_c(x))^{-1} & -B_c(x)(I_{n_o} + D_p D_c(x))^{-1} D_p \end{bmatrix} , \tag{2.12d}$$

$$C(x) = \begin{bmatrix} -(I_{n_o} + D_p D_c(x))^{-1} C_p & -(I_{n_o} + D_p D_c(x))^{-1} D_p C_c(x) \\ -D_c(x)(I_{n_o} + D_p D_c(x))^{-1} C_p & (I_{n_i} + D_c(x) D_p)^{-1} C_c(x) \\ (I_{n_o} + D_p D_c(x))^{-1} C_p & (I_{n_o} + D_p D_c(x))^{-1} D_p C_c(x) \end{bmatrix} , \tag{2.12e}$$

$$D(x) = \begin{bmatrix} (I_{n_o} + D_p D_c(x))^{-1} & -(I_{n_o} + D_p D_c(x))^{-1} & -(I_{n_o} + D_p D_c(x))^{-1} D_p \\ D_c(x)(I_{n_o} + D_p D_c(x))^{-1} & -D_c(x)(I_{n_o} + D_p D_c(x))^{-1} & (I_{n_i} + D_c(x) D_p)^{-1} \\ I_{n_o} - (I_{n_o} + D_p D_c(x))^{-1} & -(I_{n_o} + D_p D_c(x))^{-1} & (I_{n_o} + D_p D_c(x))^{-1} D_p \end{bmatrix} . \tag{2.12f}$$

The domain of A is given by $D(A) = D(A_p) \times \mathbf{R}^{n_c} \subset H$. It follows from [Paz.1, p. 76], that because, with the exception of A_p, all the operators in the matrix A are bounded, and because A_p generates a C_0-semigroup, the operator A also generates a C_0–semigroup, $\{e^{At}\}_{t \geq 0}$.

A. Frequency-Domain Performance Specifications

First, since the feedback system (2.12a,b) has 3 inputs and 3 outputs, we write its transfer function $G(x , s) = C(x)(sI - A(x))^{-1} B(x) + D(x)$, which is defined for all $s \in \rho(A(x))$, in block form, as follows:

$$G(x,s) = \begin{bmatrix} G_{11}(x,s) & G_{12}(x,s) & G_{13}(x,s) \\ G_{21}(x,s) & G_{22}(x,s) & G_{23}(x,s) \\ G_{31}(x,s) & G_{32}(x,s) & G_{33}(x,s) \end{bmatrix} . \tag{2.13}$$

We will use a "hat" to denote the Laplace transforms of various functions: e.g., $\hat{u}(s)$ is the Laplace transform of $u(t)$.

(i) Stability Constraint. Our first and most important performance requirement is closed-loop system stability. Let $z = [z_p, z_c] \in H$. Then we recall (see [Paz.1]) that the *mild solution* of (2.12a) is given by

$$z(t) = e^{A_p t} z_0 + \int_0^t e^{A_p(t-\tau)} B u(\tau) d\tau . \tag{2.14a}$$

We therefore define the *exponential stability* of the feedback system $S(P, K)$ in terms of the semigroup $\{e^{At}\}_{t \geq 0}$, as follows.

Definition 2.1: For any $\alpha \geq 0$, the feedback system $S(x)$ is α–stable if there exist $M \in (0,\infty)$ and $\alpha_0 > \alpha$ such that

$$\| e^{A(x)t} \|_H \leq M e^{-\alpha_0 t} \ \forall t \geq 0 . \tag{2.14b}$$

■

The class of plants that we can deal with are characterized by the following definition:

Definition 2.2: [Jac.1] Given an $\alpha \geq 0$, the pair (A_p, B_p) ((A_p, C_p)) is α–*stabilizable* (α–*detectable*) if there exists a bounded linear operator $K : E \to \mathbf{R}^{n_i}$ ($F : \mathbf{R}^{n_o} \to E$) such that $A_p - B_p K$ ($A_p - F C_p$) is the infinitesimal generator of an α–stable C_0–semigroup. ■

It was shown in [Jac.1] that a plant is α–stabilizable and α–detectable if and only if there exists a finite dimensional compensator with $D_c = 0$ such that the feedback system is α–stable.

For any $\alpha \geq 0$, we define the *stability region* $D_{-\alpha} \triangleq \{s \in \mathbb{C} \mid \mathrm{Re}(s) < -\alpha\}$. Let $U_{-\alpha} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq -\alpha\}$, $\partial U_{-\alpha} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) = -\alpha\}$ and let $U^o_{-\alpha} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > -\alpha\}$.

Proposition 2.1: [Jac.1] For any $\alpha \geq 0$, the plant is α–stabilizable and α–detectable if and only if there exists a decomposition of $E = E_- + E_+$, with E_+ *finite-dimensional*, which induces a decomposition of the plant (2.6a,b), of the form

$$\frac{d}{dt}\begin{bmatrix} z_{p-}(t) \\ z_{p+}(t) \end{bmatrix} = \begin{bmatrix} A_{p-} & 0 \\ 0 & A_{p+} \end{bmatrix}\begin{bmatrix} z_{p-}(t) \\ z_{p+}(t) \end{bmatrix} + \begin{bmatrix} B_{p-} \\ B_{p+} \end{bmatrix} u(t) \ , \tag{2.15a}$$

$$y(t) = [C_{p-} \ \ C_{p+}]\begin{bmatrix} z_{p-}(t) \\ z_{p+}(t) \end{bmatrix} + Du(t) \ , \tag{2.15b}$$

such that $\sigma(A_{p+}) \subset U_{-\alpha}$, (A_{p+}, B_{p+}) is controllable, (A_{p+}, C_{p+}) is observable, and A_{p-} is the infinitesimal generator of an α–stable C_0–semigroup on E_-. ■

In view of the above, we will restrict ourselves to feedback systems in which the plant is α–stabilizable and α–detectable for some $\alpha > 0$.

The relationship between α–stability of the feedback system and α–stabilizability of the plant is established in the following result:

Proposition 2.2: [Jac.1] Suppose that the plant is α–stabilizable and α–detectable for some $\alpha > 0$. Then, for any $\alpha \geq 0$, the feedback system is α–stable if and only if $U_{-\alpha}$ is contained in $\rho(A)$. ■

We are finally on the way to defining a computational stability criterion which can be expressed in the form of an inequality of the type appearing in problem (2.1a). First we define the *characteristic function* $\chi : \mathbb{C} \to \mathbb{C}$, of the feedback system $S(x)$, by

$$\chi(x, s) \triangleq \det(sI_{n_+} - A_{p+})\det(sI_{n_c} - A_c(x))\det(I_{n_i} + G_c(x, s)G_p(s)) \ , \tag{2.16}$$

where A_{p+} is defined as in (2.15a) and n_+ is the dimension of A_{p+}. Next, for any function $f : \mathbb{C} \to \mathbb{C}$, we define $Z(f(s)) \triangleq \{s \in \mathbb{C} \mid f(s) = 0\}$ to be its set of zeros. in [Har.1] we find the following pair of crucial results:

Proposition 2.3: [Har.1] The system $S(x)$ is stable if and only if $Z(\chi(x, s)) \subset D_{-\alpha}$. ■

Theorem 2.1: [Har.1] Let n_+ and n_c be the dimensions of the matrices A_{p+} in (2.15b) and $A_c(x)$ in (2.9a), respectively. Then $Z(\chi(x, s)) \subset D_{-\alpha}$ if and only if there exists an integer $N_n > 0$, and polynomials $d_0(s)$ and $n_0(s)$, of degree $N_d = N_n + n_s$ and N_n, respectively, with $n_s = n_c + n_+$, such that

$$(i) \ \ Z(d_0(s)) \subset D_{-\alpha} \ , \quad Z(n_0(s)) \subset D_{-\alpha} \ , \tag{2.17a}$$

$$(ii)\ \operatorname{Re}\left[\frac{\chi(x\,,s)n_0(s)}{d_0(s)}\right] > 0 \quad \forall s \in \partial U_{-\alpha}\,. \tag{2.17b}$$

■

In practice the test (2.17a,b) can be used only as a sufficient condition of stability, because one is forced to choose in advance the degree N_d of the polynomial $d_0(s)$. Furthermore, the polynomials $d_0(s)$ and $n_0(s)$ must be parametrized in such a way that satisfaction of (2.17a) can be ensured by satisfying a simple set of inequalities. We note that when $a\,,b \in \mathbf{R}$, $Z[(s+\alpha)+a)] \subset D_{-\alpha}$ if and only if $a > 0$, and $Z[(s+\alpha)^2 + a(s+\alpha) + b] \subset D_{-\alpha}$ if and only if $a > 0$, $b > 0$. Hence, assuming that the degree of $d_0(s)$ is odd, we set

$$d_0(s\,,y_d) \triangleq ((s+\alpha)+a_0)\prod_{i=1}^{m}((s+\alpha)^2 + a_i(s+\alpha) + b_i)\ , \tag{2.18}$$

where $y_d \triangleq [a_0\,,a_1\,,a_2\,,\cdots\,,a_m\,,b_1\,,b_2\,,\cdots\,,b_m]^T \in \mathbf{R}^{2m+1}$ and $N_d = 2m+1$. When the degree of $d_0(s)$ is even, the first factor in (2.18) is omitted. The polynomial $n_0(s)$, which is of degree $N_n = N_d - n_s$ can be parametrized similarly, with corresponding parameter vector y_n.

