FOUNDATIONS OF NONLINEAR NETWORK THEORY

PART I: PASSIVITY

by

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PART I: PASSIVITY

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ABSTRACT

An extensive discussion of the concept of passivity for nonlinear networks is given. A particular definition of passivity is proposed which does not require the existence of a state of "zero stored energy." This definition is applied to specific classes of n-ports and equivalent passivity criteria are derived. The definition of passivity proposed in this paper is shown to have various representation independence and closure properties. An equivalent view of this definition in terms of internal energy functions is presented, and these functions are used to derive a basic result regarding the passive realization of n-ports.

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I. Introduction

When we consider the central role that passivity plays in questions of network stability and network synthesis, it comes as a surprise to find that the concept has been given several conflicting definitions in the modern literature [1]-[8]. The problem seems to arise from the long period in which "network theory" meant essentially "linear network theory," since the various definitions are nearly equivalent in the linear case.

The purpose of this paper is to clarify the meaning of passivity in nonlinear network theory. It turns out to be a subtler concept that one might initially expect. We have found a number of examples which show that certain apparently reasonable definitions in the literature give odd or even nonsensical results if viewed critically. We will argue that a particular concept of passivity which does not require the existence of a state of "zero stored energy" is the proper one for nonlinear circuits. The definition of passivity adopted in this paper has been given elsewhere in the literature in various equivalent forms [1]-[5]; however, it has not been widely recognized that our definition of passivity does not require the existence of a state of "zero stored energy." Perhaps the clearest discussion of these matters was given by Willems [1], whose excellent article was the inspiration behind this paper.

We will adopt a state space point of view; and since the concept of "the state at $T = -\infty$" is problematic at best for nonlinear systems, we will only consider behavior on a time interval $[t_o,+\infty)$. Our discussion will be restricted to finite-dimensional time-invariant systems, largely for notational convenience. The assumption of time-invariance implies that we can let $\mathbb{R}^+ = [0,+\infty)$ be the time interval of interest without loss of generality, and we shall do so from now on.

Although a large number of slightly different definitions of passivity can be found in the literature, three distinct ideas can be isolated if we overlook minor differences. The first definition considered below, Passivity 1, is the concept of passivity adopted in this paper (however, see Sections II and III for a more complete discussion of our assumptions and definitions). In the context of time-invariant systems, these definitions can be stated as follows.

Passivity 1. Let Σ denote the state space of a given state space representation of an n-port. The n-port is passive if there exists a function $E : \Sigma \to \mathbb{R}^+$ such that, for every $\underset{\sim}{x}_o \in \Sigma$,

$$\int_0^T \langle \underset{\sim}{v}(t), \underset{\sim}{i}(t)\rangle\, dt + E(\underset{\sim}{x}_o) \geq 0 \tag{1-1}$$

for all $T \geq 0$ and all admissible port voltage, port current pairs $\{\underset{\sim}{v}(\cdot), \underset{\sim}{i}(\cdot)\}$ consistent with the initial state $\underset{\sim}{x}_o$, where $\langle \cdot , \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^n$.

We assume the usual associated reference convention throughout this paper; consequently, $-\int_0^T \langle \underset{\sim}{v}(t), \underset{\sim}{i}(t)\rangle\, dt$ is the energy extracted from the ports during the time interval $[0,T]$. Notice that $E(\cdot)$ is finite-valued because it is required to take values in $\mathbb{R}^+$ (recall that $\mathbb{R}^+ = [0,+\infty)$, so $+\infty$ is not an element of $\mathbb{R}^+$).[1] It follows from these observations that Passivity 1 can be stated in the following equivalent form: An n-port is passive if for each initial state there is at most a finite amount of energy available at its ports. As mentioned previously, Passivity 1 is the concept of passivity adopted in this paper. It is essentially the definition given in references [1]-[5]. Some of our reasons for adopting Passivity 1 will be given after we consider Passivity 2 and Passivity 3; other reasons will be given in Section III.

We attach no physical significance to the function $E(\cdot)$ appearing in (1-1). If such a function exists, it is obviously not unique. For any n-port, we define the <u>available energy</u> $E_A : \Sigma \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$E_A(\underset{\sim}{x}_o) \triangleq \sup \left\{ - \int_0^T \langle \underset{\sim}{v}(t), \underset{\sim}{i}(t)\rangle\, dt \right\} \tag{1-2}$$

where the supremum is taken over all $T \geq 0$ and all admissible pairs $\{\underset{\sim}{v}(\cdot), \underset{\sim}{i}(\cdot)\}$ consistent with the initial state $\underset{\sim}{x}_o$. Clearly, Passivity 1 is equivalent to the condition that $E_A(\underset{\sim}{x}_o) < +\infty$ for every $\underset{\sim}{x}_o \in \Sigma$: If $E_A(\underset{\sim}{x}_o) < +\infty$ for every $\underset{\sim}{x}_o \in \Sigma$, then it follows from the very definition of $E_A(\cdot)$ that (1-1) can be satisfied by choosing $E(\cdot) = E_A(\cdot)$; conversely, if (1-1) is satisfied, then it follows from (1-2) that $E_A(\underset{\sim}{x}_o) \leq E(\underset{\sim}{x}_o) < +\infty$ for every $\underset{\sim}{x}_o \in \Sigma$. This last relation shows that $E_A(\cdot)$ is the <u>least</u> function which satisfies (1-1). Note that it is entirely possible for $E_A(\underset{\sim}{x})$ to be non-zero for every $\underset{\sim}{x} \in \Sigma$; hence, there may not exist a state of zero available energy.

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[1]This point requires some elaboration. We <u>never</u> consider $+\infty$ or $-\infty$ to be real numbers, so if a function takes values in $\mathbb{R}$, $\mathbb{R}^+$ or $\mathbb{R}^n$, it is necessarily finite-valued; however, it will be convenient to define a mathematical object called the <u>extended real number system</u> [9], denoted $\mathbb{R}^e$, which is obtained by attaching $+\infty$ and $-\infty$ to $\mathbb{R}$. Specifically, $\mathbb{R}^e = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$.

Passivity 2. An n-port, storing no energy at $t = 0$, is passive if

$$\int_0^T \langle \underset{\sim}{v}(t), \underset{\sim}{i}(t) \rangle \, dt \geq 0 \tag{1-3}$$

for all $T \geq 0$ and all admissible pairs $\{\underset{\sim}{v}(\cdot), \underset{\sim}{i}(\cdot)\}$ [6], [7].

One difficulty with this definition is that it offers no operational rule for determining the stored energy at $t = 0$. If we mean by the term "n-port" a black box which we are not allowed to open, this is not a trivial objection. Even if we had such an operational rule, the demand that we begin with a state of zero stored energy is itself unclear. If for a given n-port we cannot find such a state, is that n-port active or does it fall outside the scope of the definition altogether? The capacitor with the constitutive relation $\hat{v}(q) = e^q$ shown in Fig. 1 is a relevant example. In Section III we will show that $E_A(q) = e^q$ for this capacitor (cf. Fig. 6), so $E_A(q) > 0$ for all q and it cannot satisfy inequality (1-3) for any initial state.

Passivity 3. An n-port is passive if whenever the state $\underset{\sim}{x}$ at time zero is $\underset{\sim}{0}$,

$$\int_0^T \langle \underset{\sim}{v}(t), \underset{\sim}{i}(t) \rangle \, dt \geq 0 \tag{1-4}$$

for all admissible pairs $\{\underset{\sim}{v}(\cdot), \underset{\sim}{i}(\cdot)\}$ and all $T \geq 0$ [7], [8], [28].

According to this concept of passivity, we only need to know the zero-state response of an n-port in order to determine if it is active or passive. For example, it would force us to classify the linear 2-port in Fig. 2 as passive. Likewise, it says that the capacitor in Fig. 3 is active even though its terminal behavior, $i = \frac{dv}{dt}$ or

$$v(t) = v(0) + \int_0^t i(\tau) d\tau \tag{1-5}$$

cannot be distinguished from that of a 1-farad capacitor by any possible voltage and current measurements. Finally, it cannot be applied at all to elements for which the origin is not an element of the state space, such as capacitors with the constitutive relations given in Fig. 4.

On a more abstract level the problem with this concept of passivity is that it singles out the origin of the state space for a special role. This is inappropriate in a general nonlinear theory. The reason is that in nonlinear problems the state space must generally be viewed as a manifold and any vector space structure it might possess cannot be used in the foundations of a nonlinear theory.

Passivity 1 is the proper definition of passivity for nonlinear n-ports because it does not single out any particular state in Σ for special attention and it does not require that a state of zero stored energy be found. As we shall see in the body of this paper, Passivity 1 classifies the 2-port in Fig. 2 as active and the 1-ports in Figs. 1 and 3 as passive. We shall express the concept of passivity embodied in Passivity 1 in the following equivalent form (cf. Def. 11): An n-port is defined to be passive if $E_A(\underset{\sim}{x}) < +\infty$ for all $\underset{\sim}{x} \in \Sigma$, where $E_A(\cdot)$ is the available energy function associated with the given state space representation of the n-port. We prefer to define passivity in terms of the available energy function because this definition is more closely related to the approach taken by Willems [1].

II. Definitions and Assumptions

It is probably best to skim this section quickly the first time through and then refer back to it as needed. The assumptions listed here are quite restrictive and they eliminate from consideration many common elements in nonlinear circuit theory. We have imposed these restrictive assumptions in order to avoid a mass of abstract concepts and unfamiliar notation. Some of these assumptions could be relaxed merely at the expense of notational convenience, while others would require a fundamental extension of the theory. We will discuss possible generalizations in Section VIII.

The n-ports dealt with in this paper are assumed to possess a state representation; this is our fundamental assumption. Roughly speaking, a state representation of an n-port (as we define it) is a state equation and two read-out maps which give the port voltages and port currents as functions of the input and state, together with a set of rules defining the class of inputs which can be applied.

Definition 1. A state representation S for an n-port is a quintuplet $\{U, \mathcal{U}, \Sigma, E, R\}$, where

(1) $U \subset \mathbb{R}^n$ is a nonempty set called the set of admissible input values.

(2) $\mathcal{U}$ is a nonempty set of functions mapping $\mathbb{R}^+$ to U called the set of admissible input waveforms.

(3) $\Sigma \subset \mathbb{R}^m$ is a nonempty set called the state space.

(4) E is a pair of equations

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{u}) \tag{2-1}$$

$$\underset{\sim}{y} = \underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{u}) \tag{2-2}$$

where $\underset{\sim}{f}(\cdot,\cdot)$ maps $\Sigma \times U \to \mathbb{R}^m$ and $\underset{\sim}{g}(\cdot,\cdot)$ maps $\Sigma \times U \to \mathbb{R}^n$. Equation (2-1) is called the <u>state equation</u> and (2-2) is called the <u>output equation</u>.

(5) R is a pair of readout maps: $\underset{\sim}{V} : \Sigma \times U \to \mathbb{R}^n$ is called the <u>port voltage readout map</u> and $\underset{\sim}{I} : \Sigma \times U \to \mathbb{R}^n$ is called the <u>port current readout map</u>.

<u>Definition 2</u>. The <u>power input function</u> $p : \Sigma \times U \to \mathbb{R}$ is defined by

$$p(\underset{\sim}{x},\underset{\sim}{u}) \triangleq \langle \underset{\sim}{V}(\underset{\sim}{x},\underset{\sim}{u}),\underset{\sim}{I}(\underset{\sim}{x},\underset{\sim}{u})\rangle .$$

<u>Definition 3</u>. A choice of input and output variables $\underset{\sim}{u}$ and $\underset{\sim}{y}$ for an n-port is called a <u>hybrid pair</u> if $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are n-dimensional and for each $k \in \{1,\dots,n\}$, either $u_k = v_k$ and $y_k = i_k$ or else $u_k = i_k$ and $y_k = v_k$, where u_k and y_k denote the k-th components of $\underset{\sim}{u}$ and $\underset{\sim}{y}$, respectively, and v_k and i_k denote the k-th port voltage and current, respectively.

Let $D \subset \mathbb{R}^p$ be an open set. Recall that a function $\underset{\sim}{h} : D \to \mathbb{R}^q$ is said to be C^0 if it is continuous, and it is said to be C^k for some positive integer k if each of its component functions possesses continuous partial derivatives of all orders up to and including k.

<u>Definition 4</u>. Let $J \subset \mathbb{R}$ be an interval (possibly unbounded). A function $\underset{\sim}{u} : J \to \mathbb{R}^n$ is said to be <u>piecewise continuous</u> (or <u>piecewise C^0</u>) if there exists a countable set $\{t_i\} \subset J$ such that $\underset{\sim}{u}(\cdot)$ is continuous at each point $t \notin \{t_i\}$, there are at most a finite number of points t_i in any bounded interval contained in J, and $\underset{\sim}{u}(\cdot)$ has finite right- and left-hand limits at each point t_i. For each positive integer k, we define a function $\underset{\sim}{u}(\cdot)$ to be piecewise C^k by the following inductive definition: $\underset{\sim}{u} : J \to \mathbb{R}^n$ is <u>piecewise C^k</u> if it can be written in the form

$$\underset{\sim}{u}(t) = \underset{\sim}{u}(t_o) + \int_{t_o}^{t} \underset{\sim}{w}(\tau)d\tau$$

where $t_o \in J$ and $\underset{\sim}{w} : J \to \mathbb{R}^n$ is piecewise $C^{(k-1)}$.

Roughly speaking, $\underset{\sim}{u} : J \to \mathbb{R}^n$ is piecewise C^k if its k-th derivative is piecewise continuous.

<u>Definition 5</u>. A function $\underset{\sim}{u}(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ is said to be <u>locally L^p</u>, $1 \leq p < +\infty$, if $\underset{\sim}{u}(\cdot)$ is measurable and for every choice of $a,b \in \mathbb{R}^+$,

$$\int_a^b (\|u(t)\|)^p dt < +\infty,$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$. We will let $L^p_{loc}(\mathbb{R}^+ \to \mathbb{R}^n)$ denote the class of all such functions.[2]

Definition 6. Given a function $\underset{\sim}{u}(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ and a real number $\tau \geq 0$, let $\underset{\sim}{u}_\tau(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ be obtained from $\underset{\sim}{u}(\cdot)$ by translating $\underset{\sim}{u}(\cdot)$ τ units to the left, i.e., $\underset{\sim}{u}_\tau(t) = \underset{\sim}{u}(t+\tau)$, $\forall t \in \mathbb{R}^+$. We say that $\mathcal{U}$ is translation invariant if $\underset{\sim}{u}(\cdot) \in \mathcal{U} \Rightarrow \underset{\sim}{u}_\tau(\cdot) \in \mathcal{U}$, $\forall \tau \geq 0$.

Definition 7. Given two functions $\underset{\sim}{u}_1(\cdot)$, $\underset{\sim}{u}_2(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ and given a real number $\tau \geq 0$, we define $\underset{\sim}{u}_{12\tau} : \mathbb{R}^+ \to \mathbb{R}^n$ by

$$\underset{\sim}{u}_{12\tau}(t) = \begin{cases} \underset{\sim}{u}_1(t), & 0 \leq t \leq \tau \\ \underset{\sim}{u}_2(t-\tau), & t > \tau. \end{cases}$$

We say that $\mathcal{U}$ is closed under concatenation if for every $\underset{\sim}{u}_1(\cdot)$, $\underset{\sim}{u}_2(\cdot) \in \mathcal{U}$ and every $\tau \geq 0$, $\underset{\sim}{u}_{12\tau}(\cdot)$ is also an element of $\mathcal{U}$. Moreover, if we redefine $\underset{\sim}{u}_{12\tau}(\cdot)$ at the point $t = \tau$ so that $\underset{\sim}{u}_{12\tau}(\tau) = \underset{\sim}{u}_2(0)$, we shall require that the resulting function also be an element of $\mathcal{U}$ when $\mathcal{U}$ is closed under concatenation.

Given an input waveform $\underset{\sim}{u}(\cdot)$, we will say that a function $\underset{\sim}{x}(\cdot) : \mathbb{R}^+ \to \Sigma$ is a solution of the state equation $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{u})$ if $\underset{\sim}{x}(\cdot)$ is absolutely continuous [9] and satisfies $\dot{\underset{\sim}{x}}(t) = \underset{\sim}{f}(\underset{\sim}{x}(t),\underset{\sim}{u}(t))$ for almost all t.

Standing Assumptions on State Representations.

(1) The functions $\underset{\sim}{f}(\cdot,\cdot)$, $\underset{\sim}{g}(\cdot,\cdot)$, $\underset{\sim}{V}(\cdot,\cdot)$, and $\underset{\sim}{I}(\cdot,\cdot)$ are continuous.

(2) For every $\underset{\sim}{x}_0 \in \Sigma$ and every $\underset{\sim}{u}(\cdot) \in \mathcal{U}$ there exists a unique solution $\underset{\sim}{x}(\cdot) : \mathbb{R}^+ \to \Sigma$ of the differential equation $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{u})$ such that $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$.

(3) If S is a state representation for an n-port and if the pair $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}$ is as described in (2), then the port voltage and port current of the n-port when the initial state is $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$ and the input is $\underset{\sim}{u}(\cdot)$ are, respectively, $\underset{\sim}{v}(t) = \underset{\sim}{V}(\underset{\sim}{x}(t),\underset{\sim}{u}(t))$ and $\underset{\sim}{i}(t) = \underset{\sim}{I}(\underset{\sim}{x}(t),\underset{\sim}{u}(t))$.

(4) For every pair $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}$ as described in (2), the function $t \to p(\underset{\sim}{x}(t),\underset{\sim}{u}(t))$ is locally L^1.

(5) The set of admissible input waveforms $\mathcal{U}$ is translation invariant and closed under concatenation, and all functions in $\mathcal{U}$ are measurable.

The second assumption implies that $\underset{\sim}{x}(\cdot)$ is defined and continuous on $\mathbb{R}^+$, so systems with finite escape times are ruled out. Since $\underset{\sim}{x}(\cdot)$ must take values in Σ, it follows that no admissible input can drive the state out of the state space.

The third assumption merely states formally what should have been evident from our notation and terminology -- the port voltage readout map $\underset{\sim}{V}(\cdot,\cdot)$ gives

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[2] These classes of functions are the same as the extended L_n^p spaces defined by Desoer and Vidyasagar [10] and denoted by L_{ne}^p.

the port voltage and the port current readout map $\underset{\sim}{I}(\cdot,\cdot)$ gives the port current. If $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are a hybrid pair, then the functions $\underset{\sim}{g}(\cdot,\cdot)$, $\underset{\sim}{V}(\cdot,\cdot)$, and $\underset{\sim}{I}(\cdot,\cdot)$ must satisfy the following condition: $p(\underset{\sim}{x},\underset{\sim}{u}) \triangleq \langle \underset{\sim}{V}(\underset{\sim}{x},\underset{\sim}{u}),\underset{\sim}{I}(\underset{\sim}{x},\underset{\sim}{u})\rangle = \langle \underset{\sim}{u},\underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{u})\rangle$ for all $(\underset{\sim}{x},\underset{\sim}{u}) \in \Sigma \times U$. Since the port voltage $\underset{\sim}{v}(\cdot)$ and the port current $\underset{\sim}{i}(\cdot)$ are the only quantities of interest, the reader might wonder why we have bothered to introduce the output equation (2-2). The reason is that in deriving passivity criteria for specific systems, some assumption on the form of the function $p(\cdot,\cdot)$ must usually be made. In these cases it is natural to introduce the output equation (2-2) and to assume that $\underset{\sim}{u}$ and $\underset{\sim}{y}$ form a hybrid pair. Our general theoretical discussion does not require that $\underset{\sim}{u}$ and $\underset{\sim}{y}$ form a hybrid pair; in fact, it does not even require that the output equation (2-2) be introduced. The output equation will be introduced only when deriving passivity criteria for specific classes of systems. In these cases it will always be assumed that $\underset{\sim}{u}$ and $\underset{\sim}{y}$ form a hybrid pair, and this assumption will be stated explicitly.

The fourth assumption implies that the input energy is finite over any finite interval of time.

The assumption that $\mathcal{U}$ is translation invariant is a natural one for time-invariant systems, and closure under concatenation means roughly that any two input waveforms which can be applied separately can be applied in sequence. Whenever the proof of a theorem or lemma requires that $\mathcal{U}$ be closed under translation or concatenation, we will repeat that assumption explicitly in the statement of the theorem or lemma. While we do not require that $\mathcal{U}$ be a vector space, all the L^p spaces of functions mapping $\mathbb{R}^+$ to $\mathbb{R}^n$ will satisfy assumption (5).

<u>Definition 8</u>. A <u>state space trajectory</u> is a function $\underset{\sim}{x} : \mathbb{R}^+ \to \Sigma$ which is a solution of $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{u})$ for some $\underset{\sim}{u}(\cdot) \in \mathcal{U}$. If $\underset{\sim}{x}(\cdot)$ is a state space trajectory with $\underset{\sim}{x}(t_1) = \underset{\sim}{x}_1$ and $\underset{\sim}{x}(t_2) = \underset{\sim}{x}_2$, $t_1 \leq t_2$, we will call the restriction of $\underset{\sim}{x}(\cdot)$ to $[t_1,t_2]$ a <u>trajectory from $\underset{\sim}{x}_1$ to $\underset{\sim}{x}_2$</u>. For convenience, the restriction of $\underset{\sim}{x}(\cdot)$ to $[t_1,t_2]$ will be denoted by $\underset{\sim}{x}(\cdot)|[t_1,t_2]$. Similar notation will be used below. An <u>input-trajectory pair</u> is a pair of functions $\underset{\sim}{u}(\cdot) \in \mathcal{U}$ and $\underset{\sim}{x} : \mathbb{R}^+ \to \Sigma$ such that $\underset{\sim}{x}(\cdot)$ is a solution of $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{u})$. If $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}$ is an input-trajectory pair with $\underset{\sim}{x}(0) = \underset{\sim}{x}'$, we call it an <u>input-trajectory pair with initial state $\underset{\sim}{x}'$</u>. If $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}$ is an input-trajectory pair with $\underset{\sim}{x}(t_1) = \underset{\sim}{x}_1$ and $\underset{\sim}{x}(t_2) = \underset{\sim}{x}_2$, $t_1 \leq t_2$, we call $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}|[t_1,t_2]$ an <u>input-trajectory pair from $\underset{\sim}{x}_1$ to $\underset{\sim}{x}_2$</u>. The <u>energy consumed</u> by $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}|[t_1,t_2]$ is the quantity

$$\int_{t_1}^{t_2} p(\underset{\sim}{x}(t),\underset{\sim}{u}(t))dt.$$

It follows from standing assumption (4) that this quantity is always finite when t_1 and t_2 are finite.

Definition 9. Let $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}$ be an input-trajectory pair. If $\underset{\sim}{y}(t) = \underset{\sim}{g}(\underset{\sim}{x}(t),\underset{\sim}{u}(t))$ for all $t \geq 0$, then $\{\underset{\sim}{u}(\cdot),\underset{\sim}{y}(\cdot)\}$ is called an input-output pair. If $\underset{\sim}{v}(t) = \underset{\sim}{V}(\underset{\sim}{x}(t),\underset{\sim}{u}(t))$ and $\underset{\sim}{i}(t) = \underset{\sim}{I}(\underset{\sim}{x}(t),\underset{\sim}{u}(t))$ for all $t \geq 0$, then $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}$ is called an admissible pair. If $\underset{\sim}{x}(0) = \underset{\sim}{x}'$, then $\{\underset{\sim}{u}(\cdot),\underset{\sim}{y}(\cdot)\}$ is called an input-output pair with initial state $\underset{\sim}{x}'$ and $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}$ is called an admissible pair with initial state $\underset{\sim}{x}'$. We will adopt the notation $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}|[0,T]$ for the restriction of $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}$ to $[0,T]$.

Note that assumption (5) implies that the class of admissible pairs is translation invariant, but it does not imply that the class of admissible pairs is closed under concatenation.

Finally, we shall often loosely use the term "system" to denote an n-port $\mathcal{N}$ along with a given state representation for $\mathcal{N}$.

III. The Proposed Definition of Passivity and Some of its Consequences

3.1 Circuits and N-Ports

When we speak of a system as a "circuit" or as an "n-port," we actually mean two slightly different things. By a circuit we mean a particular interconnection of components, and we normally assume that we can make voltage or current measurements at any node or branch. We reserve the term "n-port" for a "black box" with n electrical ports, where the port currents and voltages are related by a state representation S as described in Section II. The n-port is viewed over the time interval $\mathbb{R}^+ = [0,+\infty)$, and it is considered to be "created" at $t = 0$ with an arbitrary initial state $\underset{\sim}{x}_0 \in \Sigma$. We can investigate the n-port over the time interval $\mathbb{R}^+$ only by applying signals and making measurements at the ports. The response of the n-port to a given input depends on the initial state $\underset{\sim}{x}_0$; hence, each n-port gives rise to a family of input-output operators $\{O_{\underset{\sim}{x}_0}\}$, indexed by the initial state $\underset{\sim}{x}_0$.

The distinction between a circuit and an n-port is an important one for any serious study of passivity. If we think of passivity as a property of a circuit, then presumably a circuit should be called passive if each of its components is passive. But if we think of passivity as a property of an n-port, then any decision about activity or passivity must be based solely on experiments which could in principle be performed at the ports. Consider for example the circuit in Fig. 5(a), where the operational amplifier is modeled by an ideal linear controlled source having an infinite controlling coefficient. Although we have not yet formally defined passivity and activity, it is clear that this should be called an active circuit. But if we think of it as an n-port, the proper classification may depend on which parts of the circuit are accessible from the ports. The 1-port in Fig. 5(b) would presumably be passive since it cannot be distinguished from a 1-Ω resistor by any experiment performed at the port, but the 2-port in Fig. 5(c) should be called active.

We will adopt the point of view in this paper that passivity is an attribute of an n-port, and we will try to do so consistently. This means that all our definitions and criteria should not implicitly assume knowledge of anything more than the state representation of the n-port under study. We do not care how the n-port is realized; indeed, from our point of view the 1-port in Fig. 5(b) is identical to a 1-ohm resistor and it should be classified as a passive 1-port.

Our assumption that the n-port is "created" at t = 0 with an arbitrary initial state allows us to include uncontrollable[3] n-ports in our theory. Consider the uncontrollable 2-port shown in Fig. 2. The impedance matrix of this 2-port tells us only of its rather tame zero-state response; however, it is violently unstable when the initial conditions are nonzero. For any initial voltage v_{co} across the capacitor, the voltage at port 2 is $v_2(t) = v_{co}e^t$ regardless of the input at either port. It is easy to see that if $v_{co} \neq 0$, we can extract unlimited energy from this 2-port simply by connecting a resistor across the second port; hence, the 2-port in Fig. 2 should certainly be classified as an active 2-port.

We have framed our theory in terms of a known state representation of the n-port. Since an n-port can have many equivalent state representations, a question of consistency arises. Is it possible for an n-port to have two different state representations, one of which is active by our definition and the other passive? We will show in Section V that if two state representations really describe the port behavior of the same n-port, then this ambiguity cannot arise. For the present we will frame our definitions in terms of "an n-port $\mathcal{N}$ with state representation S," where the attribute of passivity,

[3] An n-port is said to be uncontrollable if it is not completely controllable (Def. 13).

etc., is considered to be a property of $\mathcal{N}$, but all the criteria are stated in terms of S. We will postpone justifying the consistency of this approach until Section V.

3.2 Available Energy and Passivity

Definition 10. Given an n-port $\mathcal{N}$ with a state representation S, we define the available energy $E_A : \Sigma \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$E_A(\underset{\sim}{x}) \triangleq \sup_{\substack{\underset{\sim}{x}\to \\ T\geq 0}} \left\{ -\int_0^T \langle \underset{\sim}{v}(t), \underset{\sim}{i}(t)\rangle \, dt \right\} \tag{3-1}$$

where the notation $\sup_{\substack{\underset{\sim}{x}\to \\ T\geq 0}}$ indicates that the supremum is taken over all $T \geq 0$ and all admissible pairs $\{\underset{\sim}{v}(\cdot), \underset{\sim}{i}(\cdot)\}$ with initial state $\underset{\sim}{x}$ (Def. 9).

Since we have assumed that $t \to \langle \underset{\sim}{v}(t), \underset{\sim}{i}(t)\rangle$ is locally L^1 (standing assumption (4), Section II), the integral in (3-1) always exists and is finite; however, it is possible for $E_A(\underset{\sim}{x})$ to be infinite for certain values of $\underset{\sim}{x}$. Roughly speaking, the available energy at a particular state $\underset{\sim}{x}$ is the maximum energy that can be extracted from the system when its initial state is $\underset{\sim}{x}$. Note that the above expression defines $E_A(\cdot)$ exactly, not merely to within an additive constant. Since the value T = 0 is allowed as an upper limit of the integral in (3-1), $E_A(\underset{\sim}{x})$ is the supremum of a set of numbers which includes zero. Therefore $E_A(\cdot)$ is a nonnegative function, as claimed in the definition.

Example 1. A 2-terminal capacitor characterized by a continuous function $v = \hat{v}(q)$ has the natural state representation

$$\begin{aligned} \dot{q} &= i \\ v &= \hat{v}(q) \end{aligned} \tag{3-2}$$

with $\Sigma = U = \mathbb{R}^1$. Various choices for $\mathcal{U}$ are possible. In this discussion we will let $\mathcal{U}$ be the class of all locally L^1 waveforms $i(\cdot) : \mathbb{R}^+ \to \mathbb{R}$. (This implies that $q(\cdot)$, and hence $v(\cdot)$, is bounded on every bounded interval; therefore $t \to v(t)i(t)$ is locally L^1.)

It is well known that the energy extracted in driving a 1-port capacitor from any initial state q_1 to any final state q_2 depends only on the end-points q_1 and q_2 and is given by

$$E(q_1,q_2) = -\int_{q_1}^{q_2} \hat{v}(q)dq = \int_{q_2}^{q_1} \hat{v}(q)dq. \tag{3-3}$$

Therefore when S is in the form of (3-2), Def. 10 reduces to

$$E_A(q_1) = \sup_{q_2 \in \mathbb{R}} \{E(q_1,q_2)\}. \tag{3-4}$$

Let's briefly reconsider the constitutive relations in Figs. 1 and 3. Substituting $\hat{v}(q) = e^q$ into (3-3), we have $E(q_1,q_2) = e^{q_1} - e^{q_2}$. Taking the supremum over q_2 we have $E_A(q_1) = e^{q_1}$, or $E_A(q) = e^q$ (see Fig. 6).

