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A NOTE ON THE SOLUTION OF A COMMUNICATION JAMMING PROBLEM OF ROOT

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ABSTRACT

Root recently considered some problems in binary transmission in the presence of unknown additive noise, and reduced them to infinite games. For these he derived mixed strategies which in this note are shown to be optimal, providing the values of the games.

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I. INTRODUCTION

In a recent paper,[1] Root considered two problems of binary transmission in the presence of additive noise that may either be provided by nature or by a jammer, the only restriction on the noise being that its power be bounded. In the first problem, a keyed carrier system with power detector, the communicator S wishes to decide which of the two signals

$$\begin{aligned} &\text{"MARK"}: \ s(t) = b \cos(\omega_o t + \theta) \\ & & 0 \le t < T \\ &\text{"SPACE"}: \ s(t) = 0 \end{aligned} \tag{1}$$

was sent, with the greatest possible average reliability of reception. θ being unknown, assumptions were made giving

$$\int_0^T b^2 \cos^2(\omega_o t + \theta)\, dt \sim 1 \tag{2}$$

for all θ. Also, if $z(t)$ is the noise, and hence $s(t) + z(t)$ $(0 \le t < T)$ the received wave-form, and if we assume that $z(t)$ is such that its unknown energy α is limited, i.e.,

$$\alpha = \int_0^T z^2(t)\, dt \le a = \text{constant}, \tag{3}$$

and that it is uncorrelated with $s(t)$ over $0 \le t < T$, i.e.,

$$\int_0^T s(t)\, z(t)\, dt = 0, \tag{4}$$

then the basic receiver signal-processing provides energy measurements

$$\begin{aligned} y &= 1 + \alpha \quad \text{if MARK was sent} \\ &= \alpha \qquad\ \ \text{if SPACE was sent} \ . \end{aligned} \tag{5}$$

Root's problem, then, was to determine a decision rule telling from y whether a MARK or a SPACE was sent, given that they are sent with probabilities 1/2 each. Calling the communicator S and the jammer T, let $P(\gamma, \alpha)$ be the payoff to S when S uses a pure strategy γ and T a pure strategy α. $P(\gamma, \alpha)$ is the probability of correct identification of the signal, and the players adopt maximin and minimax mixed strategies. Root showed that an upper bound on S's maximin strategy is

$$V = 1/2 + 1/2a , \tag{6}$$

and determined a strategy that achieves

$$V_1 = 1/2 + 1/2\left[a + 1\right] , \tag{7}$$

where $\left[x\right]$ is the largest integer less than or equal to x. If S's strategies are restricted to setting thresholds β and deciding on SPACE if $y < \beta$ and MARK if $y \geq \beta$, where $1 \leq \beta \leq a + 1$, then Root shows that the payoff function $P(\beta, \alpha)$ is given by

$$P(\beta, \alpha) = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq \alpha < \beta - 1 \\ 1, & \text{if } \beta - 1 \leq \alpha < \beta \\ \frac{1}{2}, & \text{if } \beta \leq \alpha < a \end{cases} \quad . \tag{8}$$

He also shows that the threshold strategy for S that guarantees the expected payoff V_1 is the choice

$$\beta = 1, 2, \ldots, [a + 1] \tag{9}$$

with equal probabilities $1/[a + 1]$. What is shown in this note is that this is indeed an optimal strategy for the infinite game, because T can choose a mixed strategy that guarantees that S cannot attain an expected payoff higher than V_1, which is hence the value of the game. Corresponding optimal strategies for T are

$$\alpha = k, k + 1, \ldots, k + [a] \tag{10}$$

with equal probabilities $1/[a + 1]$, where $0 \leq k \leq a - [a]$. These are not necessarily the only optimal strategies.

Proof: As in Root's paper, let $d_\gamma(y)$ be the decision function used by S when he employs a pure strategy γ and receives the signal y.

$d_\gamma(y) = 0$ implies the decision "SPACE"

$d_\gamma(y) = 1$ implies the decision "MARK" .

Hence,

$$P(\gamma, \alpha) = \frac{1}{2} \left(d_\gamma (\alpha + 1) + 1 - d_\gamma (\alpha) \right). \tag{11}$$

If T uses the equiprobable stragegies of Eq. (10), then the expected payoff is

$$V_k = \frac{1}{[a+1]} \sum_{\alpha = k}^{k+[a]} \int_\Gamma P(\gamma, \alpha)\, d\gamma, \tag{12}$$

where Γ is the set of all possible S-strategies γ. Hence

$$V_k = \frac{1}{2[a+1]} \sum_{\alpha = k}^{k+[a]} \int_\Gamma \left(d_\gamma (\alpha + 1) + 1 - d_\gamma (\alpha) \right) d_\gamma$$

$$= \frac{1}{2[a+1]} \left\{ [a+1] + \int_\Gamma \sum_{\alpha = k}^{k+[a]} \left(d_\gamma (\alpha + 1) - d_\gamma (\alpha) \right) d_\gamma \right\}$$

$$= \frac{1}{2} + \frac{1}{2[a+1]} \int_\Gamma \left(d_\gamma (k + [a]) - d_\gamma (k) \right) d_\gamma .$$

But $d_\gamma (k + [a]) - d_\gamma (k) \leq 1$. Therefore

$$V_k \leq \frac{1}{2} + \frac{1}{2[a+1]} , \tag{13}$$

which completes the proof.

REFERENCE

1. W. L. Root, "Communications through unspecified additive noise," Information and Cont., Vol. 4, No. 1; March 1961.