It now follows from Theorem 2.1 that the requirement of closed-loop stability reduces to solving the following set of inequality constraints:

$$\varepsilon - y_d^i \le 0, \qquad \text{for } i = 1\,,2\,,\cdots\,,N_d\,, \tag{2.19a}$$

$$\varepsilon - y_n^i \le 0, \qquad \text{for } i = 1\,,2\,,\cdots\,,N_n\,, \tag{2.19b}$$

$$\varepsilon - \operatorname{Re}\left(\frac{\chi(x\,,-\alpha+j\omega)n_0(y_n\,,-\alpha+j\omega)}{d_0(y_d\,,-\alpha+j\omega)}\right) \le 0\,, \quad \forall\omega \in [0\,,\infty)\ , \tag{2.19c}$$

where y_d^i is the i-th element of y_d, y_n^i is the i-th element of y_n and $\varepsilon > 0$ is small. If we define $\bar{x} = (x\,,y_n\,,y_d)$, and

$$\psi^1(\bar{x}) \triangleq \sup_{\omega \in [0,\infty)} \left[\varepsilon - \operatorname{Re}\left(\frac{\chi(x\,,-\alpha+j\omega)n_0(y_n\,,-\alpha+j\omega)}{d_0(y_d\,,-\alpha+jomega)}\right)\right], \tag{2.19d}$$

we see that (2.19c) is of the form

$$\psi^1(\bar{x}) \le 0\,. \tag{2.19e}$$

The evaluation of $\chi(x, -\alpha + j\omega)$ requires the evaluation of the closed-loop system frequency response. For an elementary treatment of how this computation can be carried out see the Appendix and [Wuu.1].

Now that we have established a set of inequalities ensuring stability, dealing with the remaining performance requirement is relatively easy.

(ii) Command Tracking and Output Disturbance Rejection. Suppose that the desired bandwidth for the feedback system is $[0, \omega_c]$. Both good tracking of the input u and good rejection of the output disturbance d_o, over this frequency interval, as well as reasonable behavior outside, can be achieved by making small $\| G_{11}(x, j\omega) \|$, the norm of the transfer function from the command input $\hat{u}$ to the tracking error $\hat{e}_1$. Therefore we define the performance function $\psi^2 : \mathbf{R}^n \to \mathbf{R}$ by

$$\psi^2(x) \triangleq \sup_{\omega \in [0, \omega_f]} \{ \bar{\sigma}[G_{11}(x, j\omega)] - b_{fo}(\omega) \}, \tag{2.20a}$$

where $b_{fo}(\cdot)$ is a piecewise continuous bound function and $\bar{\sigma}[A]$ denotes the largest singular value of A, and ω_f is the highest frequency over which quantitative design is required. We can now express the command tracking and output disturbance rejection requirement as an inequality:

$$\psi^2(x) \leq 0 . \tag{2.20b}$$

It remains to choose the bound function $b_{fo}(\cdot)$. Since by an extension of Bode's Integral Theorem to multivariable systems [Boy.1], it follows that for every frequency interval of nonzero measure over which the feedback system attenuates output disturbances, there must exist an interval of nonzero length over which the system amplifies output disturbances, we must let $b_{fo}(\cdot)$ exceed 1 over some frequency interval outside the system bandwidth. Therefore a simple choice for the bound function would be to set

$$\begin{aligned} 0 < b_{fo}(\omega) = b_1 << 1, \qquad & \text{if } \omega \leq \omega_c \\ b_{fo}(\omega) = b_2 > 1 \qquad & \text{if } \omega > \omega_c . \end{aligned} \tag{2.21}$$

(iii) Input Disturbance Rejection. To obtain good input disturbance rejection over the feedback system bandwidth $[0, \omega_c]$, we must keep small $\| G_{33}(x, j\omega) \|$, the norm of the transfer function from the input disturbance $\hat{d}_i(s)$ to the output $\hat{y}(s)$. Therefore we define the performance function

$\psi^3 : \mathbf{R}^n \rightarrow \mathbf{R}$ by

$$\psi^3(x) \triangleq \sup_{\omega \in [0, \omega_f]} \{ \bar{\sigma} [G_{33}(x, j\omega)] - b_{fi}(\omega) \} , \qquad (2.22a)$$

where $b_{fi}(\cdot)$ is a piecewise continuous bound function. Hence the input disturbance rejection requirement reduces to the inequality:

$$\psi^3(x) \leq 0 . \qquad (2.22b)$$

(iv) Plant-Input Saturation Avoidance. Since a large plant input, e_2, or state can drive the plant out of the operating region for which the linear model is valid, it is important to keep the plant input and state small, for otherwise deterioration of performance and instability may occur. Hence, to limit saturation effects produced by command inputs or output disturbances, we define the performance function $\psi^4 : \mathbf{R}^n \rightarrow \mathbf{R}$ by

$$\psi^4(x) \triangleq \sup_{\omega \in [0, \omega_f]} \{ \bar{\sigma}[G_{21}(x, j\omega)] - b_s \} \qquad (2.23a)$$

where $b_s > 0$ is a suitable bound for the plant input power spectrum amplitude. The plant-input saturation avoidance requirement can now be formulated as

$$\psi^4(x) \leq 0 . \qquad (2.23b)$$

(v) Stability Robustness to Plant Uncertainty. Plant models always have some uncertainty in them. Since closed loop stability is a fundamental requirement, the design process must take into account not only the *nominal* model, which was used to set up (2.19a-c), but also model uncertainty. The following result (see [Doy.2], [Che.1]) gives a characterization of plant uncertainty and a corresponding condition for stability robustness.

Theorem 2.2: Consider the feedback system in Fig. 1, and assume that the compensator has been chosen so that this system is 0-stable for the *nominal* plant transfer function $G_p(s)$. Let $l : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be a continuous, strictly positive "tolerance function" such that for some $k \in \mathbf{N}$, $\omega_0 \in \mathbf{R}_+$, $l(\omega) > 1/\omega^k$ for all $\omega > \omega_0$. Let the set

$$\mathbf{G}_l \triangleq \{ \tilde{G}_p \in \mathbf{R}(s)^{n_o \times n_i} \mid \tilde{G}_p(j\omega) = G_p(j\omega) + \delta G_p(j\omega) ,$$

$$\bar{\sigma}[\delta G_p(j\omega)] < l(\omega)\,,\ \omega \in \mathbf{R}_+\,,\ \ N_{\tilde{G}_p} = N_{G_p}\,\} \tag{2.24a}$$

where $N_{\tilde{G}_p}$, N_{G_p} denote the number of unstable poles of $\tilde{G}_p$ and G_p, respectively[4] . Then, the feedback system in Fig. 1 is 0-stable for all $\tilde{G}_p \in \mathbf{G}_l$, if and only if

$$\bar{\sigma}[G_{21}(x\,, j\omega)] \le 1/l(\omega) \quad \forall \omega \ge 0\,. \tag{2.24b}$$

■

Hence, if we define the function $\psi^5 : \mathbf{R}^n \to \mathbf{R}$ by

$$\psi^5(x) \triangleq \sup_{\omega \in [0\,,\infty)} \{\, \bar{\sigma}[G_{21}(x\,, j\omega)] - 1/l(\omega)\,\} \tag{2.25a}$$

then for all x such that

$$\psi^5(x) \le 0 \tag{2.25b}$$

holds, the compensator $K(x)$ will stabilize not only the plant G_p, but also any plant $\tilde{G}_p \in \mathbf{G}_l$.

B. Time Domain Performance Specifications

Frequency domain performance specifications are inadequate when "hard" time-domain bounds need to be satisfied at various points in the feedback loop. For example, it has been traditional to impose bounds, in terms of rise time, overshoot, and settling time specifications on feedback system zero-state step responses. In general, the satisfaction of such time-response specifications cannot be insured by shaping transfer functions in frequency domain.

We will denote the zero-state responses of the system (2.12a,b) by $e_1(t\,, u\,, d_o\,, d_i)$, $e_2(t\,, u\,, d_o\,, d_i)$, and $y(t\,, u\,, d_o\,, d_i)$, respectively. We will continue to denote components of a vector by superscripts.