The way to extract the maximum energy from such an element is to drive the charge as far negative as possible. While there is no trajectory which succeeds in reaching $q = -\infty$ in finite time and extracting all the energy possible, the supremum in (3-1) includes admissible pairs and values of T for which the extracted energy approaches the value e^q. In fact we stated Def. 10 in terms of a supremum in order to handle precisely this type of situation -- one in which no finite-time energy-optimal control exists.

For the constitutive relation in Fig. 3 we have $E(q_1,q_2) = \frac{1}{2}(q_1-1)^2 - \frac{1}{2}(q_2-1)^2$. Taking the supremum over q_2 we have $E_A(q_1) = \frac{1}{2}(q_1-1)^2$ or $E_A(q) = \frac{1}{2}(q-1)^2$, as drawn in dotted lines in Fig. 3. Clearly an energy-optimal control exists. It just drives q to the point q = 1.

Finally, let's leave the capacitive examples and reconsider the linear 2-port in Fig. 2. Considering the input to be the port current vector and the output to be the port voltage vector, the state equation and output equation become

$$\dot{v}_c = v_c \triangleq \underset{\sim}{A} v_c + \underset{\sim}{B}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} . \tag{3-5}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ v_c \end{bmatrix} + \begin{bmatrix} i_1 \\ 0 \end{bmatrix} \triangleq \underset{\sim}{C} v_c + \underset{\sim}{D}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} . \tag{3-6}$$

The impedance matrix is calculated by the usual formula to be

$$\underset{\sim}{Z}(s) = \underset{\sim}{C}\left(s\underset{\sim}{I}-\underset{\sim}{A}\right)^{-1}\underset{\sim}{B} + \underset{\sim}{D} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} . \tag{3-7}$$

This completely expresses its zero-state response. If $v_c(0) = 0$, then port 1 looks like a 1-Ω resistor and port 2 looks like a short circuit, therefore $E_A(0) = 0$. If $v_c(0) \neq 0$, then as we have shown in subsection 3.1 the available energy is infinite.

Definition 11. An n-port $\mathcal{N}$ with a state representation S is passive if for each $\underset{\sim}{x} \in \Sigma$, $E_A(\underset{\sim}{x}) < +\infty$. Otherwise $\mathcal{N}$ is active.

As shown in the Introduction, Definition 11 is equivalent to Passivity 1. If for some initial state $\underset{\sim}{x}_0$ there is no upper bound on the amount of energy which can be extracted, then $E_A(\underset{\sim}{x}_0) = +\infty$ and so $\mathcal{N}$ is active; but if no such state exists, then $\mathcal{N}$ is passive. Observe that passivity requires only that $E_A(\underset{\sim}{x})$ be finite for each $\underset{\sim}{x} \in \Sigma$, and we do not consider infinity or any point with one or more coordinates equal to $\pm\infty$ to be an element of Σ. In particular, passivity does not require that $E_A(\cdot)$ be a bounded function on Σ (cf. Fig. 3 and Fig. 6).

Definition 11 classifies the capacitors in Figs. 1 and 3 as passive and it classifies the linear 2-port in Fig. 2 as active. In the case of Fig. 2 this outcome is completely natural and needs no further explanation. After some reflection we have come to understand that any consistent theory for n-ports, as described in subsection 3.1, must classify the element in Fig. 3 as passive. The reason is that its terminal behavior is indistinguishable from that of a 1-farad capacitor, as we have pointed out in the Introduction, and a 1-farad capacitor is passive by anybody's definition.

The nonlinear capacitor in Fig. 1 violates intuitive notions of passivity because it has no state where $E_A(q) = 0$; nevertheless, it is passive by Def. 11. The capacitor in Fig. 1 has no state where $\hat{v}(q) = 0$, but it is important to realize that the condition $E_A(q) > 0$ for all q is not limited to capacitors with no state of zero voltage. As a counterexample, consider the nonlinear capacitor with constitutive relation $\hat{v}(q) = q(1-q^2)e^{-q^2/2}$. This capacitor is unbiased, in the sense that if $q(0) = 0$ and $i(t) = 0$ for $t \geq 0$, then $v(t) = 0$ for $t \geq 0$. A straightforward calculation shows that $E_A(q) = (1+q^2)e^{-q^2/2}$; therefore, this capacitor is passive by Def. 11 and $E_A(q) > 0$ for all q. This example is interesting because if $|q(t)|$ remains sufficiently small for all t, then $v(t) = \hat{v}(q(t)) \approx q(t)$; hence, this capacitor behaves locally at $q = 0$ as though it were a 1-farad linear capacitor, yet it has no state of zero available energy.

It is interesting to compare our treatment of the topic of passivity with others that have appeared in the literature. The example described in the preceding paragraph is a special case of the type of system considered by Moylan [8] and Hill [28]; they would classify this capacitor as active because they adopted the definition of passivity which we have called Passivity 3 in the Introduction (note that Passivity 3 is equivalent to the condition $E_A(\underset{\sim}{0}) = 0$). Chua and Lam [7] would classify the capacitor in Fig. 1 and the capacitor described in the preceding paragraph as active, since both examples violate Passivity 2 and Passivity 3. [Incidentally, Chua and Lam [7] erroneously claimed that Passivity 2 and Passivity 3 are equivalent for capacitors and inductors. The simple example in Fig. 3 shows that this is not the case: it does not satisfy

Passivity 3 because $E_A(q) = (q-1)^2/2$, and so $E_A(0) \neq 0$; however, it would presumably satisfy Passivity 2 if the initial charge is $q(0) = 1$. Chua and Lam [7] did not realize that in constructing a state model for an n-port, the identity of charges and fluxes as state variables is determined only to within an additive constant. A complete discussion of this point is given in subsection 5.1.] Desoer and Kuh [5] began with Passivity 1 as their fundamental definition of passivity for 1-ports. They then proceeded to "prove" that a 1-port charge-controlled capacitor with constitutive relation $v = \hat{v}(q)$ is passive if and only if $\int_0^q \hat{v}(\hat{q})d\hat{q} \geq 0$ for all q. The condition $\int_0^q \hat{v}(\hat{q})d\hat{q} \geq 0$ is equivalent to the condition $E_A(0) = 0$, so Desoer and Kuh [5] were claiming that Passivity 1 and Passivity 3 are equivalent for charge-controlled capacitors. Our capacitive examples show that their "proof" was not correct, since none of these examples satisfy $E_A(0) = 0$. Likewise, Hill and Moylan [11] erred when they claimed that Passivity 1 and Passivity 3 are equivalent for the class of systems they considered. The example in the preceding paragraph is a special case of the type of system considered by Hill and Moylan [11], and $E_A(0) \neq 0$ for this example.

The preceding comments suggest that there is some confusion in the literature over the concept of passivity for nonlinear circuits and systems. Much of this confusion arises from the intuitive notion that a passive system ought to possess a state of "zero stored energy," or more precisely, a state of zero <u>available</u> energy. For this reason, we will define in subsection 3.4 a narrower concept of passivity called <u>strong passivity</u> which is closer to engineering intuition.

Passivity requires that $E_A(\underset{\sim}{x}) < +\infty$ for <u>all</u> $\underset{\sim}{x} \in \Sigma$. Suppose we know that $E_A(\underset{\sim}{x}_0) < +\infty$ for a particular $\underset{\sim}{x}_0 \in \Sigma$. When can we conclude that $E_A(\underset{\sim}{x}) < +\infty$ for all $\underset{\sim}{x} \in \Sigma$? The following theorem and its corollaries answer this question, but first we must define reachability and complete controllability.

There are several definitions of reachability and complete controllability in the literature. The following ones are appropriate here because of our standing assumptions that the n-ports under discussion are time invariant and that $\mathcal{U}$ is translation invariant (Def. 6).

<u>Definition 12</u>. Given a state representation S, let $\underset{\sim}{x}_0, \underset{\sim}{x}_1 \in \Sigma$. The <u>state $\underset{\sim}{x}_1$</u> is said to be <u>reachable from $\underset{\sim}{x}_0$</u> if there exists a finite $T \geq 0$ and an input-trajectory pair $\{\underset{\sim}{u}(\cdot), \underset{\sim}{x}(\cdot)\}|[0,T]$ from $\underset{\sim}{x}_0$ to $\underset{\sim}{x}_1$ (Def. 8). The <u>state space Σ</u> is said to be <u>reachable from $\underset{\sim}{x}_0$</u> if every $\underset{\sim}{x} \in \Sigma$ is reachable from $\underset{\sim}{x}_0$.

Because of our standing assumption that $t \to \langle \underset{\sim}{v}(t), \underset{\sim}{i}(t)\rangle$ is locally L^1, it follows that the transfer from $\underset{\sim}{x}_0$ to $\underset{\sim}{x}_1$ can always be effected with a finite (positive or negative) amount of energy.

Definition 13. A state representation S is said to be completely controllable if Σ is reachable from $\underset{\sim}{x}$ for every $\underset{\sim}{x} \in \Sigma$.

Theorem 1. Given an n-port $\mathcal{N}$ with a state representation S, where $\mathcal{U}$ is closed under concatenation. Suppose that there exist two states $\underset{\sim}{x}_0, \underset{\sim}{x}_1 \in \Sigma$ with $\underset{\sim}{x}_1$ reachable from $\underset{\sim}{x}_0$. Under these conditions, if $E_A(\underset{\sim}{x}_0) < +\infty$, then $E_A(\underset{\sim}{x}_1) < +\infty$.

The proof is given in Appendix A, although Theorem 1 should be immediately obvious since the transfer from $\underset{\sim}{x}_0$ to $\underset{\sim}{x}_1$ can be effected with a finite amount of energy.

Corollary A for Theorem 1. Given an n-port $\mathcal{N}$ with a state representation S, where $\mathcal{U}$ is closed under concatenation. Suppose that there exists a state $\underset{\sim}{x}_0$ such that Σ is reachable from $\underset{\sim}{x}_0$. Under these conditions, $\mathcal{N}$ is passive if and only if $E_A(\underset{\sim}{x}_0) < +\infty$.

Proof. Necessity follows directly from Def. 11. Sufficiency follows from Def. 11, Def. 12, and Theorem 1. Q.E.D.

Corollary B for Theorem 1. Given an n-port $\mathcal{N}$ with a completely controllable state representation S, where $\mathcal{U}$ is closed under concatenation. Let $\underset{\sim}{x}_0 \in \Sigma$ be arbitrary and fixed. Under these conditions, $\mathcal{N}$ is passive if and only if $E_A(\underset{\sim}{x}_0) < +\infty$.

Proof. Necessity follows directly from Def. 11. Sufficiency follows from Def. 11, Def. 13, and Theorem 1. Q.E.D.

When $\mathcal{N}$ has a completely controllable state representation, Corollary B tells us that we need only determine whether $E_A(\underset{\sim}{x}_0) < +\infty$ at one arbitrary state $\underset{\sim}{x}_0 \in \Sigma$; if so, then $\mathcal{N}$ is passive.

3.3 A Mechanical Example

In this subsection we will consider the two nonlinear capacitors whose constitutive relations are shown in Fig. 4. The natural state representation for these elements is that given in Example 1, except that in this case $\Sigma = (-\infty, 0)$ and $i(\cdot)$ must be chosen[4] so that q never leaves Σ. Systems like

[4] The condition

$$q_0 + \int_0^T i(t)dt < 0, \; \forall T \geq 0$$

implies that the class of admissible input waveforms depends on q_0. This is a technical violation of our assumptions about the nature of a state representation, but it presents no problem in this case. See the discussion in subsection 8.2.

these constitute a rather severe test for any proposed definition of passivity. One reason is that neither has a relaxed state (see subsection 3.4) or any obvious candidate for a reference state. Another is that an independent line of physical reasoning, based on their mechanical analogs, strongly indicates that the dotted curve should be labelled passive. Presumably any correct definition of passivity must also classify this element as passive.

These curves are taken from the piston and cylinder arrangements shown in Figs. 7(a) and (b), with q substituted for x and v for f. The reference directions in Figs. 7(a) and (b) have been chosen so that $f\dot{x}$ is the instantaneous power <u>delivered to</u> the system, corresponding to associated reference directions for v and $\dot{q}$. These are rather idealized mechanical systems because we suppose that the cylinders are infinitely long, the walls are perfectly insulating, the pistons are massless and frictionless, the external pressure is zero, the gas behaves ideally and all motions are sufficiently slow that no internal pressure gradients arise. Beginning in any initial state we can extract a certain amount of energy from either system by allowing the gas to expand against the external vacuum and extracting useful work from the resulting motion of the piston. The question of interest to us is whether the energy we can extract is bounded or not.

Let's first consider Fig. 7(a). Since the reservoir holds the gas temperature constant, the ideal gas law requires that the pressure and volume be related by PV = NRT, where N is the number of moles of gas present and R is the gas constant [12]. Since the piston has unit area and we have chosen the origin of the x-axis to lie at the point where V = 0, the relation between force and displacement becomes PV = -fx = NRT or f = -NRT/x. Except for the positive multiplicative constant NRT, which will not turn out to be important, this is exactly analogous to the constitutive relation v = -1/q of C_1 in Fig. 4.

It is not immediately obvious whether the system in Fig. 7(a) should be called active or passive. The infinite heat reservoir itself is clearly an active element, analogous to an ideal voltage source. As the piston is moved to the left, heat is continually being extracted from it to compensate for the tendency of the gas to cool on expansion, so on purely physical grounds the possibility exists that the system is active. Writing $E(x_1,x_2)$ for the energy extracted in going from x_1 to x_2, we have

$$E(x_1,x_2) = \int_{x_1}^{x_2} -f(x)dx = \int_{x_1}^{x_2} \frac{NRT}{x}\, dx = NRT \ln (x_2/x_1). \tag{3-8}$$

Keeping in mind that x_1 is negative and that x_2 can take on any negative value, we have

$$E_A(x_1) = \sup_{x_2<0} \left\{ NRT \ln\left(\frac{x_2}{x_1}\right)\right\} = +\infty . \tag{3-9}$$

Similarly, for C_1 in Fig. 4 we have $E(q_1,q_2) = \ln(q_2/q_1)$ and $E_A(q_1) = +\infty$, so these systems are active by our definition.

Now let's consider the closed system in Fig. 7(b). In this case the gas will cool upon expansion and P will fall off more rapidly with increasing V than before. The ideal gas law predicts that $P = A/V^{5/3}$ where A is a positive constant, or $f = -A/x^{5/3}$ [12]. On purely physical grounds this system should be called passive because there can only be a finite amount of energy initially stored in the gas and there are no other energy sources involved. Direct calculation shows that

$$E(x_1,x_2) = \int_{x_1}^{x_2} \frac{A}{x^{5/3}}\, dx = \frac{3A}{2}\left[\left(\frac{1}{x_1}\right)^{2/3} - \left(\frac{1}{x_2}\right)^{2/3}\right] . \tag{3-10}$$

Keeping in mind that $(1/x)^{2/3} = (3\sqrt{x})^2 > 0$ for negative x, we have $E_A(x_1) = 3A/2x_1^{2/3}$, which is finite for each $x_1 \in \Sigma$. Similarly, for the element C_2 in Fig. 4 we have $E_A(q_1) = 3/2q_1^{2/3} < +\infty$, so both of these are passive by Def. 11, as required by physical reasoning.

3.4 Relaxed States and Strong Passivity

We are convinced, and we hope we have convinced the reader, that Def. 11 is the appropriate definition of passivity for any consistent general theory of nonlinear n-ports; however, there is no doubt that it is weaker than the concept of passivity that one would gain from experience with common circuit elements. In practice it is natural to associate with a passive element some sort of "rest state" or "relaxed state" or state of "zero stored energy." As we pointed out in the Introduction, several definitions of passivity found in the literature are based on the existence of such a state [6], [7]. The constitutive relations in Figs. 1 and 4 will produce severe problems for any such definition. While we do not wish to found a theory on the existence of such states, it is reasonable to try to incorporate relaxed states into our more general approach. To be completely consistent, however, we cannot define a relaxed state as a state of "zero stored energy." The reason is that it is not always possible to determine the energy stored in an n-port by means of measurements performed at the terminals alone. There might be, for example, a battery buried inside whose effects are not evident at the ports. But it is consistent to define a relaxed state as a state of zero available energy.

Definition 14. Given an n-port $\mathcal{N}$ with a state representation S, a point $\underset{\sim}{x} \in \Sigma$ is said to be a relaxed state if $E_A(\underset{\sim}{x}) = 0$.

In Fig. 2 the only relaxed state is $v_c = 0$, while in Fig. 3 the state $q = 1$ is relaxed. The systems in Figs. 1 and 4 do not have any relaxed states. Note that even when a relaxed state exists it need not be unique. Consider, for example, the nonlinear inductor defined by $i = I_0 \sin k\phi$; $I_0 > 0$, $k > 0$, which appears in most models of Josephson junction devices. It is not difficult to see that every state of the form $\phi = 2\pi n/k$, n an integer, is a relaxed state for this element.

Using the notion of a relaxed state, we can now define the following more restrictive concept of passivity.

Definition 15. An n-port $\mathcal{N}$ with a state representation S is strongly passive if

(1) $\mathcal{N}$ is passive Def. 11, and

(2) there exists a relaxed state $\underset{\sim}{x}^* \in \Sigma$.

The constitutive relation in Fig. 3 defines a strongly passive[5] 1-port, for example. The constitutive relation in Fig. 1 and C_2 in Fig. 4 define systems which are passive but not strongly passive. We will see in the next section, however, that passivity and strong passivity are equivalent concepts for linear, resistive, and memristive n-ports.

At first glance it might seem reasonable to say that an n-port $\mathcal{N}$ with state representation S is strongly passive if there exists a relaxed state $\underset{\sim}{x}^* \in \Sigma$, but the 2-port in Fig. 2 shows why we must explicitly require that $\mathcal{N}$ be passive in addition. The state $v_c^* = 0$ is a relaxed state for this example, but $E_A(v_c) = +\infty$ for all other values of v_c. Therefore if we wish strong passivity to be a stricter requirement than passivity, we must include condition (1) in Def. 15.

Suppose that we are given an n-port $\mathcal{N}$ with a state representation S, and suppose that there exists a relaxed state $\underset{\sim}{x}^* \in \Sigma$. If Σ is reachable from $\underset{\sim}{x}^*$, and if $\mathcal{U}$ is closed under concatenation, then condition (1) of Def. 15 is automatically satisfied; hence, $\mathcal{N}$ is strongly passive. This is stated formally in the following theorem, which also shows the relationship between strong passivity and Passivity 2 (defined in the Introduction).

Theorem 2. Given an n-port $\mathcal{N}$ with a state representation S, where $\mathcal{U}$ is closed under concatenation. Suppose that there exists a state $\underset{\sim}{x}^*$ such that Σ is reachable from $\underset{\sim}{x}^*$ (Def. 12). Under these conditions, $\mathcal{N}$ is strongly passive with relaxed state $\underset{\sim}{x}^*$ if and only if

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[5]Our use of the term "strongly passive" should not be confused with similar terminology which has appeared in the systems literature (e.g., Moylan and Hill [13]).

$$\int_0^T \langle \underset{\sim}{v}(t), \underset{\sim}{i}(t)\rangle\, dt \geq 0 \tag{3-11}$$

for all $T \geq 0$ and all admissible pairs $\{\underset{\sim}{v}(\cdot), \underset{\sim}{i}(\cdot)\}$ with initial state $\underset{\sim}{x}^*$ (Def. 9).

Proof. Clearly, $\underset{\sim}{x}^*$ is a relaxed state if and only if (3-11) is satisfied; hence, the necessity of (3-11) is immediate. Sufficiency follows from Corollary A for Theorem 1. Q.E.D.

If an n-port is strongly passive with relaxed state $\underset{\sim}{x}^*$, then by a change of coordinates we can always assume that $\underset{\sim}{x}^* = \underset{\sim}{0}$ (see the discussion in subsection 5.1). In this way we can make sense out of Passivity 3, defined in the Introduction.

IV. Necessary and Sufficient Conditions for Passivity of Several Classes of N-Ports

For certain special classes of n-ports it is possible to find necessary and sufficient conditions for passivity which can be verified directly by inspection of the state equations. The classes we will investigate here are resistive, generalized capacitive/inductive, generalized memristive,[6] and linear n-ports as well as n-ports with a 1-dimensional state space. Except for this last class the conditions we derive will be familiar, but in the generalized capacitive/inductive and linear cases they will differ in a subtle but important way from those usually given in the literature.

4.1 Resistive N-Ports

The resistive n-ports considered here are completely characterized by the relation $\underset{\sim}{y} = \underset{\sim}{g}(\underset{\sim}{u})$, where $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are a hybrid pair (Def. 3). We can afford to be quite undemanding about the details in this case, but we will present them anyway for completeness. We let U be any nonempty subset of $\mathbb{R}^n$; $\underset{\sim}{g} : U \to \mathbb{R}^n$ be any function, and $\mathcal{U}$ be the class of all functions $\underset{\sim}{u}(\cdot) : \mathbb{R}^+ \to U$ such that $t \to \langle \underset{\sim}{u}(t), \underset{\sim}{g}(\underset{\sim}{u}(t))\rangle$ is locally L^1. Note that $\mathcal{U}$ is automatically nonempty because it contains all the constant inputs.

It is unnatural to construct a state representation for a resistive element, but in order to include such elements in our theory we will give them representations of the form $\dot{\underset{\sim}{x}} = \underset{\sim}{0}$, $\underset{\sim}{y} = \underset{\sim}{g}(\underset{\sim}{u})$, with Σ taken to be any nonempty subset of $\mathbb{R}^m$. We emphasize that Σ has no real significance in this case: it has been included only to show how resistive elements can be handled within the framework of our theory. The resulting classification is obvious.

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[6] The resistive, generalized capacitive/inductive, and generalized memristive n-ports considered in this section are special cases of the so-called <u>algebraic n-ports</u> treated by Chua [14].

Theorem 3. Let $\mathcal{N}$ be a resistive n-port with a state representation S as described above. Then the following three statements are equivalent:

(i) $\langle \underset{\sim}{u}, g(\underset{\sim}{u})\rangle \geq 0$, $\forall \underset{\sim}{u} \in U$.

(ii) $E_A(\underset{\sim}{x}) = 0$, $\forall \underset{\sim}{x} \in \Sigma$.

(iii) $\mathcal{N}$ is passive according to Def. 11.

It follows immediately that passivity and strong passivity are equivalent for resistive n-ports. The proof, which is trivial, is given in Appendix B. We now have the theoretical basis for our claim that the 1-port in Fig. 5(b) is passive, since it is characterized by $v_1 = i_1$. Similarly the 2-port in Fig. 5(c) is active, since $v_1 = i_1$ and $v_2 = -10\, i_1$. We need only choose $i_1 = i_2 = 1$, for example, to see that $\langle v,i\rangle = \langle (i_1,-10i_1),(i_1,i_2)\rangle = i_1^2 - 10 i_1 i_2$ can be negative.

4.2 Generalized Capacitive/Inductive N-Ports

A generalized capacitive/inductive n-port is an n-port with state and output equations of the form

$$\begin{aligned} \dot{\underset{\sim}{x}} &= \underset{\sim}{u} \\ \underset{\sim}{y} &= \underset{\sim}{g}(\underset{\sim}{x}) \end{aligned} \tag{4-1}$$

where $\underset{\sim}{u}$ and $\underset{\sim}{y}$ form a hybrid pair. We call these generalized capacitive/inductive n-ports since they reduce to n-port capacitors if $\underset{\sim}{u} = \underset{\sim}{i}$ and $\underset{\sim}{y} = \underset{\sim}{v}$ and to n-port inductors if $\underset{\sim}{u} = \underset{\sim}{v}$ and $\underset{\sim}{y} = \underset{\sim}{i}$. The relevant technical assumptions are that $\Sigma = U = \mathbb{R}^n$, $\mathcal{U} = L^1_{loc}(\mathbb{R}^+ \to \mathbb{R}^n)$, and that $\underset{\sim}{g} : \mathbb{R}^n \to \mathbb{R}^n$ is continuous.[7]

Theorem 4. Let $\mathcal{N}$ be a generalized capacitive/inductive n-port with a state representation S as described above. Then

(1) $\mathcal{N}$ is passive $\Leftrightarrow$ $\underset{\sim}{g} = \underset{\sim}{\nabla}\psi$ where $\psi : \Sigma \to \mathbb{R}$ is a C^1 scalar function which is bounded from below.

(2) $\mathcal{N}$ is strongly passive $\Leftrightarrow$ the above conditions hold and in addition $\psi(\cdot)$ attains its lower bound, i.e., $\exists \underset{\sim}{x}^* \in \Sigma$ such that $\psi(\underset{\sim}{x}^*) \leq \psi(\underset{\sim}{x})$, $\forall \underset{\sim}{x} \in \Sigma$.

The proof is given in Appendix C. The capacitor in Fig. 3 satisfies both conditions (1) and (2); the capacitor in Fig. 1 and C_2 in Fig. 4 satisfy only (1); and C_1 in Fig. 4 satisfies neither (1) nor (2).

It is easy to see that if $\mathcal{N}$ is passive and ℓ is the greatest lower bound of $\psi(\cdot)$, then $E_A(\underset{\sim}{x}) = \psi(\underset{\sim}{x}) - \ell$, $\forall \underset{\sim}{x} \in \Sigma$. An immediate consequence of Theorem 4

[7] Note that our choice of $\mathcal{U}$ implies that $\underset{\sim}{x}(\cdot)$ is bounded on every bounded interval, and since $\underset{\sim}{g}$ is continuous it follows that $\underset{\sim}{y}(\cdot)$ is also bounded on every bounded interval. Therefore $t \to \langle \underset{\sim}{u}(t), \underset{\sim}{y}(t)\rangle$ is automatically locally L^1.

is that if $\mathcal{N}$ is passive and $g(\cdot)$ is C^1, then the Jacobian matrix $[Dg](x)$ is symmetric at each point $x \in \Sigma$. By linearizing (4-1) about any state x, it is easy to see that this symmetry condition is equivalent to reciprocity if $u = i$ or if $u = v$. But if $u \neq i$ and $u \neq v$, i.e., if u contains both voltage and currents, then symmetry of $[Dg]$ is an entirely different condition from reciprocity [14]. We have then the following simple corollary.

Corollary for Theorem 4. A passive n-port inductor or capacitor with a C^1 function $g(\cdot)$ is reciprocal.

4.3 Generalized N-Port Memristors

By a generalized n-port memristor we mean an n-port with state and output equations of the form

$$\begin{aligned} \dot{x} &= u \\ y &= [R(x)]u \end{aligned} \qquad (4\text{-}2)$$

where u and y form a hybrid pair and where $R(x)$ is an $n \times n$ real matrix which varies with x. A system of this sort is, roughly speaking, a state-dependent linear resistor. It is a special case of the class of memristive systems defined by Chua and Kang [15]. In the 1-port current-controlled case, $u = i$ and the generalized n-port memristor reduces to the 1-port memristor [16] characterized by $\phi = f(q)$; or after differentiation, $v = \dot{\phi} = f'(q)\dot{q} = f'(q)i$. An n-port memristor would be the same as Eq. (4-2) with the special assumptions that $u = i$ and that $R(x)$ is the Jacobian matrix of some vector-valued function. We have chosen the name "generalized" n-port memristor because we do not impose these last two assumptions. The relevant technical requirements in our case are that $\Sigma = \mathbb{R}^n$, the entries of $R(x)$ are continuous functions on Σ, and that[8] $\mathcal{U} = L^2_{loc}(\mathbb{R}^+ \to \mathbb{R}^n)$.

Theorem 5. A generalized n-port memristor with a state representation as described above is passive $\Leftrightarrow$ $R(x)$ is positive semidefinite at each point $x \in \Sigma$.

Proof. ($\Leftarrow$) If $R(x)$ is positive semidefinite, then for any input-output pair $\{u(\cdot), y(\cdot)\}$ and any time $t \geq 0$, $\langle u(t), y(t)\rangle = u^T(t)[R(x(t))]u(t) \geq 0$. Therefore, $E_A(x) = 0$, $\forall x \in \Sigma$.

($\Rightarrow$) Suppose $\mathcal{N}$ is passive but $R(x)$ is not positive semidefinite everywhere, i.e., $\exists x^* \in \Sigma$, $u^* \in U$ such that $u^{*T}[R(x^*)]u^* = -a < 0$. Since the entries of

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[8] As in subsection 4.2, it follows automatically that $t \to \langle u(t), y(t)\rangle$ is locally L^1.

$\underset{\sim}{R}(\underset{\sim}{x})$ are continuous functions, $\exists\varepsilon > 0$ such that $\|\underset{\sim}{x}-\underset{\sim}{x}^*\| \leq \varepsilon \Rightarrow \underset{\sim}{u}^{*T}[\underset{\sim}{R}(\underset{\sim}{x})]\underset{\sim}{u}^*$ $< -a/2 < 0$. Let $\hat{\underset{\sim}{u}}(\cdot)$ be given by $\hat{\underset{\sim}{u}}(t) = (\varepsilon/\|\underset{\sim}{u}^*\|)\underset{\sim}{u}^* \cos t$, $\forall t \geq 0$, let $\hat{\underset{\sim}{x}}(\cdot)$ be the trajectory resulting from the input $\hat{\underset{\sim}{u}}(\cdot)$ with initial state $\underset{\sim}{x}^*$, and let $\hat{\underset{\sim}{y}}(\cdot)$ be the corresponding output. Then for all $t \geq 0$

$$\|\hat{\underset{\sim}{x}}(t)-\underset{\sim}{x}^*\| = \left\|\int_0^t (\varepsilon/\|\underset{\sim}{u}^*\|)\underset{\sim}{u}^* \cos\tau \, d\tau\right\| = \varepsilon|\sin t| \leq \varepsilon,$$

so

$$\underset{\sim}{u}^{*T} \underset{\sim}{R}(\hat{\underset{\sim}{x}}(t))\underset{\sim}{u}^* < -a/2, \ \forall t \geq 0.$$

Furthermore,

$$\int_0^T -\langle\hat{\underset{\sim}{u}}(t),\hat{\underset{\sim}{y}}(t)\rangle\, dt = -\int_0^T \left(\frac{\varepsilon}{\|\underset{\sim}{u}^*\|}\right)^2 \left(\underset{\sim}{u}^{*T}[\underset{\sim}{R}(\hat{\underset{\sim}{x}}(t))]\underset{\sim}{u}^*\right) \cos^2 t \, dt$$

$$\geq \left(\frac{\varepsilon}{\|\underset{\sim}{u}^*\|}\right)^2 \left(\frac{a}{2}\right) \int_0^T \cos^2 t \, dt \xrightarrow[T\to\infty]{} +\infty \, . \tag{4-3}$$

Therefore $E_A(\underset{\sim}{x}^*) = +\infty$, contradicting our assumption that $\mathcal{N}$ was passive.

Q.E.D.

Corollary for Theorem 5. A generalized n-port memristor is passive if and only if it is strongly passive.

4.4 Linear N-Ports

In this subsection we will discuss hybrid linear n-ports. We will establish that the traditional positive real criterion for the hybrid matrix transfer function is equivalent to passivity if we assume that the n-port is completely controllable.

The following definition is standard [6] and we repeat it here merely for reference.

Definition 16. Let $\underset{\sim}{H}(\cdot)$ be an $n\times n$ matrix of rational functions of a complex variable s. Then $\underset{\sim}{H}(\cdot)$ is said to be positive real if

1) Each entry of $\underset{\sim}{H}(\cdot)$ is analytic in the open right half complex plane.

2) Each entry of $\underset{\sim}{H}(s)$ is real for all real positive s.

3) The matrix $[\underset{\sim}{H}(s)+\underset{\sim}{H}^*(s)]$ is positive semi-definite at each point s in the open right half plane, where the superscript * denotes the complex-conjugate transpose of a matrix.

The following lemma is also standard [6].

Lemma 1. Let $\underset{\sim}{H}(\cdot)$ be an $n\times n$ matrix whose entries are all rational functions of a complex variable s, with real coefficients. Then $\underset{\sim}{H}(\cdot)$ is positive real iff

1) No entry of $\underset{\sim}{H}(\cdot)$ has a pole in the open right half plane.

2) The matrix $[\underset{\sim}{H}^*(j\omega)+\underset{\sim}{H}(j\omega)]$ is positive semi-definite for each real ω, provided that $j\omega$ is not a pole of any entry of $\underset{\sim}{H}(\cdot)$.

3) If $j\omega_0$ is a pure imaginary pole of some entry of $\underset{\sim}{H}(\cdot)$, it is at most a simple pole. If we let $\underset{\sim}{K}$ denote the residue matrix of $\underset{\sim}{H}(\cdot)$ at $j\omega_0$, i.e., $\underset{\sim}{K} = \lim_{s\to j\omega_0} (s-j\omega_0)\underset{\sim}{H}(s)$ if ω_0 is finite and $K = \lim_{s\to\infty} \frac{\underset{\sim}{H}(s)}{s}$ if ω_0 is infinite, then $\underset{\sim}{K}$ is positive semi-definite and $\underset{\sim}{K} = \underset{\sim}{K}^*$.

The proof is given by Anderson and Vongpanitlerd [6].

Now consider the linear time-invariant finite dimensional state representation

$$\begin{aligned} \dot{\underset{\sim}{x}} &= \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{B}\underset{\sim}{u} \\ \underset{\sim}{y} &= \underset{\sim}{C}\underset{\sim}{x} + \underset{\sim}{D}\underset{\sim}{u} \end{aligned} \tag{4-4}$$

where $\underset{\sim}{u}$ and $\underset{\sim}{y}$ form a hybrid pair; $U = \mathbb{R}^n$ and $\Sigma = \mathbb{R}^m$; $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$, and $\underset{\sim}{D}$ are real constant matrices of appropriate dimension; and $\mathcal{U}$ is taken to be the class of all locally L^2 functions $\underset{\sim}{u}$: $\mathbb{R}^+ \to \mathbb{R}^n$. It follows immediately from the convolution integral relating $\underset{\sim}{u}(\cdot)$ and $\underset{\sim}{y}(\cdot)$ that for any initial condition $\underset{\sim}{x}_0 \in \Sigma$ and any input $\underset{\sim}{u}(\cdot) \in \mathcal{U}$, the corresponding output $\underset{\sim}{y}(\cdot)$ is defined for all positive time and is itself locally L^2. It is well known [17] that the linear system (4-4) is completely controllable (Def. 13) if and only if the matrix

$$\left[\underset{\sim}{B} \;\vdots\; \underset{\sim}{A}\underset{\sim}{B} \;\vdots\; \underset{\sim}{A}^2\underset{\sim}{B} \;---\vdots\; \underset{\sim}{A}^{m-1}\underset{\sim}{B}\right]$$

has rank m.

Theorem 6. Suppose $\mathcal{N}$ has a linear, time-invariant, finite-dimensional state representation S as in (4-4). If S is completely controllable, then the following three conditions are equivalent:

(i) $\mathcal{N}$ is passive by Def. 11.

(ii) $E_A(\underset{\sim}{0}) = 0$.

(iii) The matrix transfer function

$$\underset{\sim}{H}(s) = \underset{\sim}{C}[s\underset{\sim}{I}-\underset{\sim}{A}]^{-1}\,\underset{\sim}{B} + \underset{\sim}{D}$$

is positive real.

Slight variations on this theorem are well known [1], [18]. A complete elementary proof is given by Gannett and Chua [18]. The example in Fig. 2 shows that the assumption of complete controllability is essential: for that 2-port $\underset{\sim}{Z}(s)$ is positive real and $E_A(\underset{\sim}{0}) = 0$, but it is active nevertheless.

4.5 Systems With a 1-Dimensional State Space

The class of systems considered in this subsection are those for which the state space Σ is contained in the real line. As far as we know, the results given here are entirely new to the literature. We consider systems with a state equation of the form

$$\dot{x} = f(x,\underset{\sim}{u}) \tag{4-5}$$

and we wish to find a necessary and sufficient condition which guarantees that the available energy $E_A(x_0) = \sup_{\substack{x_0 \to \\ T \geq 0}} \left\{ -\int_0^T p(x(t),\underset{\sim}{u}(t))dt \right\}$ is finite for all $x_0 \in \Sigma$.

The technical requirements are that $\Sigma \subset \mathbb{R}^1$, U is a closed subset of $\mathbb{R}^n$, and $\mathcal{U}$ is the set of all piecewise continuous functions mapping $\mathbb{R}^+$ to U. In most examples, Σ will be $\mathbb{R}^1$ or an interval in $\mathbb{R}^1$. The functions $f(\cdot,\cdot)$ and $p(\cdot,\cdot)$ are real-valued; and recalling our assumptions in Section II, they must be continuous and $f(\cdot,\cdot)$ must satisfy enough conditions to ensure the existence and uniqueness of the solution to (4-5); it is important to realize, however, that we make no other assumption concerning the structure of these functions.

Definition 17. For each point $x \in \Sigma$, let U_x^+ be the set of all input values $\underset{\sim}{u} \in U$ such that $f(x,\underset{\sim}{u}) > 0$. Similarly, let U_x^- be all values of $\underset{\sim}{u}$ such that $f(x,\underset{\sim}{u}) < 0$.

So U_x^+ is just the set of all input values that will drive the state to the right from the point x, and U_x^- is all input values that will drive it to the left. Either U_x^+ or U_x^- or both may be empty for certain values of x.

Definition 18. We define $\underline{h}$ and $\bar{h} : \Sigma \to \mathbb{R}^e$ by

$$\underline{h}(x) \triangleq \begin{cases} \sup\limits_{\underset{\sim}{u} \in U_x^-} \dfrac{p(x,\underset{\sim}{u})}{f(x,\underset{\sim}{u})}, & U_x^- \neq \phi \\ \\ -\infty, \text{ if } U_x^- = \phi \end{cases} \tag{4-6}$$

$$\bar{h}(x) \triangleq \begin{cases} \inf\limits_{\underset{\sim}{u} \in U_x^+} \dfrac{p(x,\underset{\sim}{u})}{f(x,\underset{\sim}{u})}, & \text{if } U_x^+ \neq \phi \\ +\infty, & \text{if } U_x^+ = \phi \end{cases} \tag{4-7}$$

These functions will play a central role in what follows, so let's pause a moment to give them a physical interpretation. Consider the first line in (4-7). Before taking the infimum, the numerator is energy input per unit time and the denominator is the distance the state moves to the right per unit time; so the quotient is the input energy per unit distance Δx, in the limit as $\Delta x \to 0$ from above. Taking the infimum, we see that $\bar{h}(x)$ is the minimum energy cost per unit displacement of x to the right, with the convention that $\bar{h}(x) = +\infty$ if it is impossible to drive the state to the right from the point x. Similarly, $\underline{h}(x)$ is the maximum energy we can extract per unit displacement of the state to the left from the point x, this time as $\Delta x \to 0$ from below: the convention here is that $\underline{h}(x) = -\infty$ if we cannot move to the left from x. In general, neither $\underline{h}(\cdot)$ nor $\bar{h}(\cdot)$ will be continuous; however, $\underline{h}(\cdot)$ is lower semicontinuous and $\bar{h}(\cdot)$ is upper semicontinuous (see Appendix D).

Example 2. Consider the following memristive 1-port [15]:

$$\begin{aligned} \dot{x} &= |i|^{\alpha} \\ v &= r(x)i \end{aligned} \tag{4-8}$$

where $\Sigma = U = \mathbb{R}$, $\mathcal{U}$ consists of all piecewise continuous functions mapping $\mathbb{R}^+$ to $\mathbb{R}$, $r(\cdot)$ is a continuous function which is negative on the interval (0,1) and zero elsewhere, and α is some positive real number.

It is clear that $\underline{h}(x) = -\infty$ for all x, since $f(x,i) \triangleq |i|^{\alpha}$ is never negative; moreover, $\bar{h}(x) = 0$ for all x outside of (0,1), since $g(x,i) \triangleq r(x)i = 0$ for $x \notin (0,1)$. When $x \in (0,1)$, $\bar{h}(x)$ is the infimum over all $i \neq 0$ of $i^2 r(x)/|i|^{\alpha}$, i.e., the infimum of $r(x)|i|^{(2-\alpha)}$: if $\alpha = 2$, this is just $r(x)$; if $\alpha \neq 2$, the infimum is $-\infty$ since $r(x) < 0$.

For any state representation S and for any $\underset{\sim}{x} \in \Sigma$, let $R(\underset{\sim}{x})$ denote the set of states reachable from $\underset{\sim}{x}$ (Def. 12). In the 1-dimensional case it will be useful to define, for each $x_0 \in \Sigma$, $R^-(x_0) \triangleq \{x \in R(x_0): x < x_0\}$ and

$R^+(x_0) = \{x \in R(x_0): x > x_0\}$. In Example 2, for instance, $R^-(x_0) = \phi$ and $R^+(x_0) = (x_0,+\infty)$, for any $x_0 \in \Sigma$.

We are now in a position to state the general passivity theorem for 1-dimensional systems.

Theorem 7. Let $\mathcal{N}$ be an n-port with a state representation S as given in (4-5) and the paragraph following (4-5). $\mathcal{N}$ is passive if and only if all three of the following conditions are satisfied:

(i) $p(x,\underset{\sim}{u}) \geq 0$ at every point $(x,\underset{\sim}{u}) \in \Sigma \times U$ such that $f(x,\underset{\sim}{u}) = 0$;

(ii) $\underline{h}(x) \leq \bar{h}(x)$, $\forall x \in \Sigma$;

(iii) there exists a function $W : \Sigma \to \mathbb{R}^+$ such that, for every $x_0 \in \Sigma$,

$$\int_{x_0}^{x_1} \underline{h}(x)dx + W(x_0) \geq 0, \ \forall x_1 \in \text{int } R^-(x_0) \tag{4-9}$$

$$\int_{x_0}^{x_2} \bar{h}(x)dx + W(x_0) \geq 0, \ \forall x_2 \in \text{int } R^+(x_0). \tag{4-10}$$

The proof is given in Appendix D. Note that there is no need to actually calculate $W(x_0)$. Its existence is just another way of saying that the integrals in (4-9) and (4-10) remain bounded from below as their upper limits are allowed to vary in $R^-(x_0)$ and $R^+(x_0)$, respectively. The sense of the inequality in (4-9) may be a little confusing initially. Since $x_1 < x_0$, the integral will be positive if the integrand is everywhere negative. No such problem occurs in (4-10). Finally, if $R^-(x_0)$ is empty then (4-9) is satisfied automatically for that value of x_0; (4-10) is similar.

The reason for considering only values of x_1 and x_2 in the interior of the reachable set is connected with the existence of the integrals in (4-9) and (4-10). If $x_2 \in \text{int } R^+(x_0)$, then $\bar{h}(x) \neq +\infty$ for all $x \in [x_0,x_2]$ (Def. 18). Since $\bar{h}(\cdot)$ is upper semicontinuous and not equal to $+\infty$ at any point in $[x_0,x_2]$, it follows that $\bar{h}(\cdot)$ is bounded above on $[x_0,x_2]$ (Appendix D); hence, the integral in (4-10) has a well-defined value, although that value may be $-\infty$ (in which case, $\mathcal{N}$ is active). Analogous comments apply for the integral in (4-9): it has a well-defined value (since $x_1 \in \text{int } R^-(x_0)$), although that value may be $-\infty$.

The physical interpretation of the three conditions in Theorem 7 is straightforward and quite interesting. The first condition says that it is impossible to extract power from $\mathcal{N}$ while x stands still, i.e., while $\mathcal{N}$

remains in equilibrium. The second condition says that the maximum energy payoff per unit displacement of x to the left is less than or equal to the minimum energy cost per unit displacement to the right. This means that it is impossible to extract energy by driving the state around a closed path. The integral in (4-9) represents the minimum energy consumed while driving the system from x_0 to x_1, and the integral in (4-10) is the minimum energy consumed while driving from x_0 to x_2. If the system is to be passive, then it is clear that these quantitites must be bounded from below as x_1 and x_2 range over all states reachable from x_0. If (i) and (ii) are both satisfied, then the least function $W(\cdot)$ which satisfies (iii) is in fact the available energy $E_A(\cdot)$.

It is intuitively clear that each of the conditions of Theorem 7 is necessary for passivity. From the order properties of $\mathbb{R}$, it should also be intuitively clear that these conditions are sufficient (see Appendix D for more discussion). For a system with a 1-dimensional state space, there are evidently only three possible ways to extract an unbounded amount of energy beginning at some initial state x_0: sit still at x_0, drive x in a loop repeatedly, or move away from x_0 monotonically. If none of these strategies yields an unbounded amount of energy, then $E_A(x_0)$ is finite.

Let's reconsider Example 2. Condition (i) is always satisfied because $|i|^\alpha = 0 \Rightarrow i = 0 \Rightarrow r(x)i^2 = 0$. Since x can only move to the right, $\underline{h}(x) = -\infty$ everywhere and int $R^-(x_0) = \phi$ for all x_0; hence, condition (ii) is trivially satisfied and (4-9) always holds by default. If $\alpha = 2$, then $\bar{h}(x) = r(x)$ and (4-10) is satisfied by choosing for $W(\cdot)$ the constant function

$$\hat{W} \triangleq \int_0^1 -r(x)dx .$$

The interested reader might enjoy deriving $E_A(\cdot)$, the least function $W(\cdot)$ which would work in (4-10). If $\alpha \neq 2$, then $\bar{h}(x) = -\infty$ for all x in (0,1), (4-10) cannot be satisfied, and the system is active.

The conditions of Theorem 7 can be simplified if S is completely controllable (Def. 13).

Corollary for Theorem 7. Suppose that the n-port $\mathcal{N}$ described in Theorem 7 is completely controllable, with Σ an open interval in $\mathbb{R}$. Under these conditions, $\mathcal{N}$ is passive if and only if there exists a function $E : \Sigma \to \mathbb{R}^+$ and a measureable function $h : \Sigma \to \mathbb{R}$ which is bounded on every compact subset of Σ, such that

(i) $p(x,\underset{\sim}{u}) \geq f(x,\underset{\sim}{u}) \cdot h(x), \forall (x,\underset{\sim}{u}) \in \Sigma \times U$,

(ii) $\int_{x_0}^{x_1} h(x)dx + E(x_0) \geq 0, \forall x_0, x_1 \in \Sigma$.

The proof is given in Appendix D. Note that Σ may be an unbounded open interval, or it may be $\mathbb{R}$ itself; also, there is no assumption on the order of x_0 and x_1 in condition (ii). It may bother the reader that condition (ii) appears to be unsymmetrical in x_0 and x_1; however, condition (ii) is easily shown to be equivalent to the condition $E(x_1) \geq \int_{x_0}^{x_1} h(x)dx \geq -E(x_0), \forall x_0, x_1 \in \Sigma$.

In Theorem 7 we have answered the question of passivity for systems with a 1-dimensional state space in terms of properties of the functions $f(\cdot,\cdot)$ and $p(\cdot,\cdot)$ which can be verified by inspection. Ideally, we would like to have a similar result for multi-dimensional systems: a necessary and sufficient passivity condition in terms of easily verifiable properties of the functions $\underset{\sim}{f}(\cdot,\cdot)$ and $p(\cdot,\cdot)$ which does not make any prior assumption on the structure of these functions. The existence of such a result is a question for future research -- the proof of Theorem 7 (Appendix D) is critically dependent on the order properties of $\mathbb{R}$, therefore it cannot be generalized to multi-dimensional systems.

V. Representation Independence and Closure

5.1 Equivalent State Representations and Passivity

The definition of passivity in Section III is not based directly on the physical properties of an n-port $\mathcal{N}$, but rather on a certain function $E_A(\cdot)$ which depends on the particular state representation we have chosen for $\mathcal{N}$. The following example will help make this point clear.

Example 3. When viewed as a 1-port, a 1-farad capacitor is completely characterized by the relation $i = \frac{dv}{dt}$ or

$$v(t_2) = v(t_1) + \int_{t_1}^{t_2} i(t)dt, \forall t_2 \geq t_1.$$

Let's consider the following three state representations for such an element. In all three cases we let $\mathcal{U} = L^1_{loc}(\mathbb{R}^+ \to \mathbb{R})$.

S_1	S_2	S_3
$\dot{x}_1 = i$	$\dot{x}_2 = i$	$\dot{x}_3 = (\cos^2 x_3) i$
$v = x_1$	$v = x_2 - 1$	$v = \tan x_3$
$\Sigma_1 = \mathbb{R}$	$\Sigma_2 = \mathbb{R}$	$\Sigma_3 = (-\pi/2, \pi/2)$

To see that S_3 represents the same 1-port as S_1 and S_2, we calculate

$$\frac{dv}{dt} = \frac{dv}{dx_3}\frac{dx_3}{dt} \quad \frac{1}{\cos^2 x_3} \cos^2 x_3 \quad i = i.$$

Since S_1, S_2, and S_3 all represents a 1-farad capacitor, we certainly hope that $E_{A_1}(x_1)$, $E_{A_2}(x_2)$, and $E_{A_3}(x_3)$ will all be finite for each value of their arguments. Otherwise our definition of passivity will be dependent on the particular state representation we select. For S_1 and S_2 we have already calculated in Section III that $E_{A_1}(x_1) = x_1^2/2$ and $E_{A_2}(x_2) = (x_2-1)^2/2$, so $E_{A_1}(x_1) < +\infty$ for all $x_1 \in \Sigma_1$ and $E_{A_2}(x_2) < +\infty$ for all $x_2 \in \Sigma_2$. To calculate $E_{A_3}(x_3)$, note that if $v(0) = v_0$, then for any $T \geq 0$ and any input $i(\cdot) \in \mathcal{U}$, the energy extracted over the interval $[0,T]$ is

$$\int_0^T - v(t)i(t)dt = \int_0^T - v(t)\dot{v}(t)dt = \int_0^T - \frac{d}{dt}\left(\frac{v^2(t)}{2}\right) dt$$

$$= \frac{1}{2}(v_0^2 - v^2(T)) \leq \frac{1}{2} v_0^2 .$$

Therefore the available energy as a function of v_0 is $\frac{1}{2} v_0^2$. Writing it as a function of the initial state x_3 we have $E_{A_3}(x_3) = \frac{1}{2}(\tan x_3)^2$, which is finite for each $x_3 \in \Sigma_3$.

So at least in this example, the classification of the n-port as active or passive does not depend on which of the state representations we choose. We now want to show that this result holds in general, i.e., that with the proper definition of "equivalence," equivalent state representations of an n-port will always yield the same classification.

<u>Definition 19</u>. Two state representations of an n-port $\mathcal{N}$, S_1 and S_2, are defined to be <u>equivalent</u> if there exists a bijective map $\underset{\sim}{b} : \Sigma_1 \to \Sigma_2$ such that for each $\underset{\sim}{x} \in \Sigma_1$, the class of admissible pairs $\{\underset{\sim}{v}(\cdot), \underset{\sim}{i}(\cdot)\}$ of S_1 with initial state $\underset{\sim}{x}$

is identical to the class of admissible pairs $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}$ of S_2 with initial state $\underset{\sim}{b}(\underset{\sim}{x})$.

In Example 3, S_1, S_2, and S_3 are all equivalent state representations. For S_2 and S_1 the bijection $b_{21} : \Sigma_2 \to \Sigma_1$ is just $x_1 = b_{21}(x_2) = x_2 - 1$, and for S_3 and S_1 the bijection $b_{31} : \Sigma_3 \to \Sigma_1$ is $x_1 = b_{31}(x_3) = \tan x_3$. This definition of equivalence is stated in terms of admissible pairs $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}$ rather than input-output pairs $\{\underset{\sim}{u}(\cdot),\underset{\sim}{y}(\cdot)\}$, since passivity is defined in terms of admissible pairs (Defs. 10 and 11). It leaves open the possibility that the inputs may be chosen differently for S_1 and S_2; for example, S_1 might be voltage-controlled while S_2 is current-controlled. Our definition of equivalent state representations is very similar to the definition of equivalent <u>networks</u> given by Sangiovanni-Vincentelli and Wang [19]; however, our definition is more restrictive because it requires the existence of the bijection $\underset{\sim}{b}(\cdot)$. If the inputs for S_1 and S_2 are the same and if $\underset{\sim}{u}$ and $\underset{\sim}{y}$ form a hybrid pair (Def. 3), then it follows from Def. 19 and our standing assumption (2) that the input-output behavior of S_1 in any initial state $\underset{\sim}{x}_0$ is identical to the input-output behavior of S_2 in initial state $\underset{\sim}{b}(\underset{\sim}{x}_0)$; and under these conditions our definition of equivalence is somewhat more restrictive than that given by Desoer [17] and somewhat less restrictive than that given by Varaiya and Verma [20].

<u>Example 4</u>. Consider the two state representations whose state and output equations are given below:

$$\begin{array}{ll} \underline{S_1} & \underline{S_2} \\ \dot{\underset{\sim}{x}}_1 = \underset{\sim}{f}(\underset{\sim}{x}_1,\underset{\sim}{u}) & \dot{\underset{\sim}{x}}_2 = \hat{\underset{\sim}{f}}(\underset{\sim}{x}_2,\underset{\sim}{u}) \triangleq \underset{\sim}{f}(\underset{\sim}{x}_2+\underset{\sim}{c},\underset{\sim}{u}) \\ \underset{\sim}{y} = \underset{\sim}{g}(\underset{\sim}{x}_1,\underset{\sim}{u}) & \underset{\sim}{y} = \hat{\underset{\sim}{g}}(\underset{\sim}{x}_2,\underset{\sim}{u}) \triangleq \underset{\sim}{g}(\underset{\sim}{x}_2+\underset{\sim}{c},\underset{\sim}{u}) \end{array}$$

where $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are a hybrid pair (Def. 3) and $\underset{\sim}{c} \in \mathbb{R}^m$ is fixed. The functions $\hat{\underset{\sim}{f}}(\cdot,\cdot)$ and $\hat{\underset{\sim}{g}}(\cdot,\cdot)$ are obtained by translating $\underset{\sim}{f}(\cdot,\cdot)$ and $\underset{\sim}{g}(\cdot,\cdot)$, respectively, in the first variable. We assume that $U_1 = U_2$, $\mathcal{U}_1 = \mathcal{U}_2$, and $\Sigma_1 = \Sigma_2 + \underset{\sim}{c}$. It is clear that $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}$ is an admissible pair for S_1 with initial state $\underset{\sim}{x}_1'$ if and only if $\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\}$ is an admissible pair for S_2 with initial state $\underset{\sim}{x}_2' = \underset{\sim}{x}_1' - \underset{\sim}{c}$. It follows that S_1 and S_2 are equivalent by Def. 19, and the bijection $\underset{\sim}{b} : \Sigma_1 \to \Sigma_2$ is given by $\underset{\sim}{b}(\underset{\sim}{x}_1) = \underset{\sim}{x}_1 - \underset{\sim}{c}$.

Both state representations in Example 4 give rise to the same port behavior, so they are equally valid state models for the same n-port. In constructing a state model for an n-port, Example 4 shows that the identity of the state vector is determined only to within an additive constant. A special case of this result for charges and fluxes was mentioned in subsection 3.2. If we are given a 1-port capacitive constitutive relation $v = v_1(q)$, then for any $c \in \mathbb{R}$ the constitutive relation $v = v_2(q) \triangleq v_1(q+c)$ gives exactly the same port behavior; in other words, we can arbitrarily translate capacitive and inductive constitutive relations. This constitutes a formal basis for our argument in subsection 3.2 that the capacitors $v = q$ and $v = q-1$ (Fig. 3) are indistinguishable as 1-ports. It might seem "unnatural" to translate a constitutive relation as done in Fig. 3, and it might seem that this ambiguity could be eliminated by always requiring the constitutive relation $v = \hat{v}(q)$ to satisfy $\hat{v}(0) = 0$. The constitutive relations $v_1(q) = q(q-1)(q-2)$, $v_2(q) = e^q$, and $i_3(\phi) = \sin \phi$ show that such an approach will not work for a general nonlinear theory. In the first case, there are three points where $v_1(q) = 0$; in the second case, there are no such points; in the third case $i_3(k\pi) = 0$ for all integers k, worse yet, we can translate the constitutive relation by $2\pi k$ without changing its form in any way.

Lemma 2. Given an n-port $\mathcal{N}$ with two equivalent state representations S_1 and S_2, let $\underset{\sim}{b} : \Sigma_1 \to \Sigma_2$ be the bijection defined in Def. 19, and let $E_{A_1}(\cdot)$ and $E_{A_2}(\cdot)$ be the available energy functions for S_1 and S_2, respectively. Then $E_{A_1}(\underset{\sim}{x}) = E_{A_2}(\underset{\sim}{b}(\underset{\sim}{x}))$, $\forall \underset{\sim}{x} \in \Sigma_1$.

The proof is immediate from Defs. 10 and 19.

Theorem 8. Suppose an n-port $\mathcal{N}$ has two equivalent state representations S_1 and S_2. Then Def. 11 applied to S_1 classifies $\mathcal{N}$ as passive $\Leftrightarrow$ it classifies $\mathcal{N}$ as passive when applied to S_2.

Proof. By Lemma 2, for any $\underset{\sim}{x} \in \Sigma_1$ we have $E_{A_1}(\underset{\sim}{x}) < +\infty \Leftrightarrow E_{A_2}(\underset{\sim}{b}(\underset{\sim}{x})) < +\infty$. Since $\underset{\sim}{b}(\cdot)$ is a bijection, this concludes the proof. Q.E.D.

There is an immediate extension of Theorem 8 which shows that strong passivity is also preserved under the change from a given state representation to an equivalent one.

5.2 Distinct N-Ports Made from a Multiterminal Element

So far the discussion in this paper has been directed toward the classification of n-ports, but it is important to realize that in engineering practice one can obtain many distinct n-ports from a given multiterminal element. For example, different n-ports can be made from a given (n+1)-terminal element by choosing different datum nodes (Fig. 8); also, changing the orientation of a given n-port (i.e., reversing the roles of inputs and outputs) can produce a different n-port by modifying the state representation. It is reasonable to expect that the distinct n-ports derived from a given multiterminal element will all be passive or else all active, i.e., that the classification will be invariant under a change of excitation and observation modes. It turns out that a limited result of this sort holds, but it depends quite heavily on what happens to the state representation when the modes of excitation and observation are changed. A −1 farad capacitor with the input space restricted to the constant functions will exhibit active behavior when excited with current sources, but when excited with voltage sources it will exhibit passive behavior since $\dot{v} \equiv 0$. Note that the difficulty arises because of the lack of a state representation in the latter case. Difficulties can also arise because certain changes of excitation and observation modes may not be permissible. Consider the 2-ports shown in Fig. 2 and Fig. 5(c). In both cases the port currents can be taken as the input variables, but an arbitrary voltage waveform cannot be applied at port 2.

We begin by defining the class of allowable alterations that can be performed on the excitation and observation modes of an n-port. Note that the input-output pairs are always assumed to be <u>hybrid</u> pairs (Def. 3).

<u>Definition 20</u>. Let $(\underset{\sim}{u},\underset{\sim}{y})$ be a hybrid input-output pair for a given n-port, and let $(\hat{\underset{\sim}{u}},\hat{\underset{\sim}{y}})$ denote a new hybrid input-output pair obtained from $(\underset{\sim}{u},\underset{\sim}{y})$ by either

$$\begin{bmatrix} \hat{\underset{\sim}{u}}_1 \\ \hline \hat{\underset{\sim}{u}}_2 \\ \hline \hat{\underset{\sim}{y}}_1 \\ \hline \hat{\underset{\sim}{y}}_2 \end{bmatrix} = \left[\begin{array}{c|c|c|c} \underset{\sim}{P}_1 & \underset{\sim}{P}_2 & \underset{\sim}{0} & \underset{\sim}{0} \\ \hline \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{Q}_3 & \underset{\sim}{Q}_4 \\ \hline \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{Q}_1 & \underset{\sim}{Q}_2 \\ \hline \underset{\sim}{P}_3 & \underset{\sim}{P}_4 & \underset{\sim}{0} & \underset{\sim}{0} \end{array}\right] \begin{bmatrix} \underset{\sim}{u}_1 \\ \hline \underset{\sim}{u}_2 \\ \hline \underset{\sim}{y}_1 \\ \hline \underset{\sim}{y}_2 \end{bmatrix} \qquad (5\text{-}1)$$

or

$$\begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \left[\begin{array}{c|c|c|c} P_1 & 0 & 0 & P_2 \\ \hline P_3 & 0 & 0 & P_4 \\ \hline 0 & Q_2 & Q_1 & 0 \\ \hline 0 & Q_4 & Q_3 & 0 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \\ y_1 \\ y_2 \end{bmatrix} \tag{5-2}$$

where

$$\left[\begin{array}{c|c} P_1 & P_2 \\ \hline P_3 & P_4 \end{array}\right]^T = \left[\begin{array}{c|c} Q_1 & Q_2 \\ \hline Q_3 & Q_4 \end{array}\right]^{-1} \tag{5-3}$$

and where $u^T = (u_1^T, u_2^T)$, $y^T = (y_1^T, y_2^T)$, $\hat{u}^T = (\hat{u}_1^T, \hat{u}_2^T)$, and $\hat{y}^T = (\hat{y}_1^T, \hat{y}_2^T)$. The transformation in (5-1) will be called an Excitation-Observation Mode Transformation of Type 1 (EOMT1), and the one in (5-2) will be called an EOMT2.