(i) **Time Domain Responses.** Suppose that we are required to ensure that the step response, as measured in the output $y^1(t)$, when $u(t) = u_s(t) \triangleq (1\,, 0\,, 0 \ldots, 0)$, $d_o(t) \equiv 0$, $d_i(t) \equiv 0$, is contained in a window defined by upper and a lower, piecewise continuous bound functions, $\bar{b}(t)$, and $\underline{b}(t)$, respectively. Let $\psi^6 : \mathbf{R}^n \to \mathbf{R}$ be defined by

[4] A larger set of plant perturbations is considered in [Che.1].

$$\psi^6(x) \triangleq \max_{t \in [0, t_f]} \{ y^1(t, u_s, 0, 0) - \bar{b}(t) \} , \tag{2.26a}$$

and let $\psi^7 : \mathbf{R}^n \to \mathbf{R}$ be defined by

$$\psi^7(x) \triangleq \max_{t \in [0, t_f]} \{ \underline{b}(t) - y^1(t, u_s, 0, 0) \} \tag{2.26b}$$

Then the step response specification reduces to the pair of inequalities:

$$\psi^6(x) \leq 0 , \qquad \psi^7(x) \leq 0 . \tag{2.26c}$$

More generally, we may require that the plant output track, within a tolerance, a given command input, say $u(t) = u_d(t)$. Let $b_d(\cdot)$ be a piecewise continuous tolerance function, and let $\psi^8 : \mathbf{R}^n \to \mathbf{R}$ be defined by

$$\psi^8(x) \triangleq \max_{t \in [0, t_f]} \{ \| y(t, u_d, 0, 0) - u_d(t) \|^2 - b_d(t) \} . \tag{2.27a}$$

Then the tracking requirement becomes

$$\psi^8(x) \leq 0 . \tag{2.27b}$$

(ii) Output Disturbance Rejection. There may be a need to limit the effect of "persistent" output disturbances on the plant output (see [Vid.2], [Dah.1]). Frequency-domain disturbance specifications may be inadequate to meet this need, and one must deal with this problem in the time-domain by limiting the induced sup-norm of the operator that takes plant output disturbances into plant outputs. Clearly, the zero-state response of the feedback system in Fig. 1 to an output disturbance (assuming that all other inputs are zero) has the form

$$y(t, 0, d_o, 0) = \int_0^t K_{d_o}(x, t - \tau) d_o(\tau) d\tau . \tag{2.28a}$$

Let $\psi^9 : \mathbf{R}^n \to \mathbf{R}$ be defined by

$$\psi^9(x) \triangleq \sup_{t \in [0, \infty)} \int_0^t \| K_{d_o}(x, \tau) \| \, d\tau - b_o , \tag{2.28b}$$

where $b_o > 1$. Then, the requirement that

$$\psi^9(x) \le 0 \ , \tag{2.28c}$$

ensures that no output disturbance $d_o(\cdot)$ with norm less than 1 will produce a feedback system output of norm larger than b_o.

(iii) Plant Saturation Avoidance. The frequency-domain saturation avoidance inequality (2.23b) does not limit the time-domain amplitude of the plant input when the command input has significant spectral content outside the closed-loop system bandwidth. We can ensure that the plant input does not exceed required bounds by limiting the size of the induced sup-norm of the operator that takes the command input into the plant input. This results in an inequality similar to (2.28c).

C. Formulation of Optimal Design Problem

First of all, it is not clear that a set of specifications, such as those stated above, is consistent. Hence it may be desirable to solve first a "phase I" problem of the form:

$$\nu = \min_{x \in X} \max_{j \in \mathbf{m}} \psi^j(x) \ . \tag{2.29}$$

If the value ν turns out to be negative, all the specifications can be satisfied. If ν turns out to be positive, some compromise must be reached either by relaxing the bounds in the definitions of the $\psi^j(\cdot)$, or by increasing the compensator dimension until satisfaction is obtained. Once this has been done, then one of the performance functions can be designated as the cost function, while the others become constraint functions in a design problem of the form (2.1a). Alternatively, weights can be introduced into the minimax problem (2.29) as a way of obtaining a compromise design.

3. ALGORITHMS

Firstly, for an extensive treatment of nonsmooth optimization algorithms we refer the reader to [Pol.3]. In this paper we will content ourselves with an introduction to the subject. The easiest way to explain algorithms for solving (2.1a) and (2.2a) is to proceed in stages which take us from a *conceptual* algorithm for solving finite dimensional problems of the form (2.1a), to an *implementable* algorithm for solving finite dimensional problems of the form (2.1a), to an *implementable* algorithm which solves optimal control problems, with control and state space constraints, and either ODE or PDE dynamics, of the form (2.2a). We will not treat separately the problem (2.29) since an algorithm for it is obtained from one for solving (2.1a), by setting $\psi^0(x) \equiv -\infty$, or by observing that it is equivalent to the problem below, defined on $\mathbf{R}^{n+1}$, in which we denote vectors by $\bar{x} = (x, x^{n+1}) = (x^1, \cdots, x^{n+1})$:

$$\min\{ x^{n+1} \mid \psi^j(x) - x^{n+1} \le 0,\ j \in \mathbf{m},\ \ x \in \mathbf{X} \}. \tag{3.1}$$

Let the *steering parameter*[5] $\gamma \ge 1$ be given and let

$$\psi(x) \triangleq \max_{j \in \mathbf{m}} \psi^j(x), \tag{3.2a}$$

$$\psi_+(x) \triangleq \max\{0, \psi(x)\}, \tag{3.2b}$$

and, for any $z \in \mathbf{R}^n$, let the parametrized function $F_z(x)$ be defined by

$$F_z(x) \triangleq \max\{ \psi^0(x) - \psi^0(z) - \gamma\psi_+(z), \psi^j(x) - \psi_+(z),\ j \in \mathbf{m} \}. \tag{3.2c}$$

Our first observation is that for any $z \in \mathbf{R}^n$, $F_z(z) = 0$. Our second observation is that if x^* is a local minimizer for (2.1a), then it follows from the fact that (i) $\psi(x) > 0$ when x is infeasible for (2.1a), and (ii) $\psi^0(x) > \psi^0(x^*)$ when x is feasible, but not optimal for (2.1a), that x^* must also be a local minimizer for the problem

$$\min_{x \in \mathbf{X}} F_{x^*}(x). \tag{3.2d}$$

Hence, as we shall now show, the function $F_z(\cdot)$ provides a very useful means for obtaining a first order optimality condition for the problem (2.1a)[6].

Now, suppose that given $z \in \mathbf{R}^n$, we approximate each function $\phi^j(x, y_j)$, $j = 0, 1, 2, \dots, m$, around z by the following *first order convex* approximation:

$$\hat{\phi}^j_z(x, y_j) \triangleq \phi^j(z, y_j) + \langle \nabla_x \phi^j(z, y_j), (x - z) \rangle + \tfrac{1}{2} \| x - z \|^2. \tag{3.3a}$$

Then $\psi^j(x)$ is approximated around z by the *first order convex* approximation

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[5] An examination of (3.2c) shows that the value of γ and, in fact, the term $\psi_+(z)$ has no effect at *feasible points*. We shall see later that their inclusion enables us to construct a phase I - phase II algorithm which does not require a feasible starting point.

[6] In [Cla.1] the reader will find optimality conditions for (2.1a) in the more familiar form involving generalized gradients and multipliers, emanating from the fact that if x^* is optimal for (3.2d), then $dF_{x^*}(x^*; x - x^*) \ge 0$ must hold for all $x \in \mathbf{X}$. It is shown in [Pol.3] that the conditions given in this paper, which were derived specifically for use in algorithm construction, are equivalent to the ones in [Cla.1].

$$\hat{\psi}_z^j(x) \triangleq \max_{y_j \in Y_j} \hat{\phi}_z^j(x, y_j), \quad j = 0, 1, 2, \ldots, m, \tag{3.3b}$$

and, in turn, $F_z(x)$ is approximated around z, by the *first order convex* approximation

$$\hat{F}_z(x) \triangleq \max \{ \hat{\psi}_z^0(x) - \psi^0(z) - \gamma\psi_+(z), \hat{\psi}_z^j(x) - \psi_+(z), \ j \in \mathbf{m} \}. \tag{3.3c}$$

Referring to [Pol.3] we find the following first order optimality condition for (2.1a), where we find it convenient to replace x in (3.3c) by $x^* + h$:

Theorem 3.1 :[Pol.3] If x^* is a local minimizer for (2.1a), then

$$\theta(x^*) \triangleq \min_{h \in X - \{x^*\}} \hat{F}_{x^*}(x^* + h) = 0. \tag{3.4a}$$

■

It is shown in [Pol.3] that $\theta(\cdot)$ is continuous; it follows by inspection that $\theta(x) \leq 0$ for all $x \in \mathbf{R}^n$. Furthermore, it is shown in [Pol.3] that at any $x \in \mathbf{R}^n$ such that $\theta(x) < 0$, the vector

$$\eta(x) \triangleq \arg \min_{h \in X - \{x\}} \hat{F}_x(x + h), \tag{3.4b}$$

has the following property: if $\psi(x) > 0$, then $\eta(x)$ is a descent direction for $\psi(\cdot)$; if $\psi(x) \leq 0$, then $\eta(x)$ is a descent direction for $\psi^0(\cdot)$ along which the constraints will not be violated for some distance. The search direction function $\eta(\cdot)$ can be shown to be continuous (see [Pol.3]).