EOMT1 takes a linear combination of former inputs to create some of the new inputs, and a linear combination of former outputs to create the remaining new inputs, but never mixes former inputs with former outputs; similarly for the new outputs. EOMT2 mixes inputs and outputs, but never from the same port.

The set consisting of EOMT1 and EOMT2 is closed under matrix inversion; indeed, (5-1) and (5-3) give

$$\begin{bmatrix} u_1 \\ u_2 \\ y_1 \\ y_2 \end{bmatrix} = \left[\begin{array}{c|c|c|c} Q_1^T & 0 & 0 & Q_3^T \\ \hline Q_2^T & 0 & 0 & Q_4^T \\ \hline 0 & P_3^T & P_1^T & 0 \\ \hline 0 & P_4^T & P_2^T & 0 \end{array}\right] \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} \tag{5-4}$$

therefore the inverse of an EOMT1 is an EOMT2. Similarly, (5-2) and (5-3) give

$$\begin{bmatrix} u_1 \\ u_2 \\ y_1 \\ y_2 \end{bmatrix} = \left[\begin{array}{c|c|c|c} Q_1^T & Q_3^T & 0 & 0 \\ \hline 0 & 0 & P_2^T & P_4^T \\ \hline 0 & 0 & P_1^T & P_3^T \\ \hline Q_2^T & Q_4^T & 0 & 0 \end{array}\right] \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} \tag{5-5}$$

and so the inverse of an EOMT2 is an EOMT1.

Note that the set of EOMT's is <u>not</u> closed under the composition (matrix product) operation. This may seem inconvenient, but it can be useful for showing that passivity is preserved under more general excitation-observation mode transformations (see the comments following Corollary C for Theorem 9).

<u>Example 5</u>. The two n-ports in Fig. 8, $\mathcal{N}_1$ and $\mathcal{N}_2$, were obtained from the same (n+1)-terminal element by selecting different datum nodes. We shall call this a "datum-node transformation." For definiteness, we shall assume that both $\mathcal{N}_1$ and $\mathcal{N}_2$ are current-controlled. It is straightforward to verify the following relation:

$$\begin{bmatrix} \hat{\mathbf{i}} \\ \hline \hat{\mathbf{v}} \end{bmatrix} = \left[\begin{array}{c|c} (\mathbf{B}^{-1})^T & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{B} \end{array}\right] \begin{bmatrix} \mathbf{i} \\ \hline \mathbf{v} \end{bmatrix}$$

where

$$\mathbf{B} \triangleq \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & 1 \\ -1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

This shows that the datum-node transformation is a special case of either EOMT1 or EOMT2, with $\mathbf{Q}_1 = \mathbf{B} \in \mathbb{R}^{n\times n}$ and $\mathbf{P}_1^T = \mathbf{B}^{-1} \in \mathbb{R}^{n\times n}$.

Two classes of EOMT's which occur frequently in practice are

(i) when $\mathbf{P}_1 \in \mathbb{R}^{n\times n}$ and nonsingular, i.e., only linear combinations of former inputs are taken as new inputs. It follows that $\mathbf{Q}_1 \in \mathbb{R}^{n\times n}$ with $\mathbf{P}_1^{-1} = \mathbf{Q}_1^T$. This EOMT can be considered as either an EOMT1 or an EOMT2, and it will be called a "generalized datum-node transformation" because it generalizes the datum-node transformation described in Example 5.

(ii) when $\mathbf{P}_1 = \mathbf{I}$, $\mathbf{Q}_1 = \mathbf{I}$, $\mathbf{P}_4 = \mathbf{I}$, $\mathbf{Q}_4 = \mathbf{I}$, and all other submatrices are zero. In this case, a new orientation of the n-port is obtained with respect to some ports, namely, those described by the pair $(\mathbf{u}_2, \mathbf{y}_2)$. When $\mathbf{P}_4 = \mathbf{Q}_4 = \mathbf{I} \in \mathbb{R}^{n\times n}$, a complete reorientation of the n-port is achieved.

The EOMT described in item (ii) above can destroy the state representation of an n-port, as we saw in the case of the -1 farad capacitor and the 2-ports in Fig. 2 and Fig. 5(c). This fact necessitates the following definition.

<u>Definition 21</u>. Two n-ports, $\mathcal{N}_1$ with state representation S_1 and $\mathcal{N}_2$ with state representation S_2, are said to be <u>EOMT equivalent</u> if

(i) $\mathcal{N}_2$ is obtained from $\mathcal{N}_1$ by either an EOMT1 or an EOMT2,

(ii) there exists a bijection $\mathbf{b}(\cdot)$ which maps Σ_1 of S_1 onto Σ_2 of S_2,

(iii) $(\mathbf{u},\mathbf{y})$ is an input-output pair for $\mathcal{N}_1$ with initial state $\mathbf{x}_0 \Leftrightarrow (\hat{\mathbf{u}},\hat{\mathbf{y}})$ is an input-output pair for $\mathcal{N}_2$ with initial state $\mathbf{b}(\mathbf{x}_0)$, where $(\mathbf{u},\mathbf{y})$ is related to $(\hat{\mathbf{u}},\hat{\mathbf{y}})$ by the appropriate EOMT.

Note that Def. 21 reduces to Def. 19 when $\mathbf{P}_1 = \mathbf{Q}_1 = \mathbf{I} \in \mathbb{R}^{n\times n}$

<u>Theorem 9</u>. Suppose that $\mathcal{N}_1$ and $\mathcal{N}_2$ are EOMT equivalent. Under these conditions, $\mathcal{N}_1$ is (strongly) passive $\Leftrightarrow \mathcal{N}_2$ is (strongly) passive.

<u>Proof</u>. Since the inverse of an EOMT1 is an EOMT2 (and vice versa), we will have the proof in both directions if we have it in the same direction for EOMT1 and EOMT2.

For EOMT1 we have

$$\langle \hat{\mathbf{u}}(t),\hat{\mathbf{y}}(t)\rangle = \hat{\mathbf{u}}^T(t)\hat{\mathbf{y}}(t) = [\hat{\mathbf{u}}_1^T(t) \,\vdots\, \hat{\mathbf{y}}_2^T(t)] \begin{bmatrix} \hat{\mathbf{y}}_1(t) \\ \hline \hat{\mathbf{u}}_2(t) \end{bmatrix}$$

$$= [\mathbf{u}_1^T(t) \,\vdots\, \mathbf{u}_2^T(t)] \left[\begin{array}{c|c} \mathbf{P}_1 & \mathbf{P}_2 \\ \hline \mathbf{P}_3 & \mathbf{P}_4 \end{array}\right]^T \left[\begin{array}{c|c} \mathbf{Q}_1 & \mathbf{Q}_2 \\ \hline \mathbf{Q}_3 & \mathbf{Q}_3 \end{array}\right] \begin{bmatrix} \mathbf{y}_1(t) \\ \hline \mathbf{y}_2(t) \end{bmatrix}$$

$$= \mathbf{u}^T(t)\mathbf{y}(t) = \langle \mathbf{u}(t),\mathbf{y}(t)\rangle, \text{ for all } t \geq 0. \tag{5-6}$$

Similarly, for EOMT2 we have

$$\langle \hat{\mathbf{u}}(t),\hat{\mathbf{y}}(t)\rangle = [\hat{\mathbf{u}}_1^T(t) \,\vdots\, \hat{\mathbf{u}}_2^T(t)] \begin{bmatrix} \hat{\mathbf{y}}_1(t) \\ \hline \hat{\mathbf{y}}_2(t) \end{bmatrix}$$

$$= [\mathbf{u}_1^T(t) \,\vdots\, \mathbf{y}_2^T(t)] \left[\begin{array}{c|c} \mathbf{P}_1 & \mathbf{P}_2 \\ \hline \mathbf{P}_3 & \mathbf{P}_4 \end{array}\right]^T \left[\begin{array}{c|c} \mathbf{Q}_1 & \mathbf{Q}_2 \\ \hline \mathbf{Q}_3 & \mathbf{Q}_4 \end{array}\right] \begin{bmatrix} \mathbf{y}_1(t) \\ \hline \mathbf{u}_2(t) \end{bmatrix}$$

$$= \langle \mathbf{u}(t),\mathbf{y}(t)\rangle, \text{ for all } t \geq 0. \tag{5-7}$$

If $E_{A_1}(\cdot)$ and $E_{A_2}(\cdot)$ are the available energy functions associated with the state representations of $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively, then it follows from Def. 21, (5-6) and (5-7) that

$$E_{A_2}(\underset{\sim}{b}(\underset{\sim}{x}_0)) = \sup_{\substack{\underset{\sim}{b}(\underset{\sim}{x}_0)\to \\ T\geq 0}} \left\{ -\int_0^T \langle \hat{\underset{\sim}{u}}(t), \hat{\underset{\sim}{y}}(t)\rangle\, dt \right\}$$

$$= \sup_{\substack{\underset{\sim}{x}_0\to \\ T\geq 0}} \left\{ -\int_0^T \langle \underset{\sim}{u}(t), \underset{\sim}{y}(t)\rangle\, dt \right\} = E_{A_1}(\underset{\sim}{x}_0). \qquad \text{Q.E.D.}$$

Corollary A for Theorem 9. Suppose that the n-port $\mathcal{N}_2$ is a new orientation of the n-port $\mathcal{N}_1$ (partial or complete) and that $\mathcal{N}_1$ is EOMT equivalent to $\mathcal{N}_2$. Under these conditions, $\mathcal{N}_1$ is (strongly) passive $\Leftrightarrow \mathcal{N}_2$ is (strongly) passive.

Proof. This is the special case mentioned in item (ii) of the paragraph following Example 5; therefore, it follows directly from Theorem 9. Q.E.D.

Corollary B for Theorem 9. Suppose that the n-port $\mathcal{N}_2$ is obtained from the n-port $\mathcal{N}_1$ through a generalized datum-node transformation, i.e., $\underset{\sim}{P}_1 \in \mathbb{R}^{n\times n}$ and $\underset{\sim}{P}_1^{-1} = \underset{\sim}{Q}_1^T$. Under these conditions, $\mathcal{N}_1$ is (strongly) passive $\Leftrightarrow \mathcal{N}_2$ is (strongly) passive.

Proof. In view of Theorem 9, it is only necessary to show that $\mathcal{N}_1$ and $\mathcal{N}_2$ are EOMT equivalent. Let S_1 denote the state representation of $\mathcal{N}_1$, with state the output equations

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{u})$$
$$\underset{\sim}{y} = \underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{u}).$$

Let S_2 denote the state representation of $\mathcal{N}_2$. The státe and output equations of S_2 are then

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{Q}_1^T\hat{\underset{\sim}{u}}) \triangleq \hat{\underset{\sim}{f}}(\underset{\sim}{x},\hat{\underset{\sim}{u}})$$
$$\hat{\underset{\sim}{y}} = \underset{\sim}{Q}_1\underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{Q}_1^T\hat{\underset{\sim}{u}}) \triangleq \hat{\underset{\sim}{g}}(\underset{\sim}{x},\hat{\underset{\sim}{u}})$$

Let $\mathcal{U}_1$ denote the set of admissible inputs for $\mathcal{N}_1$; then since $\underset{\sim}{Q}_1$ is nonsingular, the set of admissible inputs $\mathcal{U}_2$ for $\mathcal{N}_2$ can be defined as follows: $\hat{\underset{\sim}{u}}(\cdot) \in \mathcal{U}_2 \Leftrightarrow \underset{\sim}{Q}_1^T\hat{\underset{\sim}{u}}(\cdot) \in \mathcal{U}_1$. It follows that $\mathcal{N}_1$ and $\mathcal{N}_2$ are EOMT equivalent, with the bijection $\underset{\sim}{b} : \Sigma_1 \to \Sigma_2$ being the identity map. Q.E.D.

Corollary C for Theorem 9. Let the n-port $\mathcal{N}_k$ with input-output pair $(\hat{\underset{\sim}{u}},\hat{\underset{\sim}{y}})$ be obtained from the n-port $\mathcal{N}$ with input-output pair $(\underset{\sim}{u},\underset{\sim}{y})$ through

$$\begin{bmatrix} \hat{\underset{\sim}{u}} \\ --- \\ \hat{\underset{\sim}{y}} \end{bmatrix} = \underset{\sim}{\mathcal{P}} \begin{bmatrix} \underset{\sim}{u} \\ --- \\ \underset{\sim}{y} \end{bmatrix}$$

where the matrix $\underset{\sim}{\mathcal{P}} = \prod_{i=1}^{k} \underset{\sim}{\mathcal{P}}_i$, each $\underset{\sim}{\mathcal{P}}_i$ being either an EOMT1 or an EOMT2. Suppose that for each i, the corresponding EOMT produces EOMT equivalent n-ports. Under these conditions, $\mathcal{N}_k$ is (strongly) passive $\Leftrightarrow \mathcal{N}$ is (strongly) passive.

Proof. Apply Theorem 9 k times. Q.E.D.

Using Corollary C, it may be possible to prove that passivity is preserved under excitation-observation mode transformations $\underset{\sim}{\mathcal{P}}$ which do not have the form of EOMT1 or EOMT2. If it is possible to write $\underset{\sim}{\mathcal{P}}$ in the form $\underset{\sim}{\mathcal{P}} = \prod_{i=1}^{k} \underset{\sim}{\mathcal{P}}_i$, and if each $\underset{\sim}{\mathcal{P}}_i$ produces EOMT equivalent n-ports, then passivity is preserved under $\underset{\sim}{\mathcal{P}}$.

5.3 Interconnections of Passive N-Ports

Definition 22. We say that an attribute of n-ports has the property of closure if it is preserved under finite interconnections, i.e., if whenever $\mathcal{N}_1,\dots,\mathcal{N}_k$ have the attribute and $\mathcal{N}$ is obtained by interconnecting $\mathcal{N}_1,\dots,\mathcal{N}_k$, then $\mathcal{N}$ must have the attribute as well.

Linearity and time-invariance, for example, possess closure. Observability and controllability do not. Does passivity have the closure property? In other words, will a finite interconnection of passive n-ports always be passive? We would certainly expect that for a reasonable definition of passivity the answer will be yes.

The purpose of this subsection is to show that passivity as defined in this paper does have the closure property, at least under certain assumptions. We let $\mathcal{N}_1,\dots,\mathcal{N}_k$ have state representations $S_1,\dots,S_k$ with state spaces $\Sigma_1,\dots,\Sigma_k$. Since we want to keep the discussion here relatively informal, we will consider only the simplest case in which $\mathcal{N}$ has a state representation S which satisfies our assumptions in Section II and has the state space $\Sigma = \Sigma_1 \times\dots\times \Sigma_k$, the Cartesian product of the individual state spaces. We will call such an interconnection admissible. Figure 9 is intended to convey this idea. The interconnection of n-ports has been studied at great length by Ikeda and Kodama [21], and we refer the reader to their paper for more details. It should be emphasized that we are discussing only the non-pathological cases here, since there are at least five ways an interconnection can fail to be admissible. These are given in Appendix E. The reason for considering only these well-behaved interconnections is that the following lemma and theorem can be proved without invoking any elaborate technical machinery.

Lemma 3. Let $\mathcal{N}$ with state representation S be an admissible interconnection of $\mathcal{N}_1,\ldots,\mathcal{N}_k$ as defined above. Let $E_{A_j} : \Sigma_j \to \mathbb{R}^+$ be the available energy for $\mathcal{N}_j$, $1 \leq j \leq k$, and $E_A : \Sigma = \Sigma_1 \times\ldots\times \Sigma_k \to \mathbb{R}^+$ be the available energy for $\mathcal{N}$. Then if $\underset{\sim}{x} = (\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_k) \in \Sigma$, we have $E_A(\underset{\sim}{x}) \leq E_{A_1}(\underset{\sim}{x}_1) +\ldots+ E_{A_k}(\underset{\sim}{x}_k)$.

Proof. By Tellegen's theorem the power leaving the ports of $\mathcal{N}$ at any instant is the sum of the power leaving the ports of $\mathcal{N}_1,\ldots,\mathcal{N}_k$, and when $\mathcal{N}_j$ has the initial state $\underset{\sim}{x}_j$, the total energy leaving its ports is bounded above by $E_{A_j}(\underset{\sim}{x}_j)$.

Q.E.D.

Theorem 10. Let $\mathcal{N}$ with state representation S be an admissible interconnection of $\mathcal{N}_1,\ldots,\mathcal{N}_k$.

(1) If $\mathcal{N}_1,\ldots,\mathcal{N}_k$ are passive, then $\mathcal{N}$ is passive.

(2) If $\mathcal{N}_1,\ldots,\mathcal{N}_k$ are strongly passive, then $\mathcal{N}$ is strongly passive.

Proof. Statement (1) follows immediately from Lemma 3. If $\mathcal{N}_1,\ldots,\mathcal{N}_k$ are strongly passive, then there exist states $\underset{\sim}{x}_j^* \in \Sigma_j$ with $E_{A_j}(\underset{\sim}{x}_j^*) = 0$. Since $\underset{\sim}{x}^* = (\underset{\sim}{x}_1^*,\ldots,\underset{\sim}{x}_k^*)$ is in Σ, we have from Lemma 3 that $E_A(\underset{\sim}{x}^*) = 0$ and statement (2) follows.

Q.E.D.

VI. Internal Energy Functions and Passive N-Ports

Most of the results in this section have been stated and proved by Willems [1]. Our purpose here is to make his work readily accessible to circuit theorists by translating it into a more appropriate language and illustrating it with a simple network example.

Definition 23. Given an n-port $\mathcal{N}$ with a state representation S, we say that a function $E_I : \Sigma \to \mathbb{R}^+$ is an internal energy function for $\mathcal{N}$ if

$$E_I\left(\underset{\sim}{x}(t_2)\right) - E_I\left(\underset{\sim}{x}(t_1)\right) \leq \int_{t_1}^{t_2} p(\underset{\sim}{x}(t),\underset{\sim}{u}(t))dt \tag{6-1}$$

for all input-trajectory pairs $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}$ (Def. 8) and all $0 \leq t_1 \leq t_2$, where $p(\cdot,\cdot)$ is the power input function (Def. 2).

In other words, an internal energy function is just a nonnegative[9] function on Σ which increases along trajectories more slowly than the rate at which energy

[9] We could equally well require only that E_I be bounded from below. The only advantage in requiring that E_I be nonnegative is that it allows us to write inequalities such as $E_A \leq E_I \leq E_{Rx^*}$ in a convenient way (see Theorem 12). Note that E_I must be finite-valued (see footnote 1).

is delivered to the ports (or decreases more rapidly than the rate at which energy is extracted from the ports). We shall show shortly (Theorem 11) that under mild technical assumptions an n-port $\mathcal{N}$ is passive if and only if there exists an internal energy function for $\mathcal{N}$. It is clear from Def. 23 that if $E_I(\cdot)$ attains its lower bound at some point $x^* \in \Sigma$, then the n-port $\mathcal{N}$ is strongly passive with relaxed state x^*.

The following lemma gives an obvious sufficient condition for the existence of a C^1 internal energy function.

Lemma 4. Given an n-port $\mathcal{N}$ with state representation S. Suppose $\Sigma \subset \mathbb{R}^m$ is open[10] and that $\psi : \Sigma \to \mathbb{R}$ is C^1, bounded below, and satisfies

$$\langle \nabla\psi(x), f(x,u)\rangle \leq p(x,u)$$

for all $(x,u) \in \Sigma \times U$; then $\psi(\cdot)-m$ is an internal energy function for $\mathcal{N}$, where m is the greatest lower bound of $\psi(\cdot)$.

Proof. For any input-trajectory pair $\{u(\cdot),x(\cdot)\}$ and any times $t_2 \geq t_1 \geq 0$, we have

$$\psi(x(t_2)) - \psi(x(t_1)) = \int_{t_1}^{t_2} \frac{d}{dt}\,\psi(x(t))dt$$

$$= \int_{t_1}^{t_2} \langle \nabla\psi(x(t)), f(x(t),u(t))\rangle\, dt$$

$$\leq \int_{t_1}^{t_2} p(x(t),u(t))dt.$$

Q.E.D.

The next lemma gives a necessary condition that a C^1 internal energy function must satisfy.

Lemma 5. Given an n-port $\mathcal{N}$ with state representation S. Suppose that $\Sigma \subset \mathbb{R}^m$ is open, that $\mathcal{U}$ contains all piecewise constant functions mapping $\mathbb{R}^+$ to U, and that $\psi : \Sigma \to \mathbb{R}^+$ is a C^1 internal energy function for $\mathcal{N}$; then

$$\langle \nabla\psi(x), f(x,u)\rangle \leq p(x,u)$$

for all $(x,u) \in \Sigma \times U$.

[10] The assumption in Lemmas 4 and 5 that Σ is open is needed so that $\nabla\psi(x)$ can be defined at each $x \in \Sigma$. More generally, we can assume that $\psi(\cdot)$ has a C^1 extension to an open set containing Σ.

Proof. Let $(\underset{\sim}{x}_0,\underset{\sim}{u}_0)$ be an arbitrary point in $\Sigma \times U$ and let $\{\underset{\sim}{u}_0,\underset{\sim}{x}(\cdot)\}$ be an input-trajectory pair with initial state $\underset{\sim}{x}_0$ (i.e., $\underset{\sim}{x}(0) = x_0$ and the input is the constant function $\underset{\sim}{u}(t) \equiv \underset{\sim}{u}_0$). It follows from Def. 23 that

$$\langle \nabla\psi(\underset{\sim}{x}_0),\underset{\sim}{f}(\underset{\sim}{x}_0,\underset{\sim}{u}_0)\rangle = \frac{d\psi(\underset{\sim}{x}(t))}{dt}\bigg|_{t=0} = \lim_{t\to 0^+} \frac{\psi(\underset{\sim}{x}(t))-\psi(\underset{\sim}{x}_0)}{t}$$

$$\leq \lim_{t\to 0^+} \frac{1}{t}\int_0^t p(\underset{\sim}{x}(\tau),\underset{\sim}{u}_0)d\tau = p(\underset{\sim}{x}_0,\underset{\sim}{u}_0).$$

The last step is simply an application of the Fundamental Theorem of Calculus and is justified because the integrand is continuous. Q.E.D.

Under the mild technical assumptions that $\Sigma \subset \mathbb{R}^m$ is open and that $\mathcal{U}$ contains all piecewise constant functions mapping $\mathbb{R}^+$ to U, it follows from Lemmas 4 and 5 that a C^1 function $\psi : \Sigma \to \mathbb{R}^+$ is an internal energy function if and only if

$$\langle \nabla\psi(\underset{\sim}{x}),\underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{u})\rangle \leq p(\underset{\sim}{x},\underset{\sim}{u}) \tag{6-2}$$

for all $(\underset{\sim}{x},\underset{\sim}{u}) \in \Sigma \times U$. Note that we have made no claim about the existence or uniqueness of such a function in general. In the case of an n-port made by interconnecting linear passive resistors, capacitors, and inductors, the total electrical energy in the storage elements would be one example of a C^1 internal energy function. But as the following example will show, there frequently exist other choices as well.

Example 6. Consider the linear 1-port shown in Fig. 10. If we choose the voltage as input, its state and output equations are

$$\dot{q} = v - (G+1)q$$

$$i = v - q$$

and we suppose that $\mathcal{U} = L^2_{loc}(\mathbb{R}^+ \to \mathbb{R})$.

Let's calculate all the possible internal energy functions of the form $\psi(q) = \alpha(q^2/2)$. Inequality (6-2) becomes in this case $\alpha q[v-(G+1)q] \leq v(v-q)$, or $v^2 - [(\alpha+1)q]v + \alpha(G+1)q^2 \geq 0$, $\forall v,q \in \mathbb{R}$. It is simple to verify that this inequality holds if and only if $(\alpha+1)^2 - 4(G+1)\alpha \leq 0$; hence, $\alpha \cdot (q^2/2)$ is an internal energy function if and only if

$$\alpha \in \left[(2G+1) - 2\sqrt{G(G+1)},\ (2G+1) + 2\sqrt{G(G+1)}\right]. \tag{6-3}$$

There are two features of this result worth noticing. The first is that there is a range of possible values of α, and hence of internal energy functions, for all $G > 0$. The second is that $\alpha = 1$ lies in this interval for any value of $G \geq 0$, as a simple calculation will verify. Therefore the electrical energy $q^2/2$ is always a valid internal energy function.

The above results can also be derived using the general theory for linear systems developed by Willems [1]. The algebraic Riccati equation for the system in Example 6 is $(\alpha+1)^2 - 4(G+1)\alpha = 0$. This scalar equation has two solutions, $\alpha^- \triangleq 2G + 1 - 2\sqrt{G(G+1)}$ and $\alpha^+ \triangleq 2G + 1 + 2\sqrt{G(G+1)}$. Willems' [1] theory shows that $\alpha(q^2/2)$ is an internal energy function if and only if $\alpha^- \leq \alpha \leq \alpha^+$, and this agrees with (6-3). But Willems' [1] theory goes beyond this: he shows how to obtain the available energy function from the solutions to the algebraic Riccati equation. When applied to Example 6, Willems' [1] theory shows that $E_A(q) = \alpha^-(q^2/2)$, that is

$$E_A(q) = \left[(2G+1) - 2\sqrt{G(G+1)}\right] \cdot (q^2/2). \tag{6-4}$$

This function is plotted in Fig. 11 for three different values of G. When $G = 0$, i.e., when the shunt resistor is an open circuit, we can extract energy from the capacitor with arbitrarily small losses by letting i be very small and the discharge time very long. Therefore $E_A(q) = q^2/2$ when $G = 0$, in agreement with (6-4).

Lemma 6. Let $\mathcal{N}$ be a passive n-port with a state representation S, and suppose that $\mathcal{U}$ is translation invariant and closed under concatenation. Under these conditions, the available energy $E_A(\cdot)$ is an internal energy function for $\mathcal{N}$.

Willems [1] has given the outline of a proof. Since it glosses over the necessity of some assumptions on $\mathcal{U}$, we have included a complete and explicit proof of Lemma 6 in Appendix F.

The following theorem shows why internal energy functions are of such importance in the study of passivity.

Theorem 11. Let $\mathcal{N}$ be an n-port with a state representation S, and suppose that $\mathcal{U}$ is translation invariant and closed under concatenation. Under these conditions, $\mathcal{N}$ is passive $\Leftrightarrow$ there exists an internal energy function $E_I(\cdot)$ defined on Σ.

Proof

(⇒) Lemma 6.

(⇐) After rearranging (6-1) and choosing $t_1 = 0$, $t_2 = T$, we have

$$E_I\big(\underset{\sim}{x}(0)\big) + \int_0^T p(\underset{\sim}{x}(t),\underset{\sim}{u}(t))dt \geq E_I\big(\underset{\sim}{x}(T)\big) \geq 0$$

for all input-trajectory pairs $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}$ and all $T \geq 0$. Therefore

$$-\int_0^T \langle \underset{\sim}{v}(t),\underset{\sim}{i}(t)\rangle\, dt \leq E_I(\underset{\sim}{x})$$

for all admissible pairs with initial state $\underset{\sim}{x}$ and all $T \geq 0$, so

$$E_A(\underset{\sim}{x}) \leq E_I(\underset{\sim}{x}) < +\infty \tag{6-5}$$

for all $\underset{\sim}{x} \in \Sigma$. Q.E.D.

In view of Theorem 11, we could just as well take the existence of an internal energy function as our definition of passivity. This is in fact the approach that Willems [1] has adopted.

As shown in Lemma 4, a nonnegative C^1 function $\psi(\cdot)$ defined on Σ is an internal energy function if it satisfies (6-2). From Theorem 11 it follows that the existence of such a function $\psi(\cdot)$ is a sufficient condition for passivity, and this fact has been put to good use by Rohrer [2]. But is it a necessary condition? Or to put it negatively, is it possible for there to exist a passive n-port satisfying the assumptions of this paper for which every internal energy function fails to be differentiable at one or more points? Even if $\underset{\sim}{f}(\cdot,\cdot)$ and $p(\cdot,\cdot)$ are C^∞, we simply do not know. We have shown in Lemma 6 that $E_A(\cdot)$ is an internal energy function, but there is no obvious reason why $E_A(\cdot)$ should always be differentiable. Example 7 below shows that $E_A(\cdot)$ can fail to be differentiable at a point, even though $\underset{\sim}{f}(\cdot,\cdot)$ and $p(\cdot,\cdot)$ are C^∞. There are often other internal energy functions besides $E_A(\cdot)$, but it is conceivable that none of them is differentiable.