Phase I - phase II methods of feasible directions, such as the ones described in [Pol.3], as well as the optimal control algorithm that we will present, can be seen as progressively more complex implementations of the following *conceptual* method, which we have derived by extension, from the Huard method of centers [Hua.1], for solving (2.1a).

Algorithm 3.1 (Phase I - Phase II Method of Centers) :

Parameters : $\gamma \geq 1$.

Data : $x_0 \in \mathbf{R}^n$,

Step 0 : Set $i = 0$.

Step 1 : Compute $x_{i+1} = \arg \min_{x \in X} F_{x_i}(x)$.

Step 2 : Set $i = i + 1$ and go to Step 1. ■

Since $F_{x_i}(x_i) = 0$, $F_{x_i}(x_{i+1}) < 0$. Hence if $\psi(x_{i_0}) \le 0$ for some i_0, then $\psi(x_i) \le 0$ for all $i \ge i_0$. When all the functions are convex, it can be shown that the value of γ controls the speed with which the above algorithm approaches the *feasible set*, $\{ x \mid \psi(x) \le 0 \}$: the larger γ, the faster the algorithm drives the iterates x_i into the feasible set.

Theorem 3.2 : Suppose that for every $x \in \mathbf{R}^n$ which is not a local minimizer of (2.1a),

$$M(x) \triangleq \min_{x' \in \mathbf{X}} F_x(x') < 0 , \tag{3.4c}$$

and that either $\mathbf{X}$ is compact or that the level sets of $F_x(\cdot)$ are compact. If $\{ x_i \}_{i=0}^{\infty}$ is an infinite sequence constructed by Algorithm 3.1, then every accumulation point x^* of $\{ x_i \}_{i=0}^{\infty}$ is a local minimizer for (2.1a).

Proof : First, referring to [Ber.1], we conclude that because of the compactness assumption, $M(\cdot)$ is continuous. Next, suppose that $\{ x_i \}_{i=0}^{\infty}$ is an infinite sequence constructed by Algorithm 3.1 which has an accumulation point x^* such that $M(x^*) < 0$, i.e., x^* is not a local minimizer for (2.1a). Then there exists an infinite subset $K \subset \mathbf{N}$ such that the subsequence $\{ x_i \}_{i \in K}$ converges to x^*, and hence, by continuity of $M(\cdot)$, there exists an i_o such that $M(x_i) \le M(x^*)/2 < 0$ for all $i \ge i_o$, $i \in K$. Now there are two possibilities.

First suppose that $\psi(x_i) > 0$ for all $i \in \mathbf{N}$. Then, because $\psi(x_{i+1}) - \psi(x_i) \le M(x_i) \le 0$ holds for all i, the sequence $\{ \psi(x_i) \}_{i=0}^{\infty}$ is monotone decreasing, and hence, since $\psi(\cdot)$ is continuous and since $\{ x_i \}_{i=0}^{\infty}$ has an accumulation point, the sequence $\{ \psi(x_i) \}_{i=0}^{\infty}$ must converge. However, this contradicts the fact that $\psi(x_{i+1}) - \psi(x_i) \le M(x_i) \le M(x^*)/2 < 0$ for all $i \ge i_o$, $i \in K$.

Next, suppose that there exists an i_1 such that $\psi(x_{i_1}) \le 0$. Then, for all $i \ge \max\{i_o, i_1\}$, we must have $\psi(x_{i+1}) \le M(x_i) \le 0$ and $\psi^0(x_{i+1}) - \psi^0(x_i) \le M(x_i) < 0$, so that the sequence $\{ \psi^0(x_i) \}_{i=0}^{\infty}$ is monotone decreasing. Since $\{ x_i \}_{i=0}^{\infty}$ has an accumulation point, the sequence $\{ \psi^0(x_i) \}_{i=0}^{\infty}$ must converge. However, this contradicts the fact that $\psi^0(x_{i+1}) - \psi^0(x_i) \le M(x_i) \le M(x^*)/2 < 0$ for all $i \ge i_o$, $i \in K$.

■

It should be obvious that the unconstrained minimax problems, in Step 1 of the Method of Centers 3.1, are hardly any easier to solve than the original problem (2.1a). However, a very efficient algorithm can be obtained by replacing the computation in Step 1 of this method by the approximation indicated in (3.4a,b) and supplementing it by an Armijo type step size rule [Arm.1], as follows[7]:

Algorithm 3.2 (Phase I - Phase II Method of Feasible Directions) :

Parameters : $\gamma > 1$, α, $\beta \in (0,1)$.

Data : $x_0 \in \mathbf{R}^n$.

Step 0 : Set $i = 0$.

Step 1 : Compute the the *optimality function* value $\theta_i = \theta(x_i)$, and the corresponding *search direction* $\eta_i = \eta(x_i)$.

Step 2 : Compute the *step size* λ_i:

$$\lambda_i = \max \{ \beta^k \mid k \in \mathbf{N},\ F_{x_i}(x_i + \beta^k \eta_i) \le \beta^k \alpha \theta_i \} . \tag{3.5a}$$

Step 3 : Set $x_{i+1} = x_i + \lambda_i \eta_i$, set $i = i + 1$ and go to Step 1. ■

The Armijo step size rule is well known to be efficient and is used in many algorithms. The sensitivity to the value of γ and the convergence properties of the Phase I - Phase II Method of Feasible Directions 3.2 are quite similar to those of the Phase I - Phase II Method of Centers 3.1 (for a proof see [Pol.9]):

Theorem 3.3 : Suppose that for every $x \in X$ such that $\psi(x) \ge 0$, $\theta(x) < 0$. If $\{ x_i \}_{i=0}^{\infty}$ is an infinite sequence constructed by Algorithm 3.2, then every accumulation point x^* of $\{ x_i \}_{i=0}^{\infty}$ satisfies the first order optimality condition, for (2.1a), $\psi(x^*) \le 0$, $\theta(x^*) = 0$.

Proof : First, we recall that it can be shown that both $\theta(\cdot)$ and $\eta(\cdot)$ are continuous. Next, suppose

[7] The phase I - phase II algorithms reported in [Pol.3] use a different step size rule when $\psi(x_i) > 0$ and when $\psi(x_i) < 0$. The simplified algorithm in this paper (see [Pol.9]) is only slightly less efficient, because it evaluates the cost function at infeasible points.

that $\{x_i\}_{i=0}^{\infty}$ is an infinite sequence constructed by Algorithm 3.2 which has an accumulation poin x^* such that $\theta(x^*) < 0$[8]. Then there exists an infinite subset $K \subset \mathbf{N}$ such that the subsequence $\{x_i\}_{i \in K}$ converges to x^*, and hence, by continuity of $\theta(\cdot)$, there exists an i_o such that $\theta(x_i) \le \theta(x^*)/2 < 0$ for all $i \ge i_o$, $i \in K$. Next, since $dF_{x^*}(x^*; \eta(x^*)) \le \theta(x^*)$, it follows that there exists a $k^* < \infty$ such that

$$F_{x^*}(x^* + \beta^{k^*}\eta(x^*)) - \beta^{k^*}\alpha\theta(x^*) < 0 . \tag{3.5b}$$

Since $\theta(\cdot)$, $\eta(\cdot)$, and $F_z(x)$ are all continuous, it follows from (3.5b) that there exists a finite $i_1 \ge i_o$, such that for all $i \in K$, $i \ge i_1$, $\lambda_i \ge \beta^{k^*}$ and hence for all $i \in K$, $i \ge i_1$,

$$F_{x_i}(x_{i+1}) \le \beta^{k^*}\alpha\theta(x^*)/2 < 0 . \tag{3.5c}$$

As in the proof of Theorem 3.1, there are now two possibilities. First suppose that $\psi(x_i) > 0$ for all $i \in \mathbf{N}$. Then, because $\psi(x_{i+1}) - \psi(x_i) \le F_{x_i}(x_{i+1}) \le 0$ holds for all i, the sequence $\{\psi(x_i)\}_{i=0}^{\infty}$ is monotone decreasing, and hence, since $\psi(\cdot)$ is continuous and since $\{x_i\}_{i=0}^{\infty}$ has an accumulation point, the sequence $\{\psi(x_i)\}_{i=0}^{\infty}$ must converge. However, this contradicts the fact that $\psi(x_{i+1}) - \psi(x_i) \le F_{x_i}(x_{i+1}) \le \beta^{k^*}\alpha\theta(x^*)/2 < 0$ for all $i \ge i_1$, $i \in K$.