Example 7. Consider the 1-dimensional voltage-controlled 1-port with state and output equations

$$\dot{x} = f(x,v) \triangleq v^2$$

$$i = g(x,v) \triangleq v\left(\frac{1}{v^2+1} + x^2\right)^{-1/4} \tanh(v^2 x)$$

where $\Sigma = U = \mathbb{R}$ and $\mathcal{U}$ is the set of piecewise continuous functions mapping

$\mathbb{R}^+$ to $\mathbb{R}$. The power input function is $p(x,v) = vg(x,v)$ $= v^2\left(\frac{1}{v^2+1} + x^2\right)^{-1/4} \tanh(v^2x)$. In the notation of subsection 4.5, $U_x^- = \phi$ and $U_x^+ = \mathbb{R}\setminus\{0\}$, for all x. It follows that $\underline{h}(x) = -\infty$ for all x, and

$$\bar{h}(x) = \inf_{v\neq 0} \frac{p(x,v)}{f(x,v)} = \inf_{v\neq 0} \left\{\left(\frac{1}{v^2+1} + x^2\right)^{-1/4} \tanh(v^2x)\right\}$$

$$= \begin{cases} \frac{-1}{\sqrt{-x}}, & x < 0 \\ 0, & x \geq 0. \end{cases}$$

It is easy to verify that the conditions of Theorem 7 are satisfied, so this is a passive system. The available energy function is seen to be

$$E_A(x) = \begin{cases} 2\sqrt{-x}, & x < 0 \\ 0, & x \geq 0. \end{cases}$$

The function $E_A(\cdot)$ in Example 7 is not differentiable at $x = 0$, and it is not even piecewise C^1 since $\frac{dE_A(x)}{dx} \to -\infty$ as $x \to 0^-$. This may not be the most convincing counterexample because (a) the system is not completely controllable (Def. 13) and (b) $E_A(\cdot)$ is not extremely pathological (it is C^1 everywhere except at $x = 0$). There may exist examples with multidimensional state spaces in which $E_A(\cdot)$ exhibits a more pathological behavior, but this remains an open question.

Although it is not known whether C^1 internal energy functions must exist for the general passive system, this question can be answered for specific classes of passive systems. Willems [1] has shown that C^1 internal energy functions always exist for passive linear systems in which the input and output form a hybrid pair, and it can be seen that C^1 internal energy functions exist for the passive nonlinear systems considered in subsections 4.1-4.3.

We should mention the results of Hill and Moylan [22] (which are a generalization of Moylan's [8] results). They considered nonlinear systems which are linear in the control. Their necessary and sufficient algebraic condition for passivity was obtained simply by applying inequality (6-2) to the class of systems considered in their paper; however, their proof of the necessity of (6-2) was based on the assumption that $E_A(\cdot)$ is C^1. The authors of this paper believe that anyone attempting a truly complete proof of the necessity of (6-2) for a class of passive systems must <u>prove</u> that C^1 internal energy functions exist, and not merely <u>assume</u> that they exist.

Next, we are going to define the required energy functions. After listing some of their properties, we will show that the required energy functions can serve as internal energy functions under certain conditions. Recall from subsection 4.5 that $R(\underset{\sim}{x})$ denotes the set of states reachable from $\underset{\sim}{x}$ (Def. 12).

<u>Definition 24</u>. Given an n-port $\mathcal{N}$ with state representation S, we define $E_R : \Sigma \times \Sigma \to \mathbb{R}^e$ by

$$E_R(\underset{\sim}{x}_0,\underset{\sim}{x}) \triangleq \begin{cases} \inf\limits_{\substack{\underset{\sim}{x}_0 \to \underset{\sim}{x} \\ T \geq 0}} \left\{ \int_0^T p(\hat{\underset{\sim}{x}}(t),\hat{\underset{\sim}{u}}(t))dt \right\}, & \text{if } \underset{\sim}{x} \in R(\underset{\sim}{x}_0) \\ +\infty, & \text{if } \underset{\sim}{x} \notin R(\underset{\sim}{x}_0) \end{cases}$$

where the notation $\inf\limits_{\substack{\underset{\sim}{x}_0 \to \underset{\sim}{x} \\ T \geq 0}}$ indicates that the infimum is taken over all $T \geq 0$ and all input-trajectory pairs $\{\hat{\underset{\sim}{u}}(\cdot),\hat{\underset{\sim}{x}}(\cdot)\}|[0,T]$ from $\underset{\sim}{x}_0$ to $\underset{\sim}{x}$ (Def. 8); and for every $\underset{\sim}{x}_0 \in \Sigma$ we define the <u>required energy</u> (from $\underset{\sim}{x}_0$), $E_{R\underset{\sim}{x}_0} : \Sigma \to \mathbb{R}^e$, by

$$E_{R\underset{\sim}{x}_0}(\underset{\sim}{x}) \triangleq E_R(\underset{\sim}{x}_0,\underset{\sim}{x}).$$

Roughly speaking, $E_R(\underset{\sim}{x}_0,\underset{\sim}{x})$ is the minimum energy required to drive the system from $\underset{\sim}{x}_0$ to $\underset{\sim}{x}$, with the convention that $E_R(\underset{\sim}{x}_0,\underset{\sim}{x}) = +\infty$ if it is impossible to drive the system from $\underset{\sim}{x}_0$ to $\underset{\sim}{x}$.

We have listed below some obvious properties of the required energy functions:

(i) If Σ is reachable from $\underset{\sim}{x}_0$ (Def. 12), then $E_{R\underset{\sim}{x}_0}(\underset{\sim}{x}) < +\infty$ for all $\underset{\sim}{x} \in \Sigma$.

(ii) If $\mathcal{N}$ is passive, then $E_R(\underset{\sim}{x}_0,\underset{\sim}{x}) > -\infty$ for all $\underset{\sim}{x}_0,\underset{\sim}{x} \in \Sigma$.

(iii) If $E_I(\cdot)$ is an internal energy function, it follows from Def. 23 and Def. 24 that

$$E_I(\underset{\sim}{x}) - E_I(\underset{\sim}{x}_0) \leq E_R(\underset{\sim}{x}_0,\underset{\sim}{x}) \qquad (6\text{-}6)$$

for all $\underset{\sim}{x}_0,\underset{\sim}{x} \in \Sigma$;

(iv) Conversely, if $\psi : \Sigma \to \mathbb{R}^+$ satisfies

$$\psi(\underset{\sim}{x}) - \psi(\underset{\sim}{x}_0) \leq E_R(\underset{\sim}{x}_0,\underset{\sim}{x})$$

for all $\underset{\sim}{x}_0,\underset{\sim}{x} \in \Sigma$, then $\psi(\cdot)$ is an internal energy function.

(v) If $\mathcal{N}$ is passive and Σ is reachable from $\underset{\sim}{x}_0$, then

$$0 \leq E_{R\underset{\sim}{x}_0}(\underset{\sim}{x}) + E_A(\underset{\sim}{x}_0) < +\infty$$

for all $\underset{\sim}{x} \in \Sigma$.

Property (ii) may require some comment. Note that the definition of the available energy function (Def. 10) can be rewritten

$$E_A(\underset{\sim}{x}_0) = -\inf_{\substack{\underset{\sim}{x}_0 \to \\ T \geq 0}} \left\{ \int_0^T p(\hat{\underset{\sim}{x}}(t), \hat{\underset{\sim}{u}}(t))dt \right\}.$$

Comparing this with Def. 24, it is then obvious that $E_R(\underset{\sim}{x}_0, x) \geq -E_A(\underset{\sim}{x}_0)$ for all $\underset{\sim}{x}_0, \underset{\sim}{x} \in \Sigma$. This inequality, along with property (i), gives property (v).

Lemma 7. Suppose that an n-port $\mathcal{N}$ with state representation S is passive, suppose that Σ is reachable from some state $\underset{\sim}{x}_0 \in \Sigma$, and suppose that $\mathcal{U}$ is translation invariant and closed under concatenation. Define $\omega_{\underset{\sim}{x}_0} : \Sigma \to \mathbb{R}^+$ by

$$\omega_{\underset{\sim}{x}_0}(\underset{\sim}{x}) = E_{R\underset{\sim}{x}_0}(\underset{\sim}{x}) + E_A(\underset{\sim}{x}_0).$$

Then $\omega_{\underset{\sim}{x}_0}(\cdot)$ is an internal energy function for $\mathcal{N}$.

Willems [1] has sketched a proof for this lemma, and we have included a complete explicit proof in Appendix G. Note that if $\underset{\sim}{x}_0$ is a relaxed state, then $\omega_{\underset{\sim}{x}_0}(\cdot) = E_{R\underset{\sim}{x}_0}(\cdot)$.

Consider Example 6 once again, and assume that $G \geq 0$. Since the state representation is completely controllable and $q = 0$ is a relaxed state, it follows from Lemma 7 that $E_{R0}(\cdot)$ is an internal energy function for this 1-port. Previously, we used Willems' [1] theory to obtain the available energy function from a solution to the algebraic Riccati equation: his theory also shows that the required energy $E_{R0}(\cdot)$ for this 1-port is $E_{R0}(q) = \alpha^+(q^2/2)$, that is

$$E_{R0}(q) = \left[(2G+1) + 2\sqrt{G(G+1)}\right](q^2/2). \tag{6-7}$$

This function is plotted in Fig. 11 for three values of G. It is intuitively reasonable that $E_{R0}(q) = q^2/2$ when $G = 0$ because we can charge the capacitor with arbitrarily small losses by charging it very slowly.

The calculation of internal energy functions from a knowledge of the state representation alone is a nontrivial problem which has been solved only for special classes of systems, but it is easy to arrive at certain basic mathematical facts about internal energy functions which are of a non-constructive nature. These are summarized in the following theorem.

Theorem 12. Let $\mathcal{N}$ be a passive n-port with state representation S. Then the following statements are true.

(a) The set of all internal energy functions $E_I(\cdot)$ for $\mathcal{N}$ is convex, and $E_A(\cdot) \leq E_I(\cdot)$ for all possible $E_I(\cdot)$.

(b) Let $\underset{\sim}{x}_0 \in \Sigma$. For any internal energy function $E_I(\cdot)$ such that $E_I(\underset{\sim}{x}_0) = 0$, $E_I(\cdot) \leq E_{R\underset{\sim}{x}_0}(\cdot)$.

(c) If $\mathcal{U}$ is translation invariant and closed under concatenation, then $E_A(\cdot)$ is an internal energy function. If, in addition, Σ is reachable from a relaxed state $\underset{\sim}{x}^*$, then $E_{R\underset{\sim}{x}^*}(\cdot)$ is also an internal energy function.

Proof.

(a) The convexity of the set of internal energy functions is an immediate consequence of (6-1), and we have shown in (6-5) that $E_A(\cdot) \leq E_I(\cdot)$.

(b) This follows from (6-6) if we substitute $E_R(\underset{\sim}{x}_0,\underset{\sim}{x}) \triangleq E_{R\underset{\sim}{x}_0}(\underset{\sim}{x})$.

(c) The statement about $E_A(\cdot)$ is just a restatement of Lemma 6, and the statement about $E_{R\underset{\sim}{x}^*}(\cdot)$ is a special case of Lemma 7. Q.E.D.

When all the assumptions under (c) hold, it follows that $E_A(\cdot)$ and $E_{R\underset{\sim}{x}^*}(\cdot)$ are extreme points [23] of the set of internal energy functions $E_I(\cdot)$ which satisfy $E_I(\underset{\sim}{x}^*) = 0$, and it follows that $E_A(\cdot) + (1-\lambda)E_{R\underset{\sim}{x}^*}(\cdot)$ is an internal energy function for all $\lambda \in [0,1]$. For example, all the internal energy functions of the form $\frac{1}{2}\alpha q^2$ for Fig. 10 can be written in this way. But this does not imply that every internal energy function is a convex linear combination of $E_A(\cdot)$ and $E_{R\underset{\sim}{x}^*}(\cdot)$. In the case that G = 1 in Fig. 10 it is easy to verify that $\psi(q) = (\frac{3}{2}q^2 + q \sin q + \cos q - 1)$ is a valid internal energy function, and $\psi(\cdot)$ is clearly not a linear combination of $E_A(\cdot)$ and $E_{R0}(\cdot)$.

VII. The Passive Realization of N-Ports

The result discussed in this section is essentially the structural result reported by Anderson and Moylan [24]. Suppose that we have a resistive (m+n)-port $\mathcal{N}_R$ characterized by

$$\begin{aligned} \underset{\sim}{w} &= -\hat{\underset{\sim}{f}}(\underset{\sim}{z},\underset{\sim}{u}) \\ \underset{\sim}{y} &= \hat{\underset{\sim}{g}}(\underset{\sim}{z},\underset{\sim}{u}) \end{aligned} \tag{7-1}$$

where the input vector $(\underset{\sim}{z},\underset{\sim}{u}) \in \mathbb{R}^m \times \mathbb{R}^n$ and the output vector $(\underset{\sim}{w},\underset{\sim}{y}) \in \mathbb{R}^m \times \mathbb{R}^n$ form a hybrid pair, and suppose that we have a generalized capacitive/inductive m-port $\mathcal{N}_{LC}$ with state and output equations

$$\begin{aligned} \dot{\underset{\sim}{x}} &= \bar{\underset{\sim}{w}} \\ \bar{\underset{\sim}{z}} &= \underset{\sim}{h}(\underset{\sim}{x}) \end{aligned} \tag{7-2}$$

where the input vector $\bar{\underset{\sim}{w}} \in \mathbb{R}^m$ and the output vector $\bar{\underset{\sim}{z}} \in \mathbb{R}^m$ form a hybrid pair. It is assumed that for each $k \in \{1,2,\ldots,m\}$, the k-th components of $\bar{\underset{\sim}{w}}$ and $\underset{\sim}{w}$ are either both voltages, or both currents. We connect the m ports of $\mathcal{N}_{LC}$ to the first m ports of $\mathcal{N}_R$ in such a way that $\bar{\underset{\sim}{w}} = -\underset{\sim}{w}$ and $\bar{\underset{\sim}{z}} = \underset{\sim}{z}$.[11] Under these conditions, it follows from (7-1) and (7-2) that the resulting n-port has state and output equations

$$\begin{aligned} \dot{\underset{\sim}{x}} &= \hat{\underset{\sim}{f}}(\underset{\sim}{h}(\underset{\sim}{x}),\underset{\sim}{u}) \\ \underset{\sim}{y} &= \hat{\underset{\sim}{g}}(\underset{\sim}{h}(\underset{\sim}{x}),\underset{\sim}{u}). \end{aligned}$$

<u>Definition 25</u>. The interconnection of $\mathcal{N}_R$ and $\mathcal{N}_{LC}$ described above is said to be a <u>realization</u> of the state representation S given in Definition 1 if $\underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{u}) = \hat{\underset{\sim}{f}}(\underset{\sim}{h}(\underset{\sim}{x}),\underset{\sim}{u})$ and $\underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{u}) = \hat{\underset{\sim}{g}}(\underset{\sim}{h}(\underset{\sim}{x}),\underset{\sim}{u})$ for all $(\underset{\sim}{x},\underset{\sim}{u}) \in \Sigma \times U$; it is said to be a <u>passive realization</u> if $\mathcal{N}_{LC}$ and $\mathcal{N}_R$ are passive, where the inputs to $\mathcal{N}_R$ are restricted to $\underset{\sim}{h}[\Sigma] \times U \subset \mathbb{R}^m \times \mathbb{R}^n$.

Note that $\underset{\sim}{h}[\Sigma]$ is the image of Σ under the mapping $\underset{\sim}{h} : \mathbb{R}^m \to \mathbb{R}^m$.

We view the multiports $\mathcal{N}_R$ and $\mathcal{N}_{LC}$ as given quantities -- we are not concerned with the difficult and unsolved problem of synthesizing these nonlinear multiports. It is clear that any state representation S has a realization in which $\mathcal{N}_{LC}$ is linear: if each port of $\mathcal{N}_{LC}$ is either a 1-farad linear capacitor or a 1-henry linear inductor, whichever is appropriate, then $\underset{\sim}{h}(\underset{\sim}{x}) \equiv \underset{\sim}{x}$ and we obtain a realization by choosing $\hat{\underset{\sim}{f}}(\cdot,\cdot) = \underset{\sim}{f}(\cdot,\cdot)$ and $\hat{\underset{\sim}{g}}(\cdot,\cdot) = \underset{\sim}{g}(\cdot,\cdot)$; in general, however, the resistive (m+n)-port $\mathcal{N}_R$ will not be passive for such a realization.

The following theorem gives a sufficient condition for the existence of a passive realization.

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[11]This can always be done. If the k-th component of $\bar{\underset{\sim}{w}}$ is a current, connect the + terminal of the k-th port of $\mathcal{N}_{LC}$ to the + terminal of the k-th port of $\mathcal{N}_R$; otherwise, connect + to −.

Theorem 13. Suppose that a state representation S (Def. 1) is passive,[12] that $\underset{\sim}{u}$ and $\underset{\sim}{y}$ form a hybrid pair, that $\Sigma = \mathbb{R}^m$, that $\mathcal{U}$ contains all piecewise constant functions mapping $\mathbb{R}^+$ to U, and suppose that there exists a C^1 internal energy function $\psi(\cdot)$ for S and functions $\hat{\underset{\sim}{f}}(\cdot,\cdot)$, $\hat{\underset{\sim}{g}}(\cdot,\cdot)$ such that

$$\underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{u}) = \hat{\underset{\sim}{f}}(\underset{\sim}{\nabla}\psi(\underset{\sim}{x}),\underset{\sim}{u})$$

$$\underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{u}) = \hat{\underset{\sim}{g}}(\underset{\sim}{\nabla}\psi(\underset{\sim}{x}),\underset{\sim}{u})$$

for all $(\underset{\sim}{x},\underset{\sim}{u}) \in \mathbb{R}^m \times U$. Then S has a passive realization.

Remark. The functions $\hat{\underset{\sim}{f}}(\cdot,\cdot)$ and $\hat{\underset{\sim}{g}}(\cdot,\cdot)$ always exist if $\underset{\sim}{\nabla}\psi(\cdot)$ is one-to-one. We define $\hat{\underset{\sim}{f}} : \underset{\sim}{\nabla}\psi[\mathbb{R}^m] \times U \to \mathbb{R}^m$ and $\hat{\underset{\sim}{g}} : \underset{\sim}{\nabla}\psi[\mathbb{R}^m] \times U \to \mathbb{R}^n$ by

$$\hat{\underset{\sim}{f}}(\underset{\sim}{z},\underset{\sim}{u}) \triangleq \underset{\sim}{f}((\underset{\sim}{\nabla}\psi)^{-1}(\underset{\sim}{z}),\underset{\sim}{u})$$

$$\hat{\underset{\sim}{g}}(\underset{\sim}{z},\underset{\sim}{u}) \triangleq \underset{\sim}{g}((\underset{\sim}{\nabla}\psi)^{-1}(\underset{\sim}{z}),\underset{\sim}{u}).$$

Proof of Theorem. The m-port $\mathcal{N}_{LC}$ has state and output equations

$$\dot{\underset{\sim}{x}} = \bar{\underset{\sim}{w}}$$

$$\bar{\underset{\sim}{z}} = \underset{\sim}{\nabla}\psi(\underset{\sim}{x})$$

where the internal energy function $\psi(\cdot)$ is C^1 and nonnegative. It follows from Theorem 4 that $\mathcal{N}_{LC}$ is passive.

The (m+n)-port $\mathcal{N}_R$ is characterized by

$$\underset{\sim}{w} = -\hat{\underset{\sim}{f}}(\underset{\sim}{z},\underset{\sim}{u})$$

$$\underset{\sim}{y} = \hat{\underset{\sim}{g}}(\underset{\sim}{z},\underset{\sim}{u})$$

where $(\underset{\sim}{z},\underset{\sim}{u}) \in \underset{\sim}{\nabla}\psi[\mathbb{R}^m] \times U$. Let $(\underset{\sim}{z}_0,\underset{\sim}{u}_0)$ be an arbitrary point in $\underset{\sim}{\nabla}\psi[\mathbb{R}^m] \times U$, and let $\underset{\sim}{x}_0 \in \mathbb{R}^m$ be such that $\underset{\sim}{z}_0 = \underset{\sim}{\nabla}\psi(\underset{\sim}{x}_0)$. Then

$$[\underset{\sim}{z}_0^T,\underset{\sim}{u}_0^T]\begin{bmatrix}-\hat{\underset{\sim}{f}}(\underset{\sim}{z}_0,\underset{\sim}{u}_0)\\ \hat{\underset{\sim}{g}}(\underset{\sim}{z}_0,\underset{\sim}{u}_0)\end{bmatrix} = \langle \underset{\sim}{u}_0,\hat{\underset{\sim}{g}}(\underset{\sim}{z}_0,\underset{\sim}{u}_0)\rangle - \langle \underset{\sim}{z}_0,\hat{\underset{\sim}{f}}(\underset{\sim}{z}_0,\underset{\sim}{u}_0)\rangle$$

$$= \langle \underset{\sim}{u}_0,\hat{\underset{\sim}{g}}(\underset{\sim}{\nabla}\psi(\underset{\sim}{x}_0),\underset{\sim}{u}_0)\rangle - \langle \underset{\sim}{\nabla}\psi(\underset{\sim}{x}_0),\hat{\underset{\sim}{f}}(\underset{\sim}{\nabla}\psi(\underset{\sim}{x}_0),\underset{\sim}{u}_0)\rangle \geq 0,$$

the last step follows from Lemma 5. It follows from Theorem 3 that $\mathcal{N}_R$ is passive.

Q.E.D.

[12] In this theorem we shall consider passivity and the existence of internal energy functions to be attributes of the state representation S, rather than attributes of an n-port $\mathcal{N}$. This is not consistent with our previous terminology, but the meaning is clear.

Theorem 13 shows that the recovery of a C^1 internal energy function from a given state representation S is an important first step toward obtaining a passive realization of S.

VIII. Concluding Remarks

8.1 Weak Passivity and Strong Activity

After defining the required energy functions in Section VI (Def. 24), we noted that for a passive n-port the function $E_R(\cdot,\cdot)$ satisfies $E_R(\underset{\sim}{x}_1,\underset{\sim}{x}_2) > -\infty$ for all $\underset{\sim}{x}_1,\underset{\sim}{x}_2 \in \Sigma$. One might be tempted to define a weaker notion of passivity based on this condition, i.e., an n-port could be defined to be weakly passive if $E_R(\underset{\sim}{x}_1,\underset{\sim}{x}_2) > -\infty$ for all $\underset{\sim}{x}_1,\underset{\sim}{x}_2 \in \Sigma$; notice, however, that a -1 farad capacitor (with the current as the input) satisfies this definition. Since the classification of a -1 farad capacitor as an active element is deeply entrenched in the literature, it seems unwise to classify it both as an active element and an element which is passive in some weaker sense. It would perhaps be better to define a stronger notion of activity based on the negation of weak passivity, i.e., an n-port could be defined to be strongly active if there exist at least two states $\hat{\underset{\sim}{x}}_1$ and $\hat{\underset{\sim}{x}}_2$ such that $E_R(\hat{\underset{\sim}{x}}_1,\hat{\underset{\sim}{x}}_2) = -\infty$. A -1 farad capacitor is then active but not strongly active. In balance, however, there appears to be very little reason to introduce these new concepts.

8.2 Possible Generalizations

We have made the standing assumptions in this paper more restrictive than necessary in order to avoid hiding our concepts under a mass of elaborate definitions and unfamiliar notation. There are at least five possible generalizations which would not require a fundamental revision of our theory:

a) Our theory could be stated in terms of an abstract dynamical system [17]. This would enable our theory to handle infinite-dimensional systems, for example. There are also obvious extensions to time-varying systems. Both of these generalizations have been carried out by Willems [1].

b) We have required that $p(\underset{\sim}{x}(t),\underset{\sim}{u}(t)) = \langle \underset{\sim}{v}(t),\underset{\sim}{i}(t)\rangle$, the instantaneous power input. Nothing essential would be altered if we merely assume that $p : \Sigma \times U \to \mathbb{R}$ is some arbitrary continuous function. This approach has been taken by Willems [1]: he simply calls $p(\cdot,\cdot)$ the "supply rate," and it might well be that a particular n-port is passive with one choice of supply rate but active with another choice. This type of generalization might prove useful for obtaining results in stability theory. Note that such a generalization would normally invalidate our result in subsection 5.3 on interconnected systems, since Tellegen's theorem applies to physical electric power but not to arbitrary functions.

c) We have only discussed systems in which problems of finite escape time do not arise. It is important to note that these problems are not restricted to active n-ports. Consider for example the 1-port formed of the series connection of a 1-farad capacitor and a resistor characterized by $vi = 1$. This is admittedly a strange 1-port because v_{in} can never equal v_C, as this would imply that $v_R = 0$, which is not an admissible voltage for this resistor. But it is surely passive by some extended definition since it is made up of passive elements, and $i \to \infty$ in finite time as we can see by solving its state equation with $v_{in} = 0$ and $q(0) \neq 0$.

We could include such systems in our theory by altering Def. 10 so that T ranges over only those nonnegative values which are less than the time of escape.

d) The capacitors in Fig. 4, which arise from the mechanical example in subsection 3.3, are a technical violation of our assumptions on state representations because the set of admissible inputs over $\mathbb{R}^+$ depends on the initial state $q_0 = q(0)$; specifically, each input $i(\cdot)$ must satisfy the relation

$$q_0 + \int_0^T i(t)dt < 0, \ \forall T \geq 0.$$

This set of functions is not translation invariant and not closed under concatenation, as simple counterexamples will show. It turns out that we can define weakened forms of translation invariance and closure under concatenation which cover this case and are sufficient for proving our lemmas and theorems which require these properties. We suppose that the set of admissible inputs depends on the initial state. Let $\mathcal{U}$ be a set of functions mapping $\mathbb{R}^+$ to U which is translation invariant and closed under concatenation as in Defs. 6 and 7, and let $\overline{\mathcal{U}}(\cdot) : \Sigma \to \mathcal{U}$ be a map which gives the set of admissible inputs for each initial state; specifically, if the initial state is $\underset{\sim}{x}_0$, then the set of admissible inputs over $\mathbb{R}^+$ is $\overline{\mathcal{U}}(\underset{\sim}{x}_0)$. Let $\{\underset{\sim}{u}_1(\cdot), \underset{\sim}{x}_1(\cdot)\}$ be an input-trajectory pair (and hence $\underset{\sim}{u}_1(\cdot) \in \overline{\mathcal{U}}(\underset{\sim}{x}_1(0))$). Let $\tau \geq 0$ and define $\underset{\sim}{u}_{1\tau}(\cdot)$ to be the translation of $\underset{\sim}{u}_1(\cdot)$ by τ units to the left, i.e., $\underset{\sim}{u}_{1\tau}(t) \triangleq u_1(t+\tau)$, $t \geq 0$. If $\underset{\sim}{u}_{1\tau}(\cdot) \in \overline{\mathcal{U}}(\underset{\sim}{x}_1(\tau))$ for every $\tau \geq 0$ and for every input-trajectory pair $\{\underset{\sim}{u}_1(\cdot), \underset{\sim}{x}_1(\cdot)\}$, then we say that the map $\overline{\mathcal{U}}(\cdot)$ is <u>weakly translation invariant</u>. Let $\{\underset{\sim}{u}_1(\cdot), \underset{\sim}{x}_1(\cdot)\}$ be an input-trajectory pair, let $\tau \geq 0$, and let $\underset{\sim}{u}_2(\cdot) \in \overline{\mathcal{U}}(\underset{\sim}{x}_1(\tau))$.

Define $\underset{\sim}{u}_{12\tau}(\cdot)$ to be the concatenation of $\underset{\sim}{u}_1(\cdot)$ and $\underset{\sim}{u}_2(\cdot)$ at τ, i.e.,

$$\underset{\sim}{u}_{12\tau}(t) \triangleq \begin{cases} \underset{\sim}{u}_1(t), & 0 \le t \le \tau \\ \underset{\sim}{u}_2(t-\tau), & t > \tau. \end{cases}$$

If $\underset{\sim}{u}_{12\tau}(\cdot) \in \mathcal{U}(\underset{\sim}{x}_1(0))$ for all $\underset{\sim}{u}_2(\cdot) \in \mathcal{U}(\underset{\sim}{x}_1(\tau))$, all $\tau \ge 0$, and for all input-trajectory pairs $\{\underset{\sim}{u}_1(\cdot), \underset{\sim}{x}_1(\cdot)\}$, then we say that the map $\mathcal{U}(\cdot)$ is <u>weakly closed under concatenation</u>. These weak forms of translation invariance and closure under concatenation are all that is required in the proofs of the various lemmas and theorems in this paper which require these properties.

e) Consider a system of the form

$$\dot{\underset{\sim}{z}}(t) = \hat{\underset{\sim}{f}}(\underset{\sim}{z}(t), \underset{\sim}{u}(t), \underset{\sim}{u}^{(1)}(t), \ldots, \underset{\sim}{u}^{(k)}(t)) \tag{8-1a}$$

$$\underset{\sim}{y}(t) = \hat{\underset{\sim}{g}}(\underset{\sim}{z}(t), \underset{\sim}{u}(t), \underset{\sim}{u}^{(1)}(t), \ldots, \underset{\sim}{u}^{(k)}(t)) \tag{8-1b}$$

where $\underset{\sim}{u}^{(j)}(\cdot)$ denotes the j-th derivative of $\underset{\sim}{u}(\cdot)$, with the convention $\underset{\sim}{u}^{(0)}(\cdot) = \underset{\sim}{u}(\cdot)$. We assume that $k \ge 1$ and that every input $\underset{\sim}{u}(\cdot)$ is piecewise C^k. An example of such a system for the case $k = 1$ would be a voltage-controlled n-port in which there is a loop consisting exclusively of capacitors and ports. Since the first (k-1) derivatives of the input $\underset{\sim}{u}(\cdot)$ must exist and be continuous, even at $t = 0$, $\underset{\sim}{u}(\cdot)$ must satisfy the condition $\underset{\sim}{u}^{(j)}(0) = \underset{\sim}{u}^{(j)}(0^-)$, $0 \le j \le k-1$, where $\underset{\sim}{u}^{(j)}(0^-)$ denotes $\lim_{\substack{t \to 0 \\ t<0}} \underset{\sim}{u}^{(j)}(t)$ (since we must know the input before $t = 0$ in order to find $\underset{\sim}{u}^{(j)}(0^-)$, we are violating our convention of viewing the system only over the time interval $\mathbb{R}^+$). Because system (8-1) requires continuous inputs, even at $t = 0$, the set of admissible inputs cannot be translation invariant or closed under concatenation as in Defs. 6 and 7; nevertheless, there is a way to handle such systems along the lines described in part d) above.[13]

Note that system (8-1) can be rewritten in the form

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{f}(\underset{\sim}{x}(t), \underset{\sim}{u}^{(k)}(t)) \tag{8-2a}$$

$$\underset{\sim}{y}(t) = \underset{\sim}{g}(\underset{\sim}{x}(t), \underset{\sim}{u}^{(k)}(t)) \tag{8-2b}$$

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[13] For a <u>linear</u> system of the form (8-1), one could use distribution theory to make some sense out of a discontinuous input; but as far as we know these concepts cannot be extended to the general <u>nonlinear</u> system of the form (8-1).

where

$$\underset{\sim}{x} \triangleq \begin{bmatrix} \underset{\sim}{x}_1 \\ \underset{\sim}{x}_2 \\ \vdots \\ \underset{\sim}{x}_k \\ \underset{\sim}{z} \end{bmatrix}, \quad \underset{\sim}{x}_i \in \mathbb{R}^n, \ i = 1,2,\ldots,k \tag{8-2c}$$

$$\underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{u}^{(k)}) \triangleq \begin{bmatrix} \underset{\sim}{x}_2 \\ \underset{\sim}{x}_3 \\ \vdots \\ \underset{\sim}{x}_k \\ \underset{\sim}{u}^{(k)} \\ \hat{\underset{\sim}{f}}(\underset{\sim}{z},\underset{\sim}{x}_1,\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_k,\underset{\sim}{u}^{(k)}) \end{bmatrix} \tag{8-2d}$$

$$\underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{u}^{(k)}) \triangleq \hat{\underset{\sim}{g}}(\underset{\sim}{z},\underset{\sim}{x}_1,\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_k,\underset{\sim}{u}^{(k)}) \tag{8-2e}$$

Let $\mathcal{U}$ denote the set of piecewise C^k functions mapping $\mathbb{R}^+$ to U and let $\Sigma' \subset \mathbb{R}^{(kn+m)}$ denote the state space of system (8-2). We define the map $\bar{\mathcal{U}}(\cdot) : \Sigma' \to \mathcal{U}$ as follows:

$$\bar{\mathcal{U}}(\underset{\sim}{x}) \triangleq \left\{ \underset{\sim}{u}(\cdot) \in \mathcal{U}: \begin{bmatrix} \underset{\sim}{u}(0) \\ \underset{\sim}{u}^{(1)}(0) \\ \vdots \\ \underset{\sim}{u}^{(k-1)}(0) \end{bmatrix} = \begin{bmatrix} \underset{\sim}{x}_1 \\ \underset{\sim}{x}_2 \\ \vdots \\ \underset{\sim}{x}_k \end{bmatrix} \right\}. \tag{8-3}$$

As in part d), $\bar{\mathcal{U}}(\cdot)$ is the map which gives the set of admissible inputs for each initial state. It is easy to verify that $\bar{\mathcal{U}}(\cdot)$ is weakly translation invariant and weakly closed under concatenation.

We illustrate the ideas of part e) with the voltage-controlled capacitor $q = \hat{q}(v)$, where $\hat{q}(\cdot)$ is C^1. The current flowing through such a capacitor is $i = q'(v)\dot{v}$. Note that this expression gives the output i directly in terms of the input v -- no state equation is necessary. But to show how such a system can be treated along the lines described in part e), we give it a state representation as in (8-2) of the form

$\dot{x} = \dot{v}$

$i = q'(x)\dot{v}$

where, with $\mathcal{U}$ denoting the set of piecewise C^1 functions mapping $\mathbb{R}^+$ to U, we define $\bar{\mathcal{U}}(\cdot) : \mathbb{R} \to \mathcal{U}$ by

$$\bar{\mathcal{U}}(x) = \{v(\cdot) \in \mathcal{U}: v(0) = x\}.$$

The five possible generalizations mentioned above would require only minor alterations. One feature they would not change is our implicit assumption that we can always choose a definite input and output. This has put elements such as the algebraic n-ports defined by Chua [14] outside the range of our theory. The ideal diode and the norator are examples of algebraic 1-ports which are neither voltage-controlled nor current-controlled. Another example of an algebraic 1-port would be a capacitor with constitutive relation $v^2+q^2 < 1$.

It is possible to include such elements by a major alteration which would eliminate state equations from our theory altogether. Although our work has been presented in terms of state equations because they are a familiar convention, the essential idea does not require them. All we really need is a set of initial states Σ_I and a map $\rho(\cdot)$ which assigns to each $\underset{\sim}{x}_0 \in \Sigma_I$ the class of all admissible pairs $\{\underset{\sim}{v}(\cdot), \underset{\sim}{i}(\cdot)\}$ which are compatible with that initial state. The available energy would be defined by

$$E_A(\underset{\sim}{x}_0) = \sup_{\substack{\forall T>0 \\ \forall\{\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot)\} \in \rho(\underset{\sim}{x}_0)}} \left\{ \int_0^T -\langle \underset{\sim}{v}(t), \underset{\sim}{i}(t)\rangle \, dt \right\}$$

and the theory could then proceed along roughly the lines we have presented in this paper.

8.3 Three Different Interpretations of Passivity

There are three different but logically equivalent ways of interpreting passivity. Since they emphasize different facets of the concept, it is worthwhile considering each in turn.

The first might be called "the thermodynamic point of view." Here we consider an n-port as a possible energy source and concern ourselves with how much energy we can hope to extract from it. The exact amount will generally depend on the initial state and is denoted by $E_A(\underset{\sim}{x}_0)$. We have written this paper from "the thermodynamic point of view," as Definitions 10 and 11 clearly reflect. This approach seems to us to present the meaning of passivity in the clearest possible light, but it does not make the potential links between passivity and stability especially obvious.

From the second perspective, which we might call the "input-output point of view," we look on an n-port as a family of operators $O_{\underset{\sim}{x}}$, one operator for each initial state $\underset{\sim}{x} \in \Sigma$. These operate on an input waveform $\underset{\sim}{u}(\cdot)$ to produce an output waveform $\underset{\sim}{y}(\cdot)$. We will assume that $\underset{\sim}{u}(\cdot)$ and $\underset{\sim}{y}(\cdot)$ are a hybrid pair and that the domain and image of $O_{\underset{\sim}{x}}$ are in $L^2_{loc}(\mathbb{R}^+ \to \mathbb{R}^n)$, since we wish to introduce the family of inner products

$$\langle \underset{\sim}{u}(\cdot), \underset{\sim}{y}(\cdot) \rangle_T = \int_0^T \langle \underset{\sim}{u}(t), \underset{\sim}{y}(t) \rangle \, dt.$$

We can then think of $O_{\underset{\sim}{x}}$ as passive if $\langle \underset{\sim}{u}(\cdot), O_{\underset{\sim}{x}} \underset{\sim}{u}(\cdot) \rangle_T$ is bounded below as $\underset{\sim}{u}(\cdot)$ varies over $\mathcal{U}$ and T varies over $\mathbb{R}^+$, and we say that the n-port is passive if $O_{\underset{\sim}{x}}$ is passive for each $\underset{\sim}{x} \in \Sigma$.

The thermodynamic and the input-output viewpoints are obviously logically equivalent. The latter is extremely important as the setting for many stability theorems from control theory which are based on functional analysis [10]. It is important to note, however, that such theorems always require conditions which are stronger than or distinct from passivity alone, e.g., strict passivity or incremental passivity.

The third way of looking at passivity might be called "the internal energy point of view." From this perspective we would say that an n-port is passive if there exists a nonnegative function on the state space which decreases along trajectories at least as rapidly as the rate at which energy leaves the ports. We have shown in Theorem 11 that under mild technical assumptions this point of view is equivalent to the first two. Many stability theorems in circuit theory take this view of passivity because the internal energy function can often serve as a Lyapunov function; it is important to note, however, that even a smooth internal energy function need not be a genuine Lyapunov function because the concept of passivity alone imposes no requirements on its shape. The internal energy function in Fig. 6 is an example which shows that additional requirements are necessary since it does not qualify as a Lyapunov function.

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APPENDIX

Appendix A: Proof of Theorem 1.

Since $\underset{\sim}{x}_1$ is reachable from $\underset{\sim}{x}_0$, there exists a finite $T \geq 0$ and an input-trajectory pair $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}|[0,T]$ from $\underset{\sim}{x}_0$ to $\underset{\sim}{x}_1$. Let $\underset{\sim}{u}'(\cdot) \in \mathcal{U}$ be an arbitrary input, and let $\{\underset{\sim}{u}'(\cdot),\underset{\sim}{x}'(\cdot)\}$ be an input-trajectory pair with initial state $\underset{\sim}{x}_1$. Define a waveform $\underset{\sim}{u}''(\cdot)$ by

$$\underset{\sim}{u}''(t) \triangleq \begin{cases} \underset{\sim}{u}(t), & 0 \leq t \leq T \\ \underset{\sim}{u}'(t-T), & t > T. \end{cases}$$

Since $\mathcal{U}$ is closed under concatenation, $\underset{\sim}{u}''(\cdot) \in \mathcal{U}$. Let $\{\underset{\sim}{u}''(\cdot),\underset{\sim}{x}''(\cdot)\}$ be an input-trajectory pair with initial state $\underset{\sim}{x}_0$; hence, $\underset{\sim}{x}''(t) = \underset{\sim}{x}(t)$ for $0 \leq t \leq T$; moreover, $\underset{\sim}{x}''(t) = \underset{\sim}{x}'(t-T)$ for $t > T$ since the state equation is time invariant. From Def. 10 we obtain

$$E_A(\underset{\sim}{x}_0) \geq -\int_0^{T+T'} p(\underset{\sim}{x}''(t),\underset{\sim}{u}''(t))dt \tag{A-1}$$

for all $T' \geq 0$. Separating the integral in (A-1) into two integrals, one over the time interval $[0,T]$ and one over the time interval $[T,T+T']$, we obtain

$$E_A(\underset{\sim}{x}_0) \geq -\int_0^{T} p(\underset{\sim}{x}(t),\underset{\sim}{u}(t))dt - \int_0^{T'} p(\underset{\sim}{x}'(t),\underset{\sim}{u}'(t))dt \tag{A-2}$$

for all $T' \geq 0$. Taking the supremum over all $\underset{\sim}{u}'(\cdot) \in \mathcal{U}$ and all $T' \geq 0$, we have

$$E_A(\underset{\sim}{x}_0) \geq -\int_0^{T} p(\underset{\sim}{x}(t),\underset{\sim}{u}(t))dt + E_A(\underset{\sim}{x}_1)\,. \tag{A-3}$$

The integral on the right-hand side of (A-3) is finite by standing assumption (4); hence, if $E_A(\underset{\sim}{x}_0) < +\infty$, then $E_A(\underset{\sim}{x}_1) < +\infty$. Q.E.D.

Appendix B: Proof of Theorem 3.

(i) $\Rightarrow$ (ii). If $\langle \underset{\sim}{u},\underset{\sim}{g}(\underset{\sim}{u})\rangle \geq 0$ everywhere, then

$$\int_0^T -\langle \underset{\sim}{u}(t),\underset{\sim}{y}(t)\rangle\, dt = \int_0^T -\langle \underset{\sim}{v}(t),\underset{\sim}{i}(t)\rangle\, dt \leq 0$$

for all initial states $\underset{\sim}{x} \in \Sigma$, all inputs $\underset{\sim}{u}(\cdot) \in \mathcal{U}$, and all $T \geq 0$. Therefore the supremum in Def. 10 is obtained by setting $T = 0$, and we have $E_A(\underset{\sim}{x}) = 0$ for all $\underset{\sim}{x} \in \Sigma$.

(ii) $\Rightarrow$ (iii). This follows immediately from the definition.

(iii) $\Rightarrow$ (i). Suppose $\mathcal{N}$ is passive but that for some value $\underset{\sim}{u}' \in U$, we have $\langle \underset{\sim}{u}', \underset{\sim}{g}(\underset{\sim}{u}') \rangle = \lambda < 0$. Then consider the constant input $\underset{\sim}{u}'(t) \equiv \underset{\sim}{u}'$. For any state $\underset{\sim}{x} \in \Sigma$ we have

$$E_A(\underset{\sim}{x}) \geq \sup_{T \geq 0} \int_0^T -\langle \underset{\sim}{u}'(t), \underset{\sim}{g}(\underset{\sim}{u}'(t)) \rangle \, dt$$

$$= \sup_{T \geq 0} \{-\lambda T\} = +\infty \, .$$

So $\mathcal{N}$ is active, contradicting our assumption. Q.E.D.

<u>Appendix C: Proof of Theroem 4</u>.

<u>Proof of (1)</u>.

($\Leftarrow$) Let ℓ be a finite lower bound for $\psi(\cdot)$, let $\underset{\sim}{x}_0 \in \Sigma$ be any state, and let $\{\underset{\sim}{u}(\cdot), \underset{\sim}{y}(\cdot)\}$ be any input-output pair with initial state $\underset{\sim}{x}_0$. Then for all $T \geq 0$ we have

$$\int_0^T -\langle \underset{\sim}{u}(t), \underset{\sim}{y}(t) \rangle \, dt = -\int_0^T \langle \dot{\underset{\sim}{x}}(t), \underset{\sim}{\nabla}\psi(\underset{\sim}{x}(t)) \rangle \, dt$$

$$= \int_0^T -\left\{ \frac{d}{dt} \psi(\underset{\sim}{x}(t)) \right\} dt = \psi(\underset{\sim}{x}_0) - \psi(\underset{\sim}{x}(T)) \leq \psi(\underset{\sim}{x}_0) - \ell \, . \qquad \text{(C-1)}$$

The third step is justified because $\underset{\sim}{x}(\cdot)$ is absolutely continuous and $\psi(\cdot)$ is C^1, which implies that $\psi \circ \underset{\sim}{x}(\cdot)$ is absolutely continuous. It follows from (C-1) that $E_A(\underset{\sim}{x}_0) \leq \psi(\underset{\sim}{x}_0) - \ell < +\infty$. Since $\underset{\sim}{x}_0$ was arbitrary, $\mathcal{N}$ is passive.

($\Rightarrow$) Assume $\mathcal{N}$ is passive. If $\underset{\sim}{g}$ is not the gradient of a scalar function which is bounded below, then two possibilities arise. The first is that $\underset{\sim}{g} = \underset{\sim}{\nabla}\psi$ where ψ is not bounded from below. The second is that $\underset{\sim}{g}$ is not the gradient of any scalar function at all.

We can easily eliminate the first possibility. If $\underset{\sim}{g} = \underset{\sim}{\nabla}\psi$ where ψ has no lower bound, then for each real number r there is a point $\underset{\sim}{x}_r \in \Sigma$ such that $\psi(\underset{\sim}{x}_r) < r$. Let the input $\underset{\sim}{u}_r(\cdot)$ be defined[14] by $\underset{\sim}{u}_r(t) = \underset{\sim}{x}_r$, $0 \leq t \leq 1$ and $\underset{\sim}{u}_r(t) = \underset{\sim}{0}$ otherwise. Let $\underset{\sim}{y}_r(\cdot)$ be the output which results from applying $\underset{\sim}{u}_r(\cdot)$ when the initial state is $\underset{\sim}{0}$. We have

[14] We are ignoring physical units here and treating all quantities as dimensionless.

$$\int_0^1 -\langle \underset{\sim}{u}_r(t), \underset{\sim}{y}_r(t)\rangle\, dt = -\int_0^1 \langle \underset{\sim}{x}_r, \underset{\sim}{g}(\underset{\sim}{x}_r t)\rangle\, dt$$

$$= -\int_0^1 \left\{\frac{d}{dt}\, \psi(\underset{\sim}{x}_r t)\right\} dt = \psi(0) - \psi(\underset{\sim}{x}_r) > \psi(0) - r\ .$$

Since r was arbitrary, $E_A(\underset{\sim}{0}) = +\infty$ contradicting our assumption that $\mathcal{N}$ was passive. This eliminates the first possibility.

Suppose that $\underset{\sim}{g}(\cdot)$ is not the gradient of any scalar function. It follows [25, Theorem 7, page 82] that there exists a point $\underset{\sim}{x}_0 \in \mathbb{R}^n$ and a piecewise C^1 function $\underset{\sim}{\gamma} : [0,1] \to \mathbb{R}^n$ such that $\underset{\sim}{\gamma}(0) = \underset{\sim}{\gamma}(1) = \underset{\sim}{x}_0$ and

$$\int_0^1 \langle \dot{\underset{\sim}{\gamma}}(t), \underset{\sim}{g}(\underset{\sim}{\gamma}(t))\rangle\, dt \neq 0 \tag{C-2}$$

(the image of [0,1] under the mapping $\underset{\sim}{\gamma}(\cdot)$ is a piecewise C^1 closed curve through $\underset{\sim}{x}_0$, and the integral in (C-2) is the line integral of $\underset{\sim}{g}(\cdot)$ around this curve). We assume without loss of generality that

$$-\int_0^1 \langle \dot{\underset{\sim}{\gamma}}(t), \underset{\sim}{g}(\underset{\sim}{\gamma}(t))\rangle\, dt \triangleq r > 0; \tag{C-3}$$

for if (C-3) is not satisfied, then we replace $\underset{\sim}{\gamma}(\cdot)$ by a new function $\hat{\underset{\sim}{\gamma}}(\cdot)$ defined by $\hat{\underset{\sim}{\gamma}}(t) \triangleq \underset{\sim}{\gamma}(1-t)$. Define an input $\underset{\sim}{u}(\cdot)$ as follows:

$$\underset{\sim}{u}(t) \triangleq \dot{\underset{\sim}{\gamma}}(t-k),\ t \in [k,k+1),\ k = 0,1,2,\ldots$$

Since $\underset{\sim}{\gamma}(\cdot)$ is piecewise C^1, $\underset{\sim}{u}(\cdot)$ is an admissible input waveform. If the initial condition is $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$, it follows that

$$\underset{\sim}{x}(t) = \underset{\sim}{\gamma}(t-k),\ t \in [k,k+1),\ k = 0,1,2,\ldots$$

For each positive integer k, the energy extracted at time T = k is

$$-\int_0^k \langle \underset{\sim}{u}(t), \underset{\sim}{g}(\underset{\sim}{x}(t))\rangle\, dt = -k\int_0^1 \langle \dot{\underset{\sim}{\gamma}}(t), \underset{\sim}{g}(\gamma(t))\rangle\, dt = kr. \tag{C-4}$$

Since r > 0 and k is an arbitrary positive integer, it follows from (C-4) that $E_A(\underset{\sim}{x}_0) = +\infty$, which contradicts the assumption that $\mathcal{N}$ is passive.

Proof of (2).

($\Leftarrow$) We need only show that $\underset{\sim}{x}^*$ is a relaxed state. To see this, let $\{\underset{\sim}{u}(\cdot), \underset{\sim}{y}(\cdot)\}$ be an input-output pair with initial state $\underset{\sim}{x}^*$. Then for any $T \geq 0$ we have

$$\int_0^T -\langle \underset{\sim}{u}(t), \underset{\sim}{y}(t)\rangle\, dt = \psi(\underset{\sim}{x}^*) - \psi(\underset{\sim}{x}(T)) \leq 0;$$

hence, the supremum in Def. 10 is obtained by setting $T = 0$, so $E_A(\underset{\sim}{x}^*) = 0$, i.e., $\underset{\sim}{x}^*$ is a relaxed state.

($\Rightarrow$) Assume $\mathcal{N}$ is strongly passive. Then $\mathcal{N}$ is passive and so the conditions of part (1) of the theorem must hold. Furthermore there exists a relaxed state $\underset{\sim}{x}' \in \Sigma$. To see that $\psi(\cdot)$ attains its minimum at $\underset{\sim}{x}'$, let $\{\underset{\sim}{u}(\cdot),\underset{\sim}{y}(\cdot)\}$ be any input-output pair with initial state $\underset{\sim}{x}'$. Then

$$\int_0^T -\langle \underset{\sim}{u}(t),\underset{\sim}{y}(t)\rangle\, dt = \psi(\underset{\sim}{x}') - \psi(\underset{\sim}{x}(T)).$$

Since $\underset{\sim}{x}'$ is a relaxed state, $E_A(\underset{\sim}{x}') = 0$ and we must have $\psi(\underset{\sim}{x}') - \psi(\underset{\sim}{x}(T)) \leq 0$, or $\psi(\underset{\sim}{x}') \leq \psi(\underset{\sim}{x}(T))$. But $\underset{\sim}{x}(T)$ could be any point of Σ, so $\psi(\cdot)$ must attain its minimum at $\underset{\sim}{x}'$. Q.E.D.

Appendix D: The 1-Dimensional System.

(D.1) Definitions. Let $\Sigma \subset \mathbb{R}$, and let the topology of Σ be the relative topology that it inherits from $\mathbb{R}$ (i.e., a subset $G \subset \Sigma$ is open if and only if there exists a set $\hat{G}$ which is open in $\mathbb{R}$ such that $G = \hat{G} \cap \Sigma$).[15] A function $f : \Sigma \to \mathbb{R}^e$ is defined to be upper semicontinuous if the set $\{x \in \Sigma : f(x) < \alpha\}$ is open (in the topology of Σ) for all $\alpha \in \mathbb{R}$. Likewise, $f : \Sigma \to \mathbb{R}^e$ is defined to be lower semicontinuous if the set $\{x \in \Sigma : f(x) > \alpha\}$ is open for all $\alpha \in \mathbb{R}$.[16] Note that $f(\cdot)$ is upper semicontinuous if and only if $-f(\cdot)$ is lower semicontinous; also, $f(\cdot)$ is continuous if and only if it is both upper and lower semicontinuous.

(D.2) Lemma. The infimum of any collection of upper semicontinuous functions is upper semicontinuous. The supremum of any collection of lower semicontinuous functions is lower semicontinuous.

Proof. Let $\{f_\beta : \beta \in B\}$ be a collection of upper semicontinuous functions, where the index set B may be finite, countable, or uncountable. Let $f \triangleq \inf\{f_\beta : \beta \in B\}$. Then $\{x : f(x) < \alpha\} = \bigcup_{\beta\in B} \{x : f_\beta(x) < \alpha\}$. Thus $\{x : f(x) < \alpha\}$

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[15] Usually, the state space Σ is an open subset of $\mathbb{R}$ (it may be $\mathbb{R}$ itself), in which case there is no reason to introduce the concept of relative topology.

[16] These definitions of semicontinuity are taken from Rudin [26]. A more restrictive definition of semicontinuity is the following [9]: $f(\cdot)$ is upper semicontinuous at $x_0 \in \Sigma$ if $f(x_0) < +\infty$ and $f(x_0) \geq \overline{\lim}_{x\to x_0} f(x)$, and $f(\cdot)$ is lower semicontinuous at x_0 if $f(x_0) > -\infty$ and $f(x_0) \leq \underline{\lim}_{x\to x_0} f(x)$; $f(\cdot)$ is upper (resp., lower) semicontinuous if it is upper (resp., lower) semicontinuous at each point $x_0 \in \Sigma$. Rudin's [26] definition is more general and better suited to our purposes.

is open for all $\alpha \in \mathbb{R}$, i.e., f is upper semicontinuous. The proof of the other assertion is similar. Q.E.D.

<u>(D.3) Lemma</u>. Let $f : \Sigma \to \mathbb{R}^e$ and let $K \subset \Sigma$ be a compact set. If $f(\cdot)$ is upper semicontinuous and $f(x) \neq +\infty$ for all $x \in K$, then $f(\cdot)$ is bounded above on K. If $f(\cdot)$ is lower semicontinuous and $f(x) \neq -\infty$ for all $x \in K$, then $f(\cdot)$ is bounded below on K.

<u>Proof</u>. Let $f : \Sigma \to \mathbb{R}^e$ be an upper semicontinuous function, and let $K \subset \Sigma$ be compact. Suppose $f(x) \neq +\infty$ for all $x \in K$. For $\alpha \in \mathbb{R}$, define

$$V_\alpha = \{x \in \Sigma : f(x) < \alpha\}.$$

Thus each set V_α is open. The collection $\{V_\alpha : \alpha \in \mathbb{R}\}$ covers K. Since K is compact, there exists a finite subcover $\{V_{\alpha_1},\ldots,V_{\alpha_n}\}$. Let $M = \max\{\alpha_1,\ldots,\alpha_n\}$. Then $f(x) < M < +\infty$ for all $x \in K$. The proof of the other assertion is similar.

Q.E.D.

<u>(D.4) Definitions</u>. Let $f(\cdot)$ be a real-valued function with domain contained in $\mathbb{R}$. Then $f(\cdot)$ is said to be <u>continuous from the right at x_0</u> if $f(x_0) = \lim_{\substack{x\to x_0\\ x>x_0}} f(x)$, and <u>continuous from the left at x_0</u> if $f(x_0) = \lim_{\substack{x\to x_0\\ x<x_0}} f(x)$. A function $s : [c,d] \subset \mathbb{R} \to \mathbb{R}$ is a <u>step function</u> if there exists a partition $c = x_0 < x_1 < \ldots < x_n = d$ of $[c,d]$ such that $s(\cdot)$ is constant on each open interval (x_{i-1},x_i), $i = 1,2,\ldots,n$, $s(\cdot)$ is continuous from right at c, continuous from the left at d, and $s(\cdot)$ is continuous from either the right or the left at x_i, $i = 1,2,\ldots,(n-1)$.

<u>(D.5) Lemma</u>. Let $f : [c,d] \subset \mathbb{R} \to \mathbb{R}^e$.

(i) Suppose $f(\cdot)$ is upper semicontinuous and $f(x) \neq +\infty$ for all $x \in [c,d]$. Let $M = \sup\{f(x) : x \in [c,d]\}$ (note that $M < +\infty$ by Lemma (D.3)). Then there exists a sequence $\langle s_n \rangle$ of upper semicontinuous step functions such that $M \geq s_1 \geq s_2 \geq \ldots \geq f$ and $\lim_{n\to\infty} s_n(x) = f(x)$ for each $x \in [c,d]$.

(ii) Suppose $f(\cdot)$ is lower semicontinuous and $f(x) \neq -\infty$ for all $x \in [c,d]$. Let $m = \inf\{f(x) : x \in [c,d]\}$ (note that $m > -\infty$ by Lemma (D.3)). Then there exists a sequence $\langle s_n \rangle$ of lower semicontinuous step functions such that $m \leq s_1 \leq s_2 \leq \ldots \leq f$ and $\lim_{n\to\infty} s_n(x) = f(x)$ for each $x \in [c,d]$.

<u>Proof</u>. Without loss of generality, assume $[c,d] = [0,1]$. If (i) is satisfied, then $M-f(\cdot)$ is nonnegative and lower semicontinuous. If (ii) is satisfied, then $f(\cdot)-m$ is nonnegative and lower semicontinuous. Hence there is

no loss of generality in assuming that $f(\cdot)$ is nonnegative and lower semicontinuous. Fix n and define $m_{kn} \triangleq \inf\left\{f(x) : x \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)\right\}$ for $k = 1,2,\ldots,(2^n-1)$ and define $m_{2^n n} = \left\{\inf f(x) : x \in \left[1 - \frac{1}{2^n}, 1\right]\right\}$. Let $p_{kn} \triangleq m_{kn}$ if $m_{kn} < +\infty$, and let $p_{kn} \triangleq n$ if $m_{kn} = +\infty$. Define

$$s_n(x) \triangleq p_{kn},\ x \in \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right),\ k = 1,2,\ldots,2^n$$

$$s_n(0) \triangleq p_{1n}$$

$$s_n(1) \triangleq p_{2^n n}$$

$$s_n\left(\frac{k}{2^n}\right) \triangleq \min\{p_{kn}, p_{(k+1)n}\},\ k = 1,2,\ldots,(2^n-1).$$

Clearly, each $s_n(\cdot)$ is lower semicontinuous and $0 \le s_1 \le s_2 \le \cdots \le f$. Fix $x_0 \in [0,1]$. Suppose first that $f(x_0) < +\infty$ and let $\varepsilon > 0$. The set $V = \{x \in [0,1] : f(x) > f(x_0) - \varepsilon\}$ is an open set (in the topology of $[0,1]$) containing x_0. Choose n so large that $\left[x_0 - \frac{1}{2^n}, x_0 + \frac{1}{2^n}\right] \cap [0,1] \subset V$. It is easy to see that $s_n(x_0) \ge f(x_0) - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows that $s_n(x_0) \to f(x_0)$. Now suppose $f(x_0) = +\infty$. For arbitrary N, define $V_N = \{x \in [0,1] : f(x) \ge N\}$. Then V_N is an open set containing x_0. Choose n so large that $n \ge N$ and $\left[x_0 - \frac{1}{2^n}, x_0 + \frac{1}{2^n}\right] \cap [0,1] \subset V_N$. Then $s_n(x_0) \ge N$. Since N was arbitrary, this shows that $s_n(x_0) \to +\infty$. Q.E.D.

<u>(D.6) Integration Conventions</u>. The integrals appearing in this appendix are Lebesgue integrals. Let $f : \mathbb{R} \to \mathbb{R}^e$ be a measurable function, and define $f^+ \triangleq \max(f,0)$, $f^- \triangleq \max(-f,0)$. Let $E \subset \mathbb{R}$ be a measurable set. The function $f(\cdot)$ is said to be <u>integrable in the extended sense over E</u>, and $\int_E f$ is assigned the value

$$\int_E f = \int_E f^+ - \int_E f^-, \tag{D-1}$$

provided that at least one of the two integrals on the right-hand side of (D-1) is finite.

<u>(D.7) Lemma (Change of Variable)</u>. Let $g : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a monotonic piecewise C^1 function. Set $c = \min\{g(a),g(b)\}$, $d = \max\{g(a),g(b)\}$. Suppose $f : [c,d] \to \mathbb{R}^e$ is either (a) upper semicontinuous and not equal to $+\infty$ at any point of $[c,d]$, or (b) lower semicontinuous and not equal to $-\infty$ at any point of $[c,d]$. Then

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(t))g'(t)dt. \tag{D-2}$$

Remarks. The integral is understood in the extended sense: $\int_{g(a)}^{g(b)} f(x)dx = +\infty$ (resp., $-\infty$) if and only if $\int_a^b f(g'(t))g'(t)dt = +\infty$ (resp., $-\infty$). The lemma holds more generally when $g(\cdot)$ is a monotonically increasing absolutely continuous function and $f(\cdot)$ is a nonnegative measurable function [9]. Another version of the lemma, familiar from basic calculus, shows that (D-2) holds if $g(\cdot)$ is continuously differentiable (C^1) and $f(\cdot)$ is continuous on the range of $g(\cdot)$ [27].