Next, suppose that there exists an i_2 such that $\psi(x_{i_2}) \le 0$. Then, for all $i \ge \max\{i_1, i_2\}$, we must have $\psi(x_{i+1}) \le F_{x_i}(x_{i+1}) \le 0$ and $\psi^0(x_{i+1}) - \psi^0(x_i) \le F_{x_i}(x_{i+1}) \le 0$, so that the sequence $\{\psi^0(x_i)\}_{i=0}^{\infty}$ is monotone decreasing. Since $\{x_i\}_{i=0}^{\infty}$ has an accumulation point, the sequence $\{\psi^0(x_i)\}_{i=0}^{\infty}$ must converge. However, this contradicts the fact that $\psi^0(x_{i+1}) - \psi^0(x_i) \le \beta^{k^*}\alpha\theta(x^*)/2 < 0$ for all $i \ge i_2$, $i \in K$ ■

When the functions $\psi^j(\cdot)$ in (2.1a) are all differentiable, Algorithm 3.2 can be used directly. However, when the functions $\psi^j(\cdot)$ are max functions, then neither these functions nor the optimality function $\theta(\cdot)$ can be evaluated exactly on a digital computer in finite time. Hence, for such problems Algorithm 3.2 must be viewed as a *conceptual algorithm.* To construct an implementable version, we make use of the theory developed in [Kle.1], which allows us to discretize the intervals $\mathbf{Y}_j$ adaptively, as follows. For $j = 1, 2, \ldots, m$, let $l_j \triangleq (b_j - a_j)$ be the length of the interval $\mathbf{Y}_j$. Next, for any positive integer q, we define the corresponding discretized versions of the functions used by Algorithm 3.2:

[8] The case where $\psi(x^*) > 0$ is eliminated by our assumption.

$$\mathbf{Y}_{jq} \triangleq \{ a_j \ , \ a_j + \frac{l_j}{q} \ , \ a_j + \frac{2l_j}{q} \ , \ \ldots \ , \ b_j \} \ , \tag{3.6a}$$

$$\psi_q^j(x) \triangleq \max_{y \in \mathbf{Y}_{jq}} \phi^j(x \, , y) \, , \tag{3.6b}$$

$$F_{q\,z}(x) \triangleq \max \{ \psi_q^0(x) - \psi_q^0(z) - \gamma\psi_{q+}(z) \, , \psi_q^j(x) - \psi_{q+}(z) \, , \ j \in \mathbf{m} \} \ , \tag{3.6c}$$

$$\hat{\psi}_{q\,z}^j(x) \triangleq \max_{y \in \mathbf{Y}_{jq}} \hat{\phi}_z^j(x \, , y) \, , \tag{3.6d}$$

$$\hat{F}_{q\,z}(x) \triangleq \max \{ \hat{\psi}_{q\,z}^0(x) - \psi_q^0(z) - \gamma\psi_{q+}(z) \, , \hat{\psi}_{q\,z}^j(x) - \psi_{q+}(z) \, , \ j \in \mathbf{m} \} \ , \tag{3.6e}$$

$$\theta_q(z) \triangleq \min_{x \in \mathbf{X}} \hat{F}_{q\,z}(x) \, , \tag{3.6f}$$

$$\eta_q(z) \triangleq \arg \min_{h \in \mathbf{X} - \{z\}} \hat{F}_{q\,z}(z + h) \ . \tag{3.6g}$$

The following easy to prove result assures that the implementable version of Algorithm 3.2, to be stated shortly, satisfies the requirements of the theory in [Kle.1].

Proposition 3.1 : [Bak.1] There exists a $K < \infty$ such that for all $z \ , \ x \in \mathbf{X}$,

$$|\psi^j(x) - \psi_q^j(x)| \le \frac{K}{q} \, , \quad j = 0 \, , 1 \, , 2 \, , \ldots \, , m \ , \tag{3.7a}$$

$$|F_z(x) - F_{q\,z}(x)| \le \frac{K}{q} \, , \tag{3.7b}$$

$$|\hat{\psi}_z^j(x) - \hat{\psi}_{q\,z}^j(x)| \le \frac{K}{q} \, , \quad j = 0 \, , 1 \, , 2 \, , \ldots \, , m \ , \tag{3.7c}$$

$$|\hat{F}_z(x) - \hat{F}_{q\,z}(x)| \le \frac{K}{q} \, , \tag{3.7d}$$

$$|\theta(x) - \theta_q(x)| \le \frac{K}{q} \, , \tag{3.7e}$$

$$\| \eta(x) - \eta_q(x) \|^2 \le \frac{K}{q} \ . \tag{3.7f}$$

We can now state an implementable version of Algorithm 3.2 which increases the discretization of the intervals $\mathbf{Y}_j$ whenever the reduction per iteration in constraint violation, or cost, as appropriate, drops below a preassigned level, which we will call ε.

Algorithm 3.3 (Implementable Phase I - Phase II Method of Feasible Directions) :

Parameters : $q \in \mathbf{N}$, $\varepsilon > 0$, $\gamma \geq 1$, α, $\beta \in (0,1)$.

Data : $x_0 \in \mathbf{R}^n$.

Step 0 : Set $i = 0$.

Step 1 : Compute the the *optimality function* value $\theta_i = \theta_q(x_i)$, and the corresponding *search direction* $\eta_i = \eta_q(x_i)$.

Step 2 : Compute the *step size* λ_i:

$$k_i = \arg\max \{ \beta^k \mid k \in \mathbf{N},\ F_{q\, x_i}(x_i + \beta^k \eta_i) \leq \beta^k \alpha \theta_i \} . \tag{3.8a}$$

If

$$F_{q\, x_i}(x_i + \beta^{k_i} \eta_i) > -\varepsilon , \tag{3.8b}$$

replace q by $2q$, ε by $\varepsilon/2$ and go to Step 1. Else set $\lambda_i = \beta^{k_i}$.

Step 3 : Set $x_{i+1} = x_i + \lambda_i \eta_i$, set $i = i + 1$ and go to Step 1. ■

When $\mathbf{X} = \mathbf{R}^n$, the search direction, η_i, in the above algorithm (as well as in Algorithm 3.2) can be computed quite efficiently using the algorithms in [Hoh.1, Hig.1]; when $\mathbf{X}$ is polyhedral, Polyak's constrained Newton algorithm [Pol.10] can be used (see [Pol.5]), after (3.5a) has been converted to dual form. Since the discretization rule that we have described satisfies the assumptions in [Kle.1], we obtain the following result.

Theorem 3.4 : Suppose that for every $x \in \mathbf{X}$ such that $\psi(x) \geq 0$, $\theta(x) < 0$. If Algorithm 3.3 jams up, cycling between Step 1 and Step 2, at a point x_k, then x_k satisfies the first order condition $\psi(x_k) \leq 0$ and $\theta(x_k) = 0$. If Algorithm 3.3 constructs an infinite sequence $\{ x_i \}_{i=0}^{\infty}$, then every accumulation point x^* of $\{ x_i \}_{i=0}^{\infty}$ satisfies the first order optimality condition, for (2.1a), $\psi(x^*) \leq 0$, $\theta(x^*) = 0$.

Proof: According to the theory in [Kle.1], because (3.7a) holds, we only need to show that for every $\hat{x} \in \mathbf{R}^n$ such that $\theta(\hat{x}) < 0$, there exist a $\hat{\rho} > 0$, a $\hat{\delta} > 0$ and a $\hat{q} \in \mathbf{N}_+$ such that for all $x \in \mathbf{B}(\hat{x}, \hat{\rho}) \triangleq \{ x \in \mathbf{X} \mid \| x - \hat{x} \| \leq \hat{\rho} \}$, if $x_i \in \mathbf{B}(\hat{x}, \hat{\rho})$, $q \geq \hat{q}$, and k_i is constructed according to

(3.8a), then

$$F_{q\,x_i}(x_i + \beta^{k_i}\eta(x_i) \leq -\hat{\delta}\,. \tag{3.9a}$$

Now, referring to the proof of Theorem 3.3, we see that for Algorithm 3.2 there exists a $\hat{\rho} > 0$, a $\hat{\delta} > 0$ and a $\hat{k} < \infty$ such that for all $x_i \in \mathbf{B}(\hat{x}\,,\hat{\rho})$,

$$F_{x_i}(x_i + \beta^{\hat{k}}\eta(x_i)) - \beta^{\hat{k}}\alpha\theta(x_i) \leq -2\hat{\delta}\,. \tag{3.9b}$$

Now, it follows from (3.7a-f) that there exists a $\hat{q} < \infty$, such that if $q \geq \hat{q}$, $x_i \in \mathbf{B}(\hat{x}\,,\hat{\rho})$ and k_i is computed according to (3.8a), as appropriate, then $k_i \leq \hat{k}$ must hold. The desired result now follows from the continuity of $\theta(\cdot)$ and (3.7a-f). ■

We will obtain an algorithm for solving optimal control problems of the form (2.2a) by formal extension of Algorithm 3.3. First, to obtain a first order optimality condition for problem (2.2a), we need to obtain an analogue of the expressions (3.2a-c), (3.3a-c) and (3.4a-b). Clearly, analogues of (3.2a-c) are obtained by replacing x in (3.2a-c) by the pair $(u\,,T)$. Next, the analogue of (3.3a) is seen to be given by

$$\hat{\phi}^j_{u',T'}(u\,,T\,,t) \triangleq g^j(z^{u',T'}(t)) + \langle \nabla g^j(z^{u',T'}(t)\,,\delta z^{u',T'}(t\,;u(t) - u'(t)\,,T - T')\rangle$$
$$+ \tfrac{1}{2}\int_0^1 \| u(t) - u'(t) \|^2 dt + \tfrac{1}{2}|T - T'|^2\,,\quad j = 0\,,1\,,2\,,\ldots\,,m\,, \tag{3.10}$$

Next, the analogues of (3.3b -c) are seen to be

$$\hat{\psi}^j_{u',T'}(u\,,T) \triangleq \max_{t \in \mathbf{Y}_j} \hat{\phi}^j_{u',T'}(u\,,T\,,t)\,,\quad j = 0\,,1\,,2\,,\ldots\,,m\,, \tag{3.11a}$$

$$\hat{F}_{u',T'}(u\,,T) \triangleq \max\{\,\hat{\psi}^0_{u',T'}(u\,,T) - \psi^0(u'\,,T') - \gamma\psi_+(u'\,,T')\,,\hat{\psi}^j_{u',T'}(u\,,T) - \psi_+(u'\,,T')\,,\ j \in \mathbf{m}\,\}\,. \tag{3.11b}$$

The analogues of (3.4a-b), defining the optimality function $\theta(\cdot)$ and search direction function $\eta(\cdot\,,\cdot)$ are:

$$\theta(u'\,,T') \triangleq \min_{(u\,,T) \in \mathbf{U} \times \mathbf{T}} \hat{F}_{u',T'}(u\,,T)\,, \tag{3.12a}$$

$$\eta(u'\,,T') \triangleq \arg \min_{(u\,,T) \in \mathbf{U} \times \mathbf{T} - \{u'\,,T'\}} \hat{F}_{u',T'}(u' + u\,,T' + T)\,. \tag{3.12b}$$

Theorem 3.1 assumes the following form for problem (2.2a):

Theorem 3.5 : If (u^* , T^*) is a local minimizer for (2.2a), then

$$\theta(u^* , T^*) = 0 . \tag{3.13}$$

■

Since $\phi^j(u', T', t) \triangleq g^j(z^{u', T}(t))$, our first observation is that expressions such as

$$d\phi^j((u', T', t) ; (u(t) - u'(t), T - T')) = \langle \nabla g^j(z^{u', T}(t)), \delta z^{u', T}(t ; u(t) - u'(t), T - T')\rangle \tag{3.14}$$

can be computed using adjoints. Our second observation is that the numerical solution of ordinary or partial differential equations requires discretization of at least one variable and hence that we cannot utilize the analogue of Algorithm 3.3 without addressing this source of difficulty. To ensure that our final implementable algorithm has the desired convergence properties, we must use discretizations in the solution of the ODEs or PDEs which guarantee that the relations (3.7a - d) are satisfied, with x replaced by u , T. Again we receive guidance from the theory in [Kle.1], where the discretizations are worked out for ODEs. Hence we will only consider the case of PDEs here. To make matters concrete, we may assume that H is the space of r-times differentiable functions, $z(s)$, from $[0, 1]$ into $\mathbb{R}^p$.

First we introduce a set of orthogonal spline functions $\{ \zeta^i_{q_s}(\cdot) \}_{i=0}^{2^{q_s}} \subset H$, for "spatial" discretization[9], write $z^{u, T}(t, s)$ in the form

$$z^{u, T}(t, s) = \sum_{i=0}^{2^{q_s}} \zeta^i_{q_s}(s) \omega^i_{q_s}(t, u, T) . \tag{3.15a}$$

and compute the projection Πz_0 of z_0 onto the subspace of H spanned by the splines. Let $\omega_{q_s} = (\omega^0_{q_s}, \dots, \omega^{2^{q_s}}_{q_s})$, and let Z_{q_s} be a matrix with columns $\zeta^i_{q_s}$, $i = 0, 1, \dots, 2^{q_s}$. Then (3.15a) can be written in the shorter form

$$z^{u, T}(t, s) = Z_{q_s}(s) \omega_{q_s}(t, u, T) . \tag{3.15b}$$

On the subspace spanned by the splines, our dynamics have the form

---

[9] For many dynamical systems, a system of second order PDEs, coupled with ODEs, is a more "natural" description than (3.10a). In that case all calculations are carried out with the original dynamics. As a result, since the weak form of a solution is used, it is often possible to use splines that are only $r/2$-times differentiable, which results in considerable computational simplification. Also, Newmark's method is then used for temporal discretization. See [Str.1] for details.

$$Z_{q_s}(s)\dot{\omega}_{q_s}(t\,,u\,,T) = T[AZ_{q_s}(s)\omega_{q_s}(t\,,u\,,T) + h(Z_{q_s}(s)\omega_{q_s}(t\,,u\,,T)\,,u(t))]\,, \tag{3.15c}$$

$$Z_{q_s}(s)\omega_{q_s}(0\,,u\,,T) = \Pi z_o(s)\ \ ,\ \ \forall s \in [0\,,1]\,. \tag{3.15d}$$

Next we use the orthogonality of the splines to set up the differential equations for the functions $\omega^i_{q_s}(t\,,u\,,T)$:

$$\dot{\omega}^i_{q_s}(t\,,u\,,T) = \int_0^1 \langle\, \zeta^i_{q_s}(s)\,, T[AZ_{q_s}(s)\omega_{q_s}(t\,,u\,,T) + h(Z_{q_s}(s)\omega_{q_s}(t\,,u\,,T)\,,u(t))]\rangle\, ds\ \ ,\ \ i = 0\,,1\,,2\,,3\,,\ldots\,,2^{q_s}\,, \tag{3.15e}$$

$$\omega^i_{q_s}(0\,,u\,,T) = \int_0^1 \langle\, \zeta^i_{q_s}(s)\,, \Pi z_o(s)\rangle\, ds\ \ ,\ \ i = 0\,,1\,,2\,,3\,,\ldots\,,2^{q_s}\,. \tag{3.15f}$$

Then (3.15e,f) can be written as a first order vector differential equation in which the function $F(\cdot\,,\cdot)$ is defined by (3.15c):

$$\dot{\omega}(t) = TF(\omega(t)\,,u(t)\,,T)\ \ ,\ \ \forall t \in [0\,,1]\,,\ \ \omega(0) = \omega_0\,. \tag{3.16a}$$

Finally, we discretize the normalized time interval $[0\,,1]$ into 2^{q_t} equal intervals, set $\Delta_{q_t} \triangleq 1/2^{q_t}$, and replace (3.16a) by the difference equation resulting from the use of the Euler method of integration:

$$\omega((k+1)\Delta_{q_t}) = \omega(k\Delta_{q_t}) + TF(\omega(k\Delta_{q_t})\,,u(k\Delta_{q_t})\,,T)\ \ ,\ \ \forall k = 0\,,1\,,2\,,\ldots\,,2^{q_t}\,,\ \ \omega(0) = \omega_0 \tag{3.16b}$$

We are now ready to relate this construction to the quantities defined in (3.6a-g). First, let

$$p_{q_t}(t) \triangleq \begin{cases} 1 \text{ for } t \in [0\,,\Delta_{q_t}] \\ 0 \text{ for } t > \Delta_{q_t} \end{cases}\ , \tag{3.17a}$$

let $\mathbf{U}_{q_t} \subset \mathbf{U}$ be the set of controls which are constant over our time grid, i.e., if $u(t) \in \mathbf{U}_{q_t}$, then for a sequence of vectors $\{\,u_k\,\}_{k=0}^{2^{q_t}-1} \subset U$,

$$u(t) = \sum_{k=0}^{2^{q_t}-1} u_k p_{q_t}(t - k\Delta_{q_t})\ , \tag{3.17b}$$

and, finally, let $\omega_{q_t}(k\Delta_{q_t}\,,u\,,T)$ denote the solution of (3.16b) corresponding to a control in $\mathbf{U}_{q_t}$. If we let $\mathbf{Y}_{jq_t} \triangleq \{\,1\,\}$ for $j = 0\,,1\,,2\,,\ldots\,,m_1$, and $\mathbf{Y}_{jq_t} = \mathbf{Y} \triangleq \{\,0\,,\frac{1}{\Delta_{q_t}}\,,\frac{2}{\Delta_{q_t}}\,,\ldots\,,1\,\}$, for $j = m_1+1\,,\ldots m$, then,

for any $u \in \mathbf{U}_{q_t}$, we can define

$$\psi^j_{q_s,q_t}(u) \triangleq \max_{t \in \mathbf{Y}_{jq_t}} g^{j-m_1}(Z_{q_s}\omega_{q_t}(t, u, T)), \ j = 0, 1, 2, \ldots, m . \tag{3.18}$$