Proof. We shall only prove the case when $g(\cdot)$ is monotonically increasing. The other case can be derived from this. Since the integration can be broken down into a finite number of intervals in $[a,b]$ where $g(\cdot)$ is C^1, it can be assumed without loss of generality that $g(\cdot)$ is C^1. Let $[c_0,d_0] \subset [c,d]$, and let $(a_0,b_0) = g^{-1}[(c_0,d_0)]$. Define a step function $s(\cdot)$ on $[c,d]$ by setting $s(x) = 1$ for $x \in [c_0,d_0]$ and $s(x) = 0$ for $x \notin [c_0,d_0]$. Then

$$\int_c^d s(x)dx = d_0 - c_0 = g(b_0) - g(a_0) = \int_{a_0}^{b_0} g'(t)dt$$

$$= \int_a^b s(g(t))g'(t)dt. \tag{D-3}$$

It follows from (D-3) that (D-2) holds for any step function. Let $\langle s_n \rangle$ be a sequence of step functions as described in Lemma (D.5). Then since $g'(t) \geq 0$ for all $t \in [a,b]$, the function given by $s_n(g(t))g'(t)$ approaches monotonically the function given by $f(g(t))g'(t)$. It follows from (D-3) and the Monotone Convergence Theorem [9] that

$$\int_c^d f(x)dx = \lim_{n\to\infty} \int_c^d s_n(x)dx = \lim_{n\to\infty} \int_c^d s_n(g(t))g'(t)dt$$

$$= \int_c^d \lim_{n\to\infty} s_n(g(t))g'(t)dt$$

$$= \int_c^d f(g(t))g'(t)dt.$$

Q.E.D.

(D.8) Lemma. $\underline{h}(\cdot)$ is lower semicontinuous and $\bar{h}(\cdot)$ is upper semicontinuous (see Def. 18 and Def. (D.1)).

Proof. We shall prove that $\bar{h}(\cdot)$ is upper semicontinuous. The proof of the other assertion is similar. For every $\underset{\sim}{u} \in U$, define $\bar{h}_{\underset{\sim}{u}} : \Sigma \to \mathbb{R}^e$ by

$$\bar{h}_{\underset{\sim}{u}}(x) \triangleq \begin{cases} \dfrac{p(x,\underset{\sim}{u})}{f(x,\underset{\sim}{u})}, & \underset{\sim}{u} \in U_x^+ \\ +\infty, & \underset{\sim}{u} \in U \setminus U_x^+ . \end{cases}$$

It is clear that $\bar{h}(x) = \inf\{\bar{h}_{\underset{\sim}{u}}(x) : \underset{\sim}{u} \in U\}$. Let $\underset{\sim}{u} \in U$ and $\alpha \in \mathbb{R}$ be fixed. Then

$$\{x \in \Sigma : \bar{h}_{\underset{\sim}{u}}(x) < \alpha\} = \{x \in \Sigma : f(x,\underset{\sim}{u}) > 0\} \cap \{x \in \Sigma : p(x,\underset{\sim}{u}) - \alpha f(x,\underset{\sim}{u}) < 0\}. \tag{D-4}$$

From the continuity of the functions $f(\cdot,\cdot)$ and $p(\cdot,\cdot)$, both sets on the right-hand side of (D-4) are open, and thus their intersection is open. Hence $\bar{h}(\cdot)$ is the infimum of a collection of upper semicontinuous functions. It follows from Lemma (D.2) that $\bar{h}(\cdot)$ is upper semicontinuous. Q.E.D.

The functions $\underline{h}(\cdot)$ and $\bar{h}(\cdot)$ are continuous in the special case when $\underline{h}(x) = \bar{h}(x)$ for all $x \in \Sigma$ (this follows from Lemma (D.8) and the comments in Def. (D.1)). In general, however, neither $\underline{h}(\cdot)$ nor $\bar{h}(\cdot)$ will be continuous. The following example shows that these functions can be quite bizarre.

(D.9) Example. Let $\langle r_n \rangle_{n=1}^{\infty}$ be any enumeration of the rational numbers. Consider the system

$$\dot{x} = u^3$$

$$y = u^2 \operatorname{sgn}(u) \sum_{n=1}^{\infty} \frac{1}{2^n} \exp(-u^2(x-r_n)^2),$$

where u and y form a hybrid pair and where $\operatorname{sgn}(u) \triangleq \dfrac{u}{|u|}$, for $u \neq 0$, and $\operatorname{sgn}(0) \triangleq 0$. This system is passive because $uy \geq 0$ always. It follows from (4-6) and (4-7) that

$$\underline{h}(x) = \begin{cases} 0, & x \text{ irrational} \\ -\dfrac{1}{2^n}, & x = r_n \end{cases}$$

since the supremum in (4-6) is found by letting $u \to -\infty$, and

$$\bar{h}(x) = \begin{cases} 0, & x \text{ irrational} \\ \dfrac{1}{2^n}, & x = r_n \end{cases}$$

since the infimum in (4-7) can be found by letting $u \to +\infty$. Notice that both $\bar{h}(\cdot)$ and $\underline{h}(\cdot)$ are discontinuous at each rational number. It can be verified that $\bar{h}(\cdot)$ is upper semicontinuous and $\underline{h}(\cdot)$ is lower semicontinuous; moreover, note that $\underline{h}(x) \leq \bar{h}(x)$ for all x (this is one of the conditions for passivity listed in Theorem 7).

(D.10) Definitions. For every $x_0 \in \Sigma$, let $R(x_0)$ denote the set of states reachable from x_0. Let $R^-(x_0) \triangleq \{x \in R(x_0) : x < x_0\}$ and $R^+(x_0) \triangleq \{x \in R(x_0) : x > x_0\}$. Note that $R(x_0) = R^-(x_0) \cup \{x_0\} \cup R^+(x_0)$, a disjoint union. There are four possibilities for $R^-(x_0)$: $R^-(x_0) = (-\infty, x_0)$; $R^-(x_0) = \phi$; $R^-(x_0) = (a, x_0)$; and $R^-(x_0) = [a, x_0)$; where $-\infty < a < x_0$. We emphasize that we are considering only states which are reachable in finite time. The example $\dot{x} = -u^2 x^{1/3}$ shows that the fourth possibility, a closed end point, can indeed occur (this system can be driven to the singular point $x = 0$ in finite time). Likewise, $R^+(x_0)$ can have one of four forms: $R^+(x_0) = (x_0, +\infty)$; $R^+(x_0) = \phi$; $R^+(x_0) = (x_0, b)$; and $R^+(x_0) = (x_0, b]$; where $x_0 < b < +\infty$.

For any set $B \subset \mathbb{R}$, let int B denote the set of interior points in B.

(D.11) Lemma. Let $\underline{h}(\cdot)$ and $\bar{h}(\cdot)$ be as defined in Def. 18. Then for each $x_0 \in \Sigma$,

$$\text{(i)} \quad \inf_{\substack{x_0 \to x_1 \\ \dot{x}<0 \\ \hat{t}>0}} \left\{ \int_0^{\hat{t}} p(x(t), \underset{\sim}{u}(t))dt \right\} = \int_{x_0}^{x_1} \underline{h}(x)dx, \quad \text{for all } x_1 \in \text{int } R^-(x_0)$$

$$\text{(ii)} \quad \inf_{\substack{x_0 \to x_2 \\ \dot{x}>0 \\ \hat{t}>0}} \left\{ \int_0^{t} p(x(t), \underset{\sim}{u}(t))dt \right\} = \int_{x_0}^{x_2} \bar{h}(x)dx, \quad \text{for all } x_2 \in \text{int } R^+(x_0)$$

where the infimum in (i) is taken over all $\hat{t} > 0$ and all input-trajectory pairs $\{\underset{\sim}{u}(\cdot), x(\cdot)\}|[0,\hat{t}]$ from x_0 to x_1 with $\dot{x}(t) < 0$ for almost all $t \in [0,\hat{t}]$, and the infimum in (ii) is taken over all $\hat{t} > 0$ and all input-trajectory pairs $\{\underset{\sim}{u}(\cdot), x(\cdot)\}|[0,\hat{t}]$ from x_0 to x_2 with $\dot{x}(t) > 0$ for almost all $t \in [0,\hat{t}]$.

Remarks. The phrase "almost all $t \in [0,\hat{t}]$" means for all $t \in [0,\hat{t}]$ except for some t which lie in a set of Lebesgue measure zero.

The integral on the right-hand side of (i) may be slightly confusing. Since $x_1 < x_0$, $\int_{x_0}^{x_1} \underline{h}$ will be negative if $\underline{h}(\cdot)$ is positive on the interval $[x_1, x_0]$.

Proof. We shall prove (ii) only. The proof of (i) is similar. Let $x_0 \in \Sigma$ and let $x_2 \in \text{int } R^+(x_0)$. By Lemma (D.8), $\bar{h}(\cdot)$ is upper semicontinuous. Since x_2 is in the interior of $R^+(x_0)$, it follows from Def. 18 that $\bar{h}(x) \neq +\infty$ for all $x \in [x_0, x_2]$. It follows from Lemma (D.3) that $\bar{h}(\cdot)$ is bounded above on the compact interval $[x_0,x_2]$. Thus $\int_{x_0}^{x_2} \bar{h}(x)dx$ exists in the extended sense (see Integration Conventions (D.6)).

Now let $\{\underset{\sim}{u}(\cdot),x(\cdot)\}|[0,\hat{t}]$ be an input-trajectory pair from x_0 to x_2 such that $\dot{x}(t) > 0$ for almost all $t \in [0,\hat{t}]$. It follows from Def. 18 that

$$p(x(t),\underset{\sim}{u}(t)) \geq \bar{h}(x(t))\dot{x}(t), \text{ almost all } t \in [0,\hat{t}]. \tag{D-5}$$

Thus

$$\int_0^{\hat{t}} p(x(t),\underset{\sim}{u}(t))dt \geq \int_0^{\hat{t}} \bar{h}(x(t))\dot{x}(t)dt = \int_{x_0}^{x_2} \bar{h}(x)dx. \tag{D-6}$$

The use of the change of variables formula in (D-6) is justified by Lemma (D.7), since $x(\cdot)$ is monotonically increasing and piecewise C^1, and $\bar{h}(\cdot)$ is upper semicontinuous and not equal to $+\infty$ at any point on the interval $[x_0,x_2]$. The fact that $x(\cdot)$ is piecewise C^1 follows since U is closed, $f(\cdot,\cdot)$ is continuous on $\Sigma\times U$, and the elements of $\mathcal{U}$ are piecewise continuous. In light of (D-6), it only remains to show the following:

Case (a). $\int_{x_0}^{x_2} \bar{h}(x)dx > -\infty$. Then given $\varepsilon > 0$, there must exist an input-trajectory pair $\{\underset{\sim}{u}(\cdot),x(\cdot)\}|[0,\hat{t}]$ from x_0 to x_2 such that $\dot{x}(t) > 0$ for almost all $t \in [0,\hat{t}]$, which satisfies

$$\int_0^{\hat{t}} p(x(t),\underset{\sim}{u}(t))dt \leq \int_{x_0}^{x_2} \bar{h}(x)dx + \varepsilon.$$

Case (b). $\int_{x_0}^{x_2} \bar{h}(x)dx = -\infty$. Then given $K > 0$, there must exist an input-trajectory pair $\{\underset{\sim}{u}(\cdot),x(\cdot)\}|[0,\hat{t}]$ from x_0 to x_2 such that $\dot{x}(t) > 0$ for almost $t \in [0,\hat{t}]$, which satisfies

$$\int_0^{\hat{t}} p(x(t),\underset{\sim}{u}(t))dt \leq -K.$$

We shall prove Cases (a) and (b) simultaneously by constructing a piecewise constant "nearly optimal" control law.

Let $\langle s_n \rangle$ be a sequence of upper semicontinuous step functions as in Lemma (D.5) which satisfy $M \geq s_1 \geq s_2 \geq \dots \geq \bar{h}$ and $\lim_{n\to\infty} s_n(x) = \bar{h}(x)$ for each $x \in [x_0,x_2]$, where $M = \sup\{\bar{h}(x) : x \in [x_0,x_2]\}$. It follows from the Monotone Convergence Theorem [9] that

$$\int_{x_0}^{x_2} \bar{h}(x)dx = \lim_{n\to\infty} \int_{x_0}^{x_2} s_n(x)dx. \tag{D-7}$$

Let $\varepsilon > 0$. If Case (a) holds, choose N so large that

$$\int_{x_0}^{x_2} s_N(x)dx \leq \int_{x_0}^{x_2} \bar{h}(x)dx + \varepsilon. \tag{D-8}$$

Let $K > 0$. If Case (b) holds, choose N so large that

$$\int_{x_0}^{x_2} s_N(x)dx \leq -K. \tag{D-9}$$

Let $\hat{x} \in [x_0,x_2]$ be such that $s_N(\cdot)$ is continuous at $\hat{x}$. Then there exists a $\underset{\sim}{u}_{\hat{x}} \in U^+_{\hat{x}}$ and an open interval $I_{\hat{x}}$ containing $\hat{x}$ such that

$$f(x,\underset{\sim}{u}_{\hat{x}}) > 0 \quad \text{for all } x \in I_{\hat{x}} \cap [x_0,x_2] \tag{D-10a}$$

$$\frac{p(x,\underset{\sim}{u}_{\hat{x}})}{f(x,\underset{\sim}{u}_{\hat{x}})} < s_N(x) + \varepsilon \quad \text{for all } x \in I_{\hat{x}} \cap [x_0,x_2]. \tag{D-10b}$$

Thus we have associated an interval $I_{\hat{x}}$ and an input value $\underset{\sim}{u}_{\hat{x}}$ with each point $\hat{x} \in [x_0,x_2]$ where $s_N(\cdot)$ is continuous. Now consider the points where $s_N(\cdot)$ is not continuous. Let $\{a_i : 0 \leq i \leq p\}$ denote the partition of $[x_0,x_2]$ associated with $s_N(\cdot)$. Specifically, $x_0 = a_0 < a_1 < \ldots < a_p = x_2$, $s_N(\cdot)$ is constant on each interval (a_{i-1},a_i), and $s_N(\cdot)$ is continuous from either the left or the right at each end point a_i. Let b_i denote the value of $s_N(\cdot)$ on (a_{i-1},a_i), $i = 1,2,\ldots,p$, and let $\Delta = \min\{a_i-a_{i-1}, i = 1,2,\ldots,p\}$. To each point a_i associate an interval of the form $I_{a_i} \triangleq (a_i-\alpha\varepsilon, a_i+\alpha\varepsilon)$, where $\alpha > 0$. If $\alpha > 0$ is chosen sufficiently small, then the following conditions can be satisfied: $\frac{1}{2}\Delta > \alpha\varepsilon > 0$, and for each a_i, $i = 1,2,\ldots,p-1$, there exists $\underset{\sim}{u}_{a_i} \in U^+_{a_i}$ such that

$$f(x,\underset{\sim}{u}_{a_i}) > 0 \quad \text{for all } x \in I_{a_i} \triangleq (a_i-\alpha\varepsilon, a_i+\alpha\varepsilon) \tag{D-11a}$$

$$\frac{p(x,\underset{\sim}{u}_{a_i})}{f(x,\underset{\sim}{u}_{a_i})} < \max\{b_i,b_{i+1}\} + \varepsilon, \text{ for all } x \in I_{a_i}. \tag{D-11b}$$

Thus we have associated an open interval $I_{\hat{x}}$ and an input value $\underset{\sim}{u}_{\hat{x}} \in U^+_{\hat{x}}$ with every point $\hat{x} \in [x_0,x_2]$. The collection $\{I_{\hat{x}} : \hat{x} \in [x_0,x_2]\}$ is an open cover of the compact interval $[x_0,x_2]$. Hence there exists a finite subcover $\{I_{\hat{x}_1}, I_{\hat{x}_2}, \ldots, I_{\hat{x}_q}\}$. We may assume that $\hat{x}_1 < \hat{x}_2 < \ldots < \hat{x}_q$ and that

$$\begin{aligned} I_{\hat{x}_i} \cap I_{\hat{x}_j} &\neq \phi \text{ for } |i-j| = 1 \\ &= \phi \text{ for } |i-j| > 1. \end{aligned}$$

Now choose $c_i \in I_{\hat{x}_i} \cap I_{\hat{x}_{i+1}}$, $i = 1,2,\ldots,q-1$, and set $c_0 = x_0$, $c_q = x_2$. Define

$$\underset{\sim}{w}(x) \triangleq \underset{\sim}{u}_{\hat{x}_i}, x \in [c_{i-1}, c_i),\ i = 1,2,\ldots,q. \tag{D-12}$$

The piecewise constant control law $\underset{\sim}{w}(\cdot)$ transfers the state from x_0 at $t = 0$ to x_2 at some finite time $t = \hat{t}$. Using (D-10) and (D-11) gives

$$\int_{\substack{0 \\ x_0 \to x_2}}^{\hat{t}} p(x(t), \underset{\sim}{w}(x(t)))dt$$

$$= \int_{\substack{0 \\ x_0 \to x_2}}^{\hat{t}} \frac{p(x(t), \underset{\sim}{w}(x(t)))}{f(x(t), \underset{\sim}{w}(x(t)))} \dot{x}(t)dt$$

$$< \int_{x_0}^{a_1-\alpha\varepsilon} [s_N(x)+\varepsilon]dx + \int_{a_1+\alpha\varepsilon}^{a_2-\alpha\varepsilon} [s_N(x)+\varepsilon]dx + \ldots$$

$$\ldots + \int_{a_{p-1}+\alpha\varepsilon}^{x_2} [s_N(x)+\varepsilon]dx + \sum_{i=1}^{p-1} \int_{a_i-\alpha\varepsilon}^{a_i+\alpha\varepsilon} [\max\{b_i, b_{i+1}\}+\varepsilon]dx$$

$$= \int_{x_0}^{x_2} [s_N(x)+\varepsilon]dx + \sum_{i=1}^{p-1} \int_{a_i-\alpha\varepsilon}^{a_i+\alpha\varepsilon} [\max\{b_i, b_{i+1}\}-s_N(x)]dx. \tag{D-13}$$

Note that

$$\int_{a_i-\alpha\varepsilon}^{a_i+\alpha\varepsilon} [\max\{b_i, b_{i+1}\}-s_N(x)]dx \leq \varepsilon\alpha(M-m), \tag{D-14}$$

where $M = \sup\{\bar{h}(x) : x \in [x_0, x_2]\}$ and $m = \min\{s_N(x) : x \in [x_0, x_2]\}$. Combining (D-13) and (D-14) gives

$$\int_{\substack{0 \\ x_0 \to x_2}}^{\hat{t}} p(x(t), \underset{\sim}{w}(x(t)))dt$$

$$< \int_{x_0}^{x_2} [s_N(x)+\varepsilon]dx + \alpha\varepsilon(M-m)(p-1)$$

$$= \int_{x_0}^{x_2} s_N(x)dx + \varepsilon[(x_2-x_0) + \alpha(M-m)(p-1)]. \tag{D-15}$$

If Case (a) is satisfied, then it follows from (D-8) and (D-15) that

$$\int_{0}^{\hat{t}} p(x(t),\underset{\sim}{w}(x(t)))dt < \int_{x_0}^{x_2} \bar{h}(x)dx + \varepsilon[1+(x_2-x_0)+\alpha(M-m)(p-1)]. \tag{D-16}$$
$$x_0 \to x_2$$

Since $\varepsilon > 0$ was arbitrary and $\alpha > 0$ can be chosen arbitrarily small, this proves Case (a). If Case (b) is satisfied, then it follows from (D-9) and (D-15) that

$$\int_{0}^{\hat{t}} p(x(t),\underset{\sim}{w}(x(t)))dt < -K + \varepsilon[(x_2-x_0)+\alpha(M-m)(p-1)]. \tag{D-17}$$
$$x_0 \to x_2$$

Since $\varepsilon > 0$ and $K > 0$ were arbitrary, and since $\alpha > 0$ can be chosen arbitrarily small, this proves Case (b). Q.E.D.

<u>(D.12) Lemma</u>.

(i) For each $x_0 \in \Sigma$ and each $x_1 \in R^-(x_0)$, define

$$E(x_0,x_1) = \inf_{\substack{x_0 \to x_1 \\ \dot{x}<0 \\ \hat{t}>0}} \left\{\int_0^{\hat{t}} p(x(t),\underset{\sim}{u}(t))dt\right\}$$

where the infimum is taken over all $\hat{t} > 0$ and all input-trajectory pairs $\{\underset{\sim}{u}(\cdot),x(\cdot)\}|[0,\hat{t}]$ from x_0 to x_1 with $\dot{x}(t) < 0$ for almost all $t \in [0,\hat{t}]$. Then for each $x_0 \in \Sigma$,

$$\inf\{E(x_0,x_1): x_1 \in R^-(x_0)\} = \inf\{E(x_0,x_1): x_1 \in \text{int } R^-(x_0)\}. \tag{D-18}$$

(ii) For each $x_0 \in \Sigma$ and each $x_2 \in R^+(x_0)$, define

$$E(x_0,x_2) \triangleq \inf_{\substack{x_0 \to x_2 \\ \dot{x}>0 \\ \hat{t}>0}} \left\{\int_0^{t} p(x(t),\underset{\sim}{u}(t))dt\right\}$$

where the infimum is taken over all $\hat{t} > 0$ and all input-trajectory pairs $\{\underset{\sim}{u}(\cdot),x(\cdot)\}|[0,\hat{t}]$ from x_0 to x_2 with $\dot{x}(t) > 0$ for almost all $t \in [0,\hat{t}]$. Then for each $x_0 \in \Sigma$,

$$\inf\{E(x_0,x_2): x_2 \in R^+(x_0)\} = \inf\{E(x_0,x_2):x_2 \in \text{int } R^+(x_0)\}. \tag{D-19}$$

<u>Proof</u>. We shall prove (ii) only. The proof of (i) is similar. Clearly, the left-hand side of (D-19) is smaller than or equal to the right-hand side. We need only show the opposite inequality. Suppose $R^+(x_0) = (x_0,b]$, where $x_0 < b < +\infty$. Let $\{\underset{\sim}{u}(\cdot),x(\cdot)\}|[0,\hat{t}]$ be an input-trajectory pair from x_0 to b with $\dot{x}(t) > 0$ for almost all $t \in [0,\hat{t}]$. Then $x(\cdot)$ is strictly monotonic over the interval $[0,\hat{t}]$. Define, for all $t \in [0,\hat{t}]$,

$$M_{\underset{\sim}{u}(\cdot)}(x_0,x(t)) \triangleq \int_0^t p(x(\tau),\underset{\sim}{u}(\tau))d\tau . \tag{D-20}$$

Note that $M_{\underset{\sim}{u}(\cdot)}(x_0,\cdot)$ is a well-defined function on the interval $(x_0,b]$. For every $x_2 \in (x_0,b]$,

$$E(x_0,x_2) \leq M_{\underset{\sim}{u}(\cdot)}(x_0,x_2); \tag{D-21}$$

moreover, $M_{\underset{\sim}{u}(\cdot)}(x_0,\cdot)$ is a continuous function mapping $(x_0,b]$ to $\mathbb{R}$; thus it follows from (D-21) that

$$\overline{\lim_{x_2 \to b^-}} E(x_0,x_2) \leq M_{\underset{\sim}{u}(\cdot)}(x_0,b). \tag{D-22}$$

Taking the infimum over all such inputs $\underset{\sim}{u}(\cdot)$ gives

$$\overline{\lim_{x_2 \to b^-}} E(x_0,x_2) \leq E(x_0,b). \tag{D-23}$$

Q.E.D.

(D.13) Observation. If the system described in subsection (4.5) is passive, then

$$\oint_0^{\hat{t}} p(x(t),\underset{\sim}{u}(t))dt \geq 0,$$

where the notation $\oint_0^{\hat{t}}$ indicates that the input-trajectory pair $\{\underset{\sim}{u}(\cdot),x(\cdot)\}$ satisfies $x(0) = x(\hat{t})$. For if this were not true, then we could extract an unbounded amount of energy by driving the system repeatedly in a loop. Obviously, this observation remains valid for the general m-dimensional system.

(D.14) Proof of Theorem 7. (Necessity). Suppose that the system is passive. To prove (i), suppose that there exists some $(x_0,\underset{\sim}{u}_0) \in \Sigma \times U$ such that $f(x_0,\underset{\sim}{u}_0) = 0$. Now $\{\underset{\sim}{u}_0,x_0\}|[0,t]$ is valid input-trajectory pair (from x_0 to x_0) for all $t \geq 0$. Thus, from Observation (D.13),

$$0 \leq \oint_0^t p(x_0,\underset{\sim}{u}_0)d\tau = p(x_0,\underset{\sim}{u}_0)t, \text{ for } t \geq 0. \tag{D-24}$$

It follows from (D-24) that $p(x_0,\underset{\sim}{u}_0) \geq 0$.

Note that (ii) is trivially satisfied anywhere that $U_x^- = \phi$ and/or $U_x^+ = \phi$. To obtain a contradiction, assume that there exists an $x_0 \in \Sigma$ such that $U_{x_0}^- \neq \phi$, $U_{x_0}^+ \neq \phi$, and

$$\underline{h}(x_0) > \bar{h}(x_0).$$

Then there exists $\underset{\sim}{u}_1 \in U^+_{x_0}$ and $\underset{\sim}{u}_2 \in U^-_{x_0}$ such that

$$\frac{p(x_0,\underset{\sim}{u}_2)}{f(x_0,\underset{\sim}{u}_2)} > \frac{p(x_0,\underset{\sim}{u}_1)}{f(x_0,\underset{\sim}{u}_1)} .$$

By continuity, there exists $\delta > 0$ such that

$$f(x,\underset{\sim}{u}_1) > 0 \quad \text{for every } x \in [x_0,x_0+\delta]$$

$$f(x,\underset{\sim}{u}_2) < 0 \quad \text{for every } x \in [x_0,x_0+\delta]$$

$$\frac{p(x,\underset{\sim}{u}_2)}{f(x,\underset{\sim}{u}_2)} > \frac{p(x,\underset{\sim}{u}_1)}{f(x,\underset{\sim}{u}_1)} \quad \text{for every } x \in [x_0,x_0+\delta].$$

Hence the constant input value $\underset{\sim}{u}_1$ transfers the system from state x_0 at $t = 0$ to state $x_0 + \delta$ at some finite time $t_1 > 0$. The constant input value $\underset{\sim}{u}_2$ transfers the system from state $x_0 + \delta$ at time t_1 to state x_0 at some finite time $t_2 > t_1$. Define an input $\underset{\sim}{u}(\cdot)$ as follows:

$$\underset{\sim}{u}(t) \triangleq \underset{\sim}{u}_1, \; t \in [0,t_1]$$

$$\qquad \triangleq \underset{\sim}{u}_2, \; t \in (t_1,t_2].$$

Then if $x(0) = x_0$, $\{\underset{\sim}{u}(\cdot),x(\cdot)\}|[0,t_2]$ is an input-trajectory pair from x_0 to x_0. We have

$$\oint_0^{t_2} p(x(t),\underset{\sim}{u}(t))dt = \int_0^{t_1} \frac{p(x(t),\underset{\sim}{u}_1)}{f(x(t),\underset{\sim}{u}_1)}\dot{x}(t)dt + \int_{t_1}^{t_2} \frac{p(x(t),\underset{\sim}{u}_2)}{f(x(t),\underset{\sim}{u}_2)}\dot{x}(t)dt$$

$$= \int_{x_0}^{x_0+\delta} \frac{p(x,\underset{\sim}{u}_1)}{f(x,\underset{\sim}{u}_1)} dx + \int_{x_0+\delta}^{x_0} \frac{p(x,\underset{\sim}{u}_2)}{f(x,\underset{\sim}{u}_2)} dx = \int_{x_0}^{x_0+\delta} \left[\frac{p(x,\underset{\sim}{u}_1)}{f(x,\underset{\sim}{u}_1)} - \frac{p(x,\underset{\sim}{u}_2)}{f(x,\underset{\sim}{u}_2)}\right] dx < 0.$$

The last inequality follows because the integrand is strictly negative on the interval $[x_0,x_0+\delta]$. Observation (D.13) shows that the system is active. This contradiction shows that condition (ii) must be satisfied.

Finally, if we choose $W(x_0) = E_A(x_0)$, the available energy function (Def. 10), then condition (iii) follows trivially from Lemma (D.11).

(Sufficiency) Suppose that conditions (i), (ii), and (iii) are satisfied. It follows from conditions (i), (ii) and Def. 18 that

$$p(x,\underset{\sim}{u}) \geq \underline{h}(x)f(x,\underset{\sim}{u}) \tag{D-25a}$$

$$p(x,\underset{\sim}{u}) \geq \bar{h}(x)f(x,\underset{\sim}{u}) \tag{D-25b}$$

where we use the convention $0 \cdot (\pm\infty) = 0$ when multiplying extended real numbers. We shall call an input-trajectory pair $\{\underset{\sim}{u}(\cdot),x(\cdot)\}|[0,\hat{t}]$ a <u>simple loop between x_0 and x_1</u> if $x(0) = x(\hat{t}) = x_0$, $x(\cdot)$ is monotonic on the intervals $[0,t_1]$ and $[t_1,\hat{t}]$ for some $t_1 \in [0,\hat{t}]$, and $x(t_1) = x_1$. Suppose that there exist states $x_0,x_1 \in \Sigma$ such that $x_1 \in R^+(x_0)$ and $x_0 \in R^-(x_1)$. Hence there exists a simple loop between x_0 and x_1. Moreover, from Lemma (D.3) and condition (ii), both $\underline{h}(\cdot)$ and $\bar{h}(\cdot)$ are bounded above and below on $[x_0,x_1]$. Let $\{\underset{\sim}{u}(\cdot),x(\cdot)\}|[0,\hat{t}]$ be a simple loop between x_0 and x_1 with $x(t_1) = x_1$. From (D-25) we obtain

$$\int_0^{\hat{t}} p(x(t),\underset{\sim}{u}(t))dt = \int_0^{t_1} p(x(t),\underset{\sim}{u}(t))dt + \int_{t_1}^{\hat{t}} p(x(t),\underset{\sim}{u}(t))dt$$

$$\geq \int_0^{t_1} \bar{h}(x(t))\dot{x}(t)dt + \int_{t_1}^{\hat{t}} \underline{h}(x(t))\dot{x}(t)dt = \int_{x_0}^{x_1} \bar{h}(x)dx + \int_{x_1}^{x_0} \underline{h}(x)dx$$

$$= \int_{x_0}^{x_1} [\bar{h}(x)-\underline{h}(x)]dx \geq 0.$$

The use of the change of variables is justified by Lemma (D.7) because $x(\cdot)$ is piecewise C^1 and monotonic on $[0,t_1]$ and $[t_1,\hat{t}]$. The fourth step is justified because $\bar{h}(\cdot)$ and $\underline{h}(\cdot)$ are bounded on $[x_0,x_1]$. The final step follows from condition (ii). We have shown that the energy consumed along a simple loop is nonnegative.