Next, the sensitivities of the difference equation (3.16b) to perturbations in the control and scale factor are given by the solution $\delta\omega^{u,T}_{q_t}(t, u, T)$, of the linearized difference equation:

$$\delta\omega((k+1)\Delta_{q_t}) = \delta\omega(k\Delta_{q_t}) + T\frac{\partial F(\omega(k\Delta_{q_t}), u(k\Delta_{q_t}), T)}{\partial\omega}\delta\omega(k\Delta_{q_t})$$

$$+ T\frac{\partial F(\omega(k\Delta_{q_t}), u(k\Delta_{q_t}), T)}{\partial u}\delta u(k\Delta_{q_t})$$

$$+ F(\omega(k\Delta_{q_t}), u(k\Delta_{q_t}), T)\delta T, \ \forall k = 0, 1, 2, \ldots, 2^{q_t}, \ \delta\omega(0) = 0 . \tag{3.19}$$

Hence, given any $(u', T'), (u, T) \in \mathbf{U}_{q_t} \times \mathbf{T}$, we define

$$\hat{\phi}^j_{q_s,q_t\,(u',T')}(u, T, t) \triangleq g^j(Z_{q_s}\omega^{u',T'}_{q_t}(t)) + \langle \nabla g^j(Z_{q_s}\omega^{u',T'}_{q_t}(t)), Z_{q_s}\delta\omega^{u',T'}_{q_t}(t; u(t) - u'(t), T - T')\rangle$$

$$+ \tfrac{1}{2}\int_0^1 \| u(t) - u'(t) \|^2 dt + \tfrac{1}{2}|T - T'|^2, t \in \mathbf{Y}_{q_t} \ \ j = 0, 1, 2, \ldots, m , \tag{3.20}$$

In turn, these definitions lead to the following ones: For any $(u, T) \in \mathbf{U}_{q_t} \times \mathbf{T}$,

$$\hat{\psi}^j_{q_s,q_t\,(u',T')}(u, T) \triangleq \max_{t \in \mathbf{Y}_{jq_t}} \hat{\phi}^j_{q_s,q_t\,(u',T')}(u, T, t), \tag{3.21a}$$

$$\hat{F}_{q_s,q_t\,(u',T')}(u, T) \triangleq \max \{ \hat{\psi}^0_{q_s,q_t\,(u',T')}(u, T) - \psi^0_{q_s,q_t}(u', T') - \gamma\psi_{q_s,q_t+}(u', T'),$$

$$\hat{\psi}^j_{q_s,q_t\,(u',T')}(u, T) - \psi_{q_s,q_t+}(u', T'), \ j \in \mathbf{m} \} , \tag{3.21b}$$

$$\theta_{q_s,q_t}(u', T') \triangleq \min_{(u,T) \in \mathbf{U}_{q_t} \times \mathbf{T}} \hat{F}_{q_s,q_t\,(u',T')}(u, T), \tag{3.21c}$$

$$\eta_{q_s,q_t}(u', T') \triangleq \arg \min_{(u,T) \in \mathbf{U}_{q_t} \times \mathbf{T} - \{u',T'\}} \hat{F}_{q_s,q_t\,(u',T')}(u' + u, T' + T) . \tag{3.21d}$$

It takes some work to show that the following result is true (see [Bak.1]):

Proposition 3.2 : There exists a $K < \infty$ such that for any positive integer q, if $\max\{q_s, q_t\} > q$, then for all $(u', T'),\ (u, T) \in \mathbf{U}_{q_t} \times \mathbf{T}$,

$$|\psi^j(u, T) - \psi^j_{q_s, q_t}(u, T)| \le \frac{K}{q}, \quad j = 0, 1, 2, \dots, m, \tag{3.22a}$$

$$|\hat{\psi}^j_{(u', T')}(u, T) - \hat{\psi}^j_{q_s, q_t\ (u', T')}(u, T)| \le \frac{K}{q}, \quad j = 0, 1, 2, \dots, m, \tag{3.22b}$$

$$|\theta(u, T) - \theta_{q_s, q_t}(u, T)| \le \frac{K}{q}, \tag{3.22c}$$

$$\| \eta(u, T) - \eta_{q_s, q_t}(u, T) \|^2 \le \frac{K}{q}. \tag{3.22d}$$

■

With these developments out of the way, we can now state our implementable optimal control algorithm. A close examination will show that the algorithm below constructs a finite dimensional problem, in which the design vector is the sequence of vector coefficients $\{ u_k \}_{k=0}^{2^{q_t}-1}$ which defines a control in $\mathbf{U}_{q_t}$, to be solved by Algorithm 3.2 until the discretization test requires that the discretization be refined.

Algorithm 3.4 (Implementable Phase I - Phase II Optimal Control Algorithm) :

Parameters : $q_s,\ q_t \in \mathbf{N}$, $\varepsilon > 0$, $\gamma > 1$, $\alpha,\ \beta \in (0,1)$.

Data : A vector coefficient sequence $u_0 = (u_0^0, \dots, u^{2^{q_t}-1}) \in \mathbf{R}^{p2^{q_t}}$, defining the control $u_0(t)$ via (3.17b), and a scaling parameter T_0.

Step 0 : Set $i = 0$.

Step 1 : Compute the the *optimality function* value $\theta_i = \theta_{q_s, q_t}(u_i, T_i)$, and the corresponding *search direction* $\eta_i = \eta_{q_s, q_t}(u_i, T_i)$.[10]

Step 2 : Compute the *step size* λ_i:

$$k_i = \arg\max \{ \beta^k \mid k \in \mathbf{N},\ F_{q_s, q_t}((u_i, T_i) + \beta^k \eta_i) \le \beta^k \alpha \theta_i \}. \tag{3.23a}$$

If

[10] See [Bak.1] for an efficient procedure, based on Polyak's constrained Newton method [Pol.4], for computing both θ_i and η_i.

$$F_{q_s, q_t}((u_i, T_i) + \beta^{k_i}\eta_i) > -\varepsilon , \tag{3.23b}$$

replace q_s, q_t by $2q_s$, $2q_t$, respectively, replace ε by $\varepsilon/2$ and go to Step 1. Else set $\lambda_i = \beta^{k_i}$.

Step 3 : Set $(u_{i+1}, T_{i+1}) = (u_i, T_i) + \lambda_i \eta_i$, set $i = i + 1$ and go to Step 1. ■

The convergence properties of the above optimal control algorithm are quite analogous to those of Algorithm 3.3 and depend on Proposition 3.2:

Theorem 3.6 : Suppose that for every $(u, T) \in \mathbf{U} \times \mathbf{R}_+$ such that $\psi(u, T) \geq 0$, $\theta(u, T) < 0$. If Algorithm 3.4 jams up, cycling between Step 1 and Step 2, at a point u_k, T_k, then u_k, T_k satisfies the first order condition $\psi(u_k, T_k) \leq 0$ and $\theta(u_k, T_k) = 0$. If Algorithm 3.4 constructs an infinite sequence $\{(u_i, T_i)\}_{i=0}^{\infty}$, then every accumulation point (u^*, T^*) of $\{(u_i, T_i)\}_{i=0}^{\infty}$ satisfies the first order optimality condition, for (2.1a), $\psi(u^*, T^*) \leq 0$, $\theta(u^*, T^*) = 0$. ■

We recall that optimal control problems, such as (3.11a) do not necessarily have solutions in U. Similarly, the sequence of controls $u_i(t)$ constructed by Algorithm 3.4 need not have accumulation points in U. This difficulty can be resolved by showing that the conclusions of Theorem 3.5 are valid in the space of relaxed controls (for a proof of this fact see [Bak.1]). Alternatively, one may resort to arguments involving infimizing sequences, as in [Pol.6].

4. CONCLUSION

We have shown that nonsmooth optimization algorithms can be used for solving both open-loop and closed-loop complex optimal control problems involving both open-loop and closed-loop systems. By comparison with other methods in the literature, the design procedure that we have presented for closed-loop systems has the advantage that it can deal with time- and frequency-domain specifications simultaneously, including L^1 -type specifications. Furthermore, it makes possible design by selection and tuning of bounds on responses, which is a much more direct process than the use of weights common to such methods as linear quadratic regulator theory. Of particular significance to the design of finite dimensional controllers for flexible structures is the fact that our procedure does not require modal truncation of partial differential equation models and that it therefore avoids destabilizing "spill-over" effects which plague many other approaches.

5. APPENDIX: EVALUATION OF THE CHARACTERISTIC FUNCTION

The design of a feedback system by means of a nonsmooth optimization algorithm, such as Algorithm 3.3, requires a large number of evaluations of the characteristic function $\chi(x, -\alpha + j\omega)$ and of its partial derivatives with respect to x^i for many values of ω. Hence it is important to perform these operations as efficiently as possible. In the discussion below, we follow the presentation in [Pol.2, Har.1].

Referring to (2.16), we see that the evaluation of $\chi(x, s)$, involves the evaluation of the determinants $\det(sI_{n_c} - A_c(x))$ and $\det(I_{n_i} + G_c(x, s)G_p(s))$. The simplest situation occurs when the matrix $A_c(x)$ is diagonalizable, i.e., when there exists a matrix of eigenvectors $V(x)$ such that $\Lambda(x) = V(x)^{-1}A_c(x)V(x)$, where $\Lambda(x) = diag(\lambda_1(x), \dots, \lambda_{n_c}(x))$, with the $\lambda_j(x)$ the eigenvalues of the matrix $A_c(x)$. In this case, considerable computational savings result from the use of the two formulae

$$\det[sI_{n_c} - A_c(x)] = \det[sI_{n_c} - \Lambda(x)] = \prod_{j=1}^{n_c}[s - \lambda_j(x)], \tag{5.1a}$$

$$G_c(x, s) = C_c(x)V(x)[sI_{n_c} - \Lambda_c(x)]^{-1}V^{-1}(x)B_c(x) + D_c(x). \tag{5.1b}$$

When diagonalization cannot be used, one can simplify the computation of the required determinants by first reducing $A_c(x)$ to upper Hessenberg form $H_c(x)$ by means of an orthogonal similarity transformation: $H_c(x) = U(x)^T A_c(x)U(x)$, where $U(x)$ is a Hermitian matrix. This results in

$$\det[sI_{n_c} - A_c(x)] = \det[sI_{n_c} - H(x)], \tag{5.2a}$$

$$G_c(x, s) = C_c(x)U(x)(sI_{n_c} - H(x))^{-1}U(x)^T B_c(x) + D_c(x). \tag{5.2b}$$

Next we need to deal with the evaluation of the plant matrix transfer function $G_p(-\alpha + j\omega)$ for frequencies ω. Since we do not wish to expose ourselves to spillover effects resulting from modal truncation, we propose to evaluate this matrix transfer function by solving two-point boundary value problems which are most conveniently produced by Laplace transformation of the original partial differential equations describing the plant, and thus bypassing a transcription into the form (2.6a,b). We shall illustrate this process by an example.

The planar bending motion of a flexible beam of unit length, which is fixed at one end and carries a particle with mass M attached to the other end is described by (see [Har.1]), can be described by a partial differential equation of the form,

$$m\frac{\partial^2 w(t,x)}{\partial t^2} + cI\frac{\partial^5 w(t,x)}{\partial x^4 \partial t} + EI\frac{\partial^4 w(t,x)}{\partial x^4} = \sum_{j=1}^{n_i} f^j(t)\zeta^j(x,x^j), \quad t \geq 0, \quad 0 \leq x \leq 1, \tag{5.3a}$$

with boundary conditions

$$w(t,0) = 0, \quad \frac{\partial w}{\partial x}(t,0) = 0, \tag{5.3b}$$

$$J\frac{\partial^3 w}{\partial x \partial t^2}(t,1) + cI\frac{\partial^3 w}{\partial x^2 \partial t}(t,1) + EI\frac{\partial^2 w}{\partial x^2}(t,1) = 0, \quad M\frac{\partial^2 w}{\partial t^2}(t,1) - cI\frac{\partial^4 w}{\partial x^3 \partial t}(t,1) - EI\frac{\partial^3 w}{\partial x^3}(t,1) = 0 \tag{5.3c}$$

where x is the distance along the undeformed-beam centroidal line, $w(t,x)$ is the vibration along the cross section principal axis (y-axis), $f^j(t)$ is a control force, $\zeta^j(x,x^j)$ is the influence function of the j-th actuator which is located at x^j, m is the distributed mass per unit length of the beam, c is the material viscous damping coefficient, E is Young's modulus, M is the end mass, I is the beam sectional moment of inertia with respect to y-axis, EI is the beam flexural stiffness in the direction of y-axis, J is the inertia of the end mass in the direction of y-axis, and n_i is the number of inputs.

The output sensors can be assumed to satisfy

$$y^i(t) = \int_0^1 \kappa^i(\nu, z^i) w(t,\nu) d\nu, \quad t \geq 0, \quad 1 \leq i \leq n_o, \tag{5.4}$$

where n_o is the number of the sensors, and $\kappa^i(\nu, z^i)$ is the distribution function of the i-th sensor and z_i is the location of the i-th sensor.

It can be shown that the plant described by (5.3a-c), (5.4) can be transcribed into the form (2.6a,b) with the associated hypotheses satisfied [Gib.1]. In fact, the corresponding operator A_p generates an analytic semigroup [Hua.2].

Taking the Laplace transforms of the partial differential equations (5.3a) - (5.3c) and (5.4) with respect to time, we obtain, for each value of $s = -\alpha + j\omega$, the two-point boundary value problem involving an *ordinary differential equation*:

$$(cIs + EI)\frac{d^4W(x\ ,s)}{dx^4} + ms^2W(x\ ,s) = \sum_{j=1}^{n_i} F^j(s)\zeta^j(x\ ,x^j),\ \ 0 \le x \le 1\ , \tag{5.5a}$$

with boundary conditions

$$W(s\ ,0) = 0\ ,\ \ \frac{dW}{dx}(s,0) = 0\ , \tag{5.5b}$$

$$(cI\ s + EI)\frac{d^2W}{dx^2}(s\ ,1) + J\ s^2\frac{dW}{dx}(s\ ,1) = 0\ , \qquad (cI\ s + EI)\frac{d^3W}{dx^3}(s\ ,1) - M\ s^2W(s\ ,1) \tag{5.5c}$$

The Laplace transforms of the outputs are given by

$$Y^i(s) = \int_0^1 \kappa^i(\nu\ ,z^i)W(s\ ,\nu)d\nu\ ,\ \ \ 1 \le i \le n_o, \tag{5.6}$$

where $W(x\ ,s)$, $F^j(s)$ and $Y^i(s)$ are the Laplace transforms of $w(t\ ,x)$, $f^{\,j}(t)$ and $y^i(t)$, respectively. Hence the $(k\ ,l)$-th element of the matrix $G_p(s)$ can be obtained by setting $F^l(s) = 1$ and $F^j(s) = 0$ for all other j, then solving (5.5a) - (5.5d), and evaluating (5.6) for $i = k$. The boundary value problem (5.5a) - (5.5d) can be solved by means of shooting methods (see [Kel.1], [Pol.8]).

Next, we turn to the computation of the partial derivatives of $\chi(x\ ,s)$. This requires the calculation of the partial derivatives of $\det[s - A_c(x)]$ and $\det[I_{n_i} + G_c(x)G_p(s)]$. When the eigenvalues $\lambda_j(x)$ of $A_c(x)$ are distinct, they are differentiable [Kat.1] and their partial derivatives are given by

$$\frac{\partial\lambda_j(x)}{\partial x^i} = \langle u_j\ ,\frac{\partial A_c(x)}{\partial x^i}v_j\rangle\ /\ \langle u_j\ ,v_j\rangle\ , \tag{5.7a}$$

where v_j and u_j are the right and left eigenvectors, respectively, of $A_c(x)$, corresponding to the eigenvalue $\lambda_j(x)$. In this case, the partial derivatives of $\det[sI_{n_c} - A_c(x)]$ can be computed making use of the following formula [Pol.2]:

$$\frac{\partial\det[sI_{n_c} - A_c(x)]}{\partial x^i} = \sum_{j=1}^{n_c}\{-\frac{\partial\lambda_j(x)}{\partial x^i}\prod_{\substack{k=1\\k\neq j}}^{n_c}[s - \lambda_k(p_c)]\} = \det[sI_{n_c} - A_c(x)]\sum_{j=1}^{n_c} - \frac{\partial\lambda_j(x)}{\partial x^i}\frac{1}{s - \lambda_j(x)} \tag{5.7b}$$

When the eigenvalues of $A_c(x)$ are not distinct, the computation of its partial derivative requires a more general formula which can be found in [Pol.2]. The computation of the partial derivatives of $\det[I_{n_i} + G_c(x\ ,s)G_p(s)]$ can also be carried out by making use of a formula analogous to (5.7b), provided that the matrix $\left[I_{n_i} + G_c(x\ ,s)G_p(s)\right]$ has distinct eigenvalues. When the eigenvalues of

$(I_{n_i} + G_c(x, s)G_p(s))$ are not distinct, the computation of its partial derivative becomes considerably more difficult. Fortunately, this is not very likely to be the case in practice.

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