Now consider the available energy function (Def. 10):

$$E_A(x_0) = \sup_{\substack{x_0\to \\ \hat{t}\geq 0}} \left\{- \int_0^{\hat{t}} p(x(t),\underset{\sim}{u}(t))dt\right\} = -\inf_{\substack{x_0\to \\ \hat{t}\geq 0}}\left\{\int_0^{\hat{t}} p(x(t),\underset{\sim}{u}(t))dt\right\}. \tag{D-26}$$

If $R^-(x_0) = R^+(x_0) = \phi$, then $f(x_0,\underset{\sim}{u}) = 0$ for all $\underset{\sim}{u} \in U$. It follows trivially from condition (i) that $E_A(x_0) = 0$. Now suppose that $R^-(x_0) \neq \phi$ and/or $R^+(x_0) \neq \phi$. To calculate $E_A(x_0)$, the argument above shows that it is only necessary to consider input-trajectory pairs $\{\underset{\sim}{u}(\cdot),x(\cdot)\}$ for which the state trajectory $x(\cdot)$ is monotonic, because a nonmonotonic state trajectory would necessarily contain simple loops, and it has been shown that simple loops give a nonnegative contribution to the energy integral. Moreover, we may assume that $\dot{x}(t) \neq 0$ for almost all t, since condition (i) shows that $p(x(t),\underset{\sim}{u}(t)) \geq 0$ wherever $\dot{x}(t) = 0$. Therefore, if $E(\cdot,\cdot)$ is the function defined in Lemma (D.12), then

$$E_A(x_0) = \max\left\{\sup\left\{-E(x_0,x_1): x_1 \in R^-(x_0)\right\}, \sup\left\{-E(x_0,x_2): x_2 \in R^+(x_0)\right\}\right\}. \tag{D-27}$$

Using the conclusion of Lemma (D.12) in (D-27) gives

$$E_A(x_0) = \max\left\{\sup\left\{-E(x_0,x_1): x_1 \in \text{int } R^-(x_0)\right\}, \sup\left\{-E(x_0,x_2): x_2 \in \text{int } R^+(x_0)\right\}\right\}. \tag{D-28}$$

Using Lemma (D.11) in (D-28) gives

$$E_A(x_0) = \max\left\{\sup\left\{-\int_{x_0}^{x_1} \underline{h}(x)dx: x_1 \in \text{int } R^-(x_0)\right\}, \sup\left\{-\int_{x_0}^{x_2} \bar{h}(x)dx: x_2 \in \text{int } R^+(x_0)\right\}\right\}. \tag{D-29}$$

Finally, substituting condition (iii) into (D-29) gives $E_A(x_0) \leq W(x_0) < +\infty$. Therefore the system is passive. Q.E.D.

(D.15) Proof of the Corollary for Theorem 7.

(Sufficiency). Condition (i) trivially implies condition (i) of Theorem 7. Also, condition (i) and Def. 18 imply that $\underline{h}(x) \leq h(x) \leq \bar{h}(x)$, which gives condition (ii) of Theorem 7. Finally, the condition $\underline{h}(x) \leq h(x) \leq \bar{h}(x)$ along with (ii) gives condition (iii) of Theorem (7), with $W(x_0) = E(x_0)$. Therefore the system is passive.

(Necessity). Suppose that the system is passive. Since Σ is open and the system is completely controllable, it follows from Lemma (D.3) and condition (ii) of Theorem 7 that $\underline{h}(\cdot)$ and $\bar{h}(\cdot)$ are bounded above and below on every compact set in Σ. Fix $z_0 \in \Sigma$ and define $W_1 : \Sigma \to \mathbb{R}^+$ by

$$W_1(x_0) \triangleq \begin{cases} \sup\left\{\int_z^{x_0} \bar{h}(x)dx : z \in [z_0,x_0]\right\}, & x_0 \geq z_0 \\ -\inf\left\{\int_{x_0}^{z} \underline{h}(x)dx: z \in [x_0,z_0]\right\}, & x_0 \leq z_0. \end{cases} \tag{D-30}$$

Now define $h : \Sigma \to \mathbb{R}$ as follows:

$$h(x) \triangleq \begin{cases} \underline{h}(x), & \text{for } x < z_0 \\ \bar{h}(x), & \text{for } x \geq z_0, \end{cases} \tag{D-31}$$

and define $E : \Sigma \to \mathbb{R}^+$ by

$$E(x_0) \triangleq W(x_0) + W(z_0) + W_1(x_0) \tag{D-32}$$

where $W_1(\cdot)$ is defined in (D-30) and $W(\cdot)$ is the function appearing in Theorem 7. From Def. 18 and conditions (i) and (ii) of Theorem 7, it follows that $p(x,\underset{\sim}{u}) \geq f(x,\underset{\sim}{u})\underline{h}(x)$ and $p(x,\underset{\sim}{u}) \geq f(x,\underset{\sim}{u})\bar{h}(x)$. Therefore $h(\cdot)$, as defined in (D-31), satisfies condition (i). It is straightforward to verify that $E(\cdot)$, defined in (D-32), satisfies condition (ii). Q.E.D.

Appendix E: Five Ways that an Interconnection Can Fail to be Admissible

1) Some interconnections are forbidden altogether, e.g., a parallel connection of a 1-volt source and a 2-volt source.

2) Some interconnections produce state constraints so that Σ will not be all of $\Sigma_1 \times \ldots \times \Sigma_k$. For example, $\mathcal{N}$ might contain a loop of 1-farad capacitors, each of which is viewed as a 1-port with state q_i, $1 \leq i \leq j$, $j \leq k$. Then by KVL a state $(q_1,\ldots,q_j)$ can occur only if $q_1 + \ldots + q_j = 0$.

3) The most natural state representation for $\mathcal{N}$ may lump together distinct states in $\Sigma_1 \times \ldots \times \Sigma_k$. Suppose for example that $\mathcal{N}_1$ and $\mathcal{N}_2$ are 1-farad capacitors and $\mathcal{N}$ is the 1-port consisting of $\mathcal{N}_1$ and $\mathcal{N}_2$ in series. Then it is natural to consider $\mathcal{N}$ as a single capacitor with a 1-dimensional state space by identifying (q_1,q_2) and (q_1',q_2') in $\Sigma_1 \times \Sigma_2$ if $q_1 + q_2 = q_1' + q_2'$.

4) It is possible that the interconnection makes sense as an electric circuit but simply has no state representation at all. Suppose for example that $\mathcal{N}_1$ is the active resistor characterized by $v = i^3 - i$, $\mathcal{N}_2$ is a 1-farad capacitor, and $\mathcal{N}$ is the 1-port consisting of $\mathcal{N}_1$ and $\mathcal{N}_2$ in parallel. Then no current-controlled or voltage-controlled state equations exist for $\mathcal{N}$.

5) An interconnection of n-ports, all of which satisfy the standing assumptions in Section II, might violate those assumptions by, for example, having finite escape time.

Appendix F: Proof of Lemma 6. It follows from Def. 10 that $E_A(\cdot)$ is nonnegative. Since $\mathcal{U}$ is translation invariant and since the state equations are time-invariant, it suffices to show that for any input-trajectory pair $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}$ and any $T \geq 0$,

$$E_A(\underset{\sim}{x}(0)) \geq \int_0^T -p(\underset{\sim}{x}(t),\underset{\sim}{u}(t))dt + E_A(\underset{\sim}{x}(T)). \tag{F-1}$$

Let $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}$ be any input-trajectory pair, let $\underset{\sim}{x}_0 = \underset{\sim}{x}(0)$, and let $\underset{\sim}{x}_1 = \underset{\sim}{x}(T)$. It follows from Def. 10 that for each $\varepsilon > 0$ there exists an input $\underset{\sim}{u}'(\cdot) \in \mathcal{U}$ and a $T' \geq 0$ such that

$$\int_0^{T'} -p(\underset{\sim}{x}'(t),\underset{\sim}{u}'(t))dt \geq E_A(\underset{\sim}{x}_1) - \varepsilon \tag{F-2}$$

where $\{\underset{\sim}{u}'(\cdot),\underset{\sim}{x}'(\cdot)\}$ is an input-trajectory pair with initial state $\underset{\sim}{x}_1$. Let the waveform $\underset{\sim}{u}''(\cdot)$ be defined by

$$u''(t) = \begin{cases} \underset{\sim}{u}(t), & 0 \leq t \leq T \\ u'(t-T), & T < t. \end{cases}$$

Then $\underset{\sim}{u}''(\cdot) \in \mathcal{U}$, since $\mathcal{U}$ is closed under concatenation. Let $\underset{\sim}{x}''(\cdot)$ be the state-space trajectory produced by applying $\underset{\sim}{u}''(\cdot)$ with initial state $\underset{\sim}{x}_0$. Then

$$E_A(\underset{\sim}{x}_0) \geq \int_0^{T+T'} -p(\underset{\sim}{x}''(t),\underset{\sim}{u}''(t))dt = \int_0^T -p(\underset{\sim}{x}(t),\underset{\sim}{u}(t))dt + \int_0^{T'} -p(\underset{\sim}{x}'(t),\underset{\sim}{u}'(t))dt$$

$$\geq \int_0^T -p(\underset{\sim}{x}(t),\underset{\sim}{u}(t))dt + E_A(\underset{\sim}{x}_1) - \varepsilon. \tag{F-3}$$

Since (F-3) holds for each $\varepsilon > 0$, this proves (F-1). Q.E.D.

Appendix G: Proof of Lemma 7.

It follows from property (v) listed in the paragraph preceding Lemma 7 that $\omega_{\underset{\sim}{x}_0}(\cdot)$ is nonnegative. Since $\omega_{\underset{\sim}{x}_0}(\underset{\sim}{x}_2) - \omega_{\underset{\sim}{x}_0}(\underset{\sim}{x}_1) = E_{R\underset{\sim}{x}_0}(\underset{\sim}{x}_2) - E_{R\underset{\sim}{x}_0}(\underset{\sim}{x}_1)$, since $\mathcal{U}$ is translation invariant, and since the state equations are time invariant, it suffices to show that for any input-trajectory pair $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}$ and any $T \geq 0$,

$$E_{R\underset{\sim}{x}_0}(\underset{\sim}{x}(T)) - E_{R\underset{\sim}{x}_0}(\underset{\sim}{x}(0)) \leq \int_0^T p(x(t),u(t)dt. \tag{G-1}$$

Let $\{\underset{\sim}{u}(\cdot),\underset{\sim}{x}(\cdot)\}$ be any input-trajectory pair and $T \geq 0$ be any time. To avoid confusion later, let $\underset{\sim}{x}_1$ stand for $\underset{\sim}{x}(0)$ and $\underset{\sim}{x}_2$ for $\underset{\sim}{x}(T)$. By Def. 24, for any $\varepsilon > 0$ there exists an input-trajectory pair $\{\underset{\sim}{u}'(\cdot),\underset{\sim}{x}'(\cdot)\}$ and a time $T' \geq 0$ such that $\underset{\sim}{x}'(0) = \underset{\sim}{x}_0, \underset{\sim}{x}'(T') = \underset{\sim}{x}_1$, and

$$\int_0^{T'} p(\underset{\sim}{x}'(t),\underset{\sim}{u}'(t))dt \leq E_{R\underset{\sim}{x}_0}(\underset{\sim}{x}_1) + \varepsilon. \tag{G-2}$$

Let the waveform $\underset{\sim}{u}''(\cdot)$ be given by

$$\underset{\sim}{u}''(t) = \begin{cases} \underset{\sim}{u}'(t), & 0 \leq t \leq T' \\ \underset{\sim}{u}(t-T'), & T' < t. \end{cases}$$

Then $\underset{\sim}{u}''(\cdot) \in \mathcal{U}$, since $\mathcal{U}$ is closed under concatenation. Let $\underset{\sim}{x}''(\cdot)$ be the trajectory produced by applying the input $\underset{\sim}{u}''(\cdot)$ when the system is in initial state $\underset{\sim}{x}_0$. Then $\underset{\sim}{x}''(T') = \underset{\sim}{x}_1$, $\underset{\sim}{x}''(T'+T) = \underset{\sim}{x}_2$, and

$$E_{R\underset{\sim}{x}_0}(\underset{\sim}{x}_2) \leq \int_0^{T'+T} p(\underset{\sim}{x}''(t),\underset{\sim}{u}''(t))dt = \int_0^{T'} p(\underset{\sim}{x}'(t),\underset{\sim}{u}'(t))dt$$

$$+ \int_0^{T} p(\underset{\sim}{x}(t),\underset{\sim}{u}(t))dt \leq E_{R\underset{\sim}{x}_0}(\underset{\sim}{x}_1) + \varepsilon + \int_0^{T} p(\underset{\sim}{x}(t),\underset{\sim}{u}(t))dt. \qquad \text{(G-3)}$$

Since (G-3) holds for each $\varepsilon > 0$, this proves (G-1). Q.E.D.

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FIGURE CAPTIONS

Fig. 1. The constitutive relation $v(q) = e^q$ for a nonlinear capacitor.

Fig. 2. Judging from its impedance matrix alone, this 2-port would appear to be passive. But is violently unstable and in any nonzero initial state it can furnish unlimited energy to the outside world.

Fig. 3. The capacitive constitutive relation $v(q) = q-1$. The energy available from such an element is given by $E_A(q) = (q-1)^2/2$.

Fig. 4. Two capacitive constitutive relations defined only for negative values of q. While such elements would probably never arise in electronics, their mechanical analogs in Fig. 7 are quite natural.

Fig. 5. (a) A simple active circuit based on an ideal operational amplifier.
(b) A passive 1-port.
(c) An active 2-port.

Fig. 6. The available energy for the capacitive element in Fig. 1.

Fig. 7. (a) A cylinder of ideal gas held at constant temperature through contact with an infinite heat reservoir. The coordinate x measures the distance from the piston face to the end of the cylinder. Since the piston has unit area, the volume of the gas is the negative of this coordinate, i.e., $V = -x$, and the pressure is equal to the force required to hold the piston in place, i.e., $P = f$. The ideal gas law $P = NRT/V$ becomes $f = -NRT/x$ in this case, producing a constitutive relation analogous to that of C_1 in Fig. 4.
(b) With insulating walls on the cylinder and the heat reservoir removed, the gas heats up when compressed and cools when it expands. In this case the pressure varies as $P = A/V^{5/3}$, where A is a positive constant. The system's constitutive relation is now $f = -A/x^{5/3}$, analogous to that of C_2 in Fig. 4.

Fig. 8. Two different n-ports which can be constructed from a single (n+1)-terminal element $\mathcal{E}$.

Fig. 9. A composite n-port $\mathcal{N}$ produced by interconnecting $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{N}_3$. We assume that $\mathcal{N}$ can be characterized by state equations $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{u})$, $\underset{\sim}{y} = \underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{u})$ where $\underset{\sim}{x} = (\underset{\sim}{x}_1,\underset{\sim}{x}_2,\underset{\sim}{x}_3) \in \Sigma_1 \times \Sigma_2 \times \Sigma_3$.

Fig. 10. The value of the shunt resistor is $R = (1/G)$ ohms. The 1-port is of course strongly passive provided $G \geq 0$.

Fig. 11. Available energy and required energy for the 1-port in Fig. 10, plotted for several values of G. The solid line in the center represents the function $q^2/2$. Notice that $E_A(q) < q^2/2 < E_{R0}(q)$ unless $G = 0$, i.e., unless the shunt resistor is an open circuit.

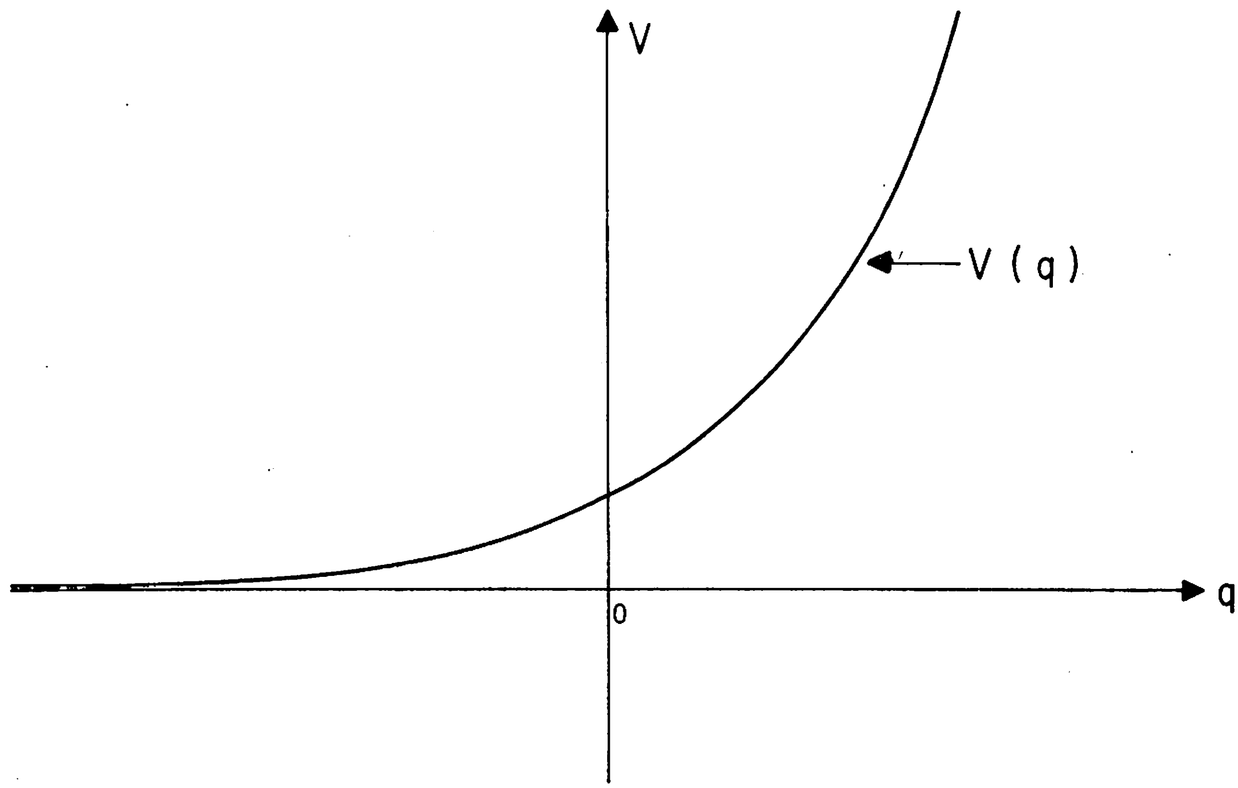
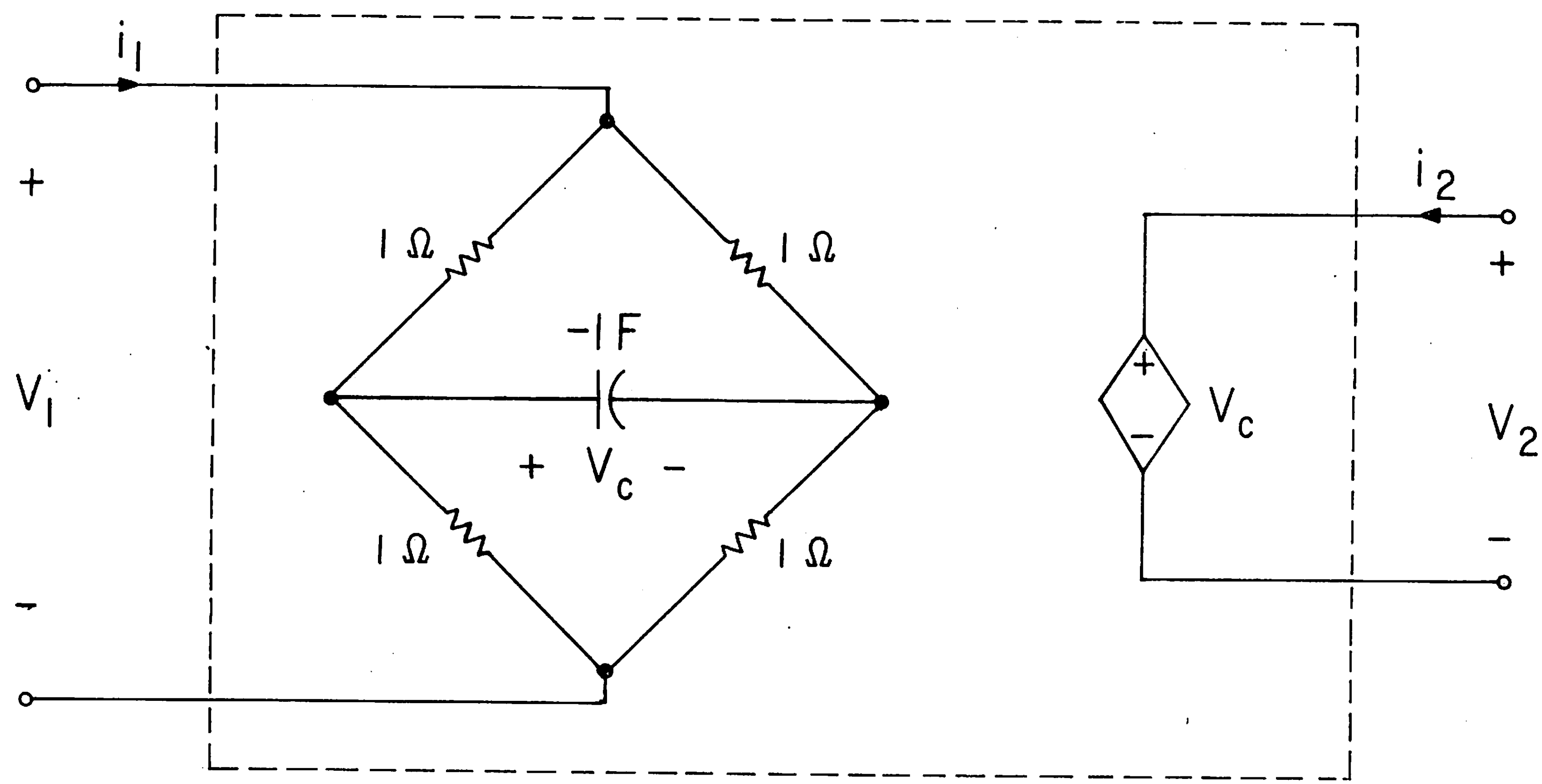


Figure 1



$$\underset{\sim}{Z}(s) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Figure 2

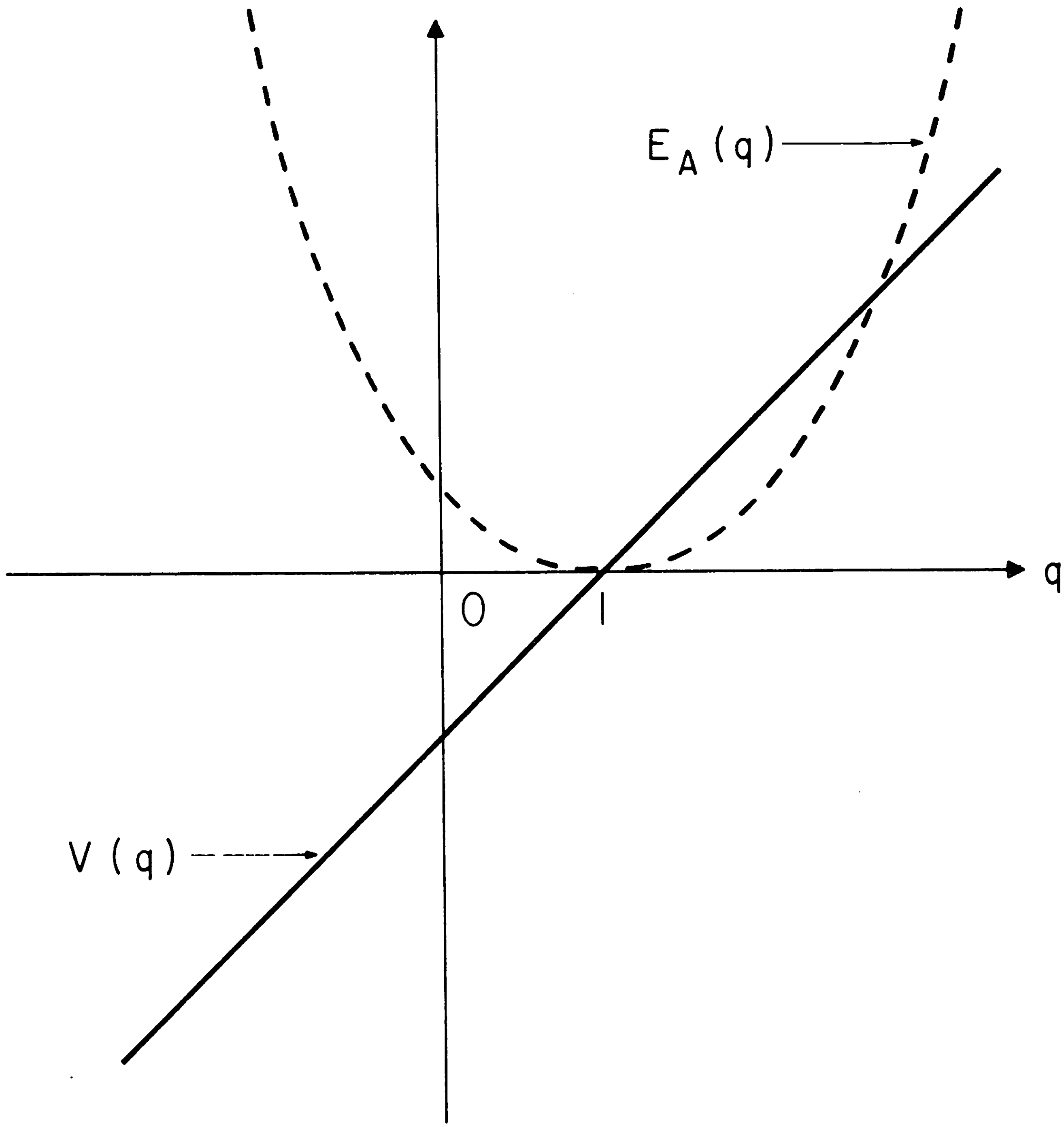


Figure 3

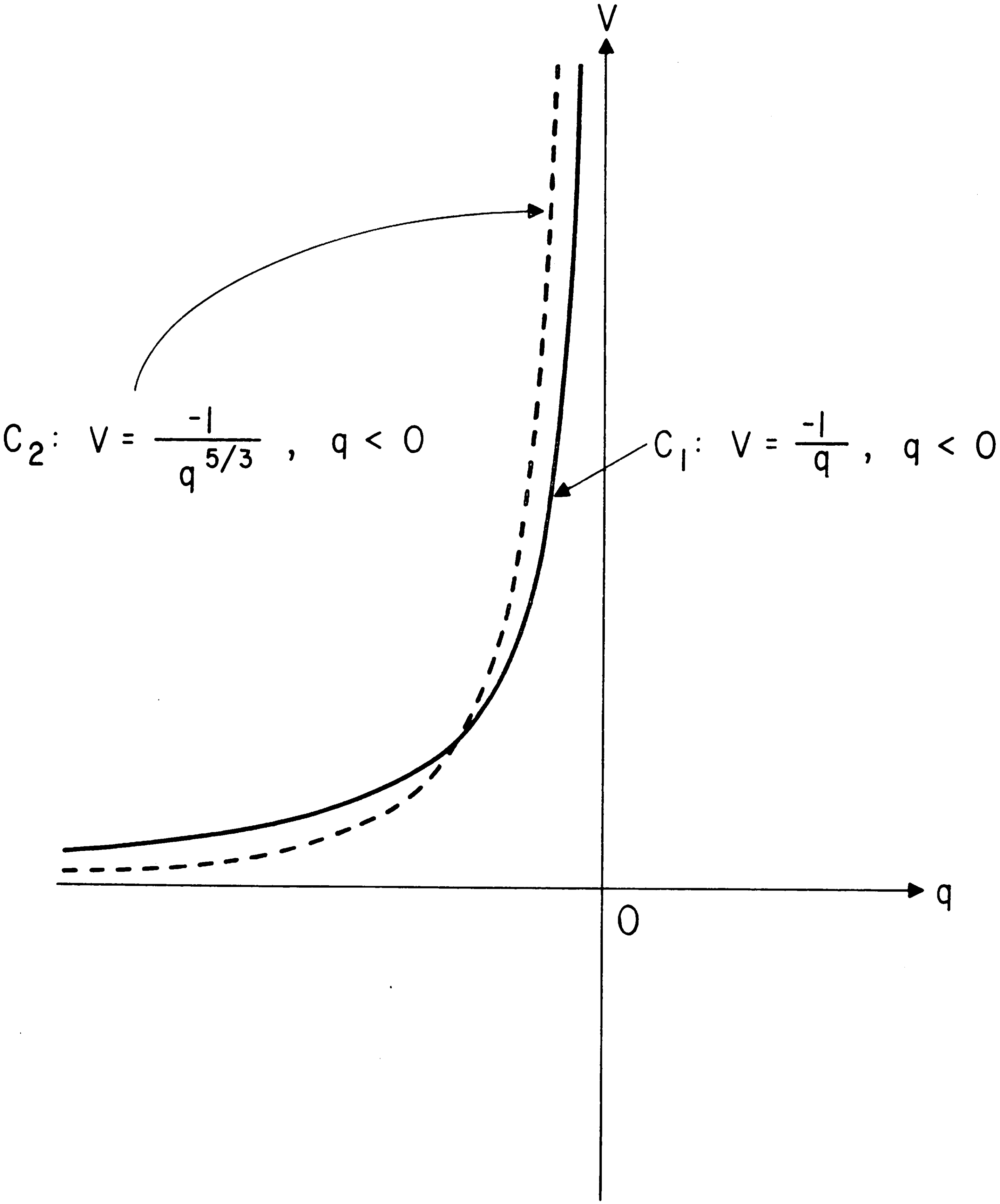


Figure 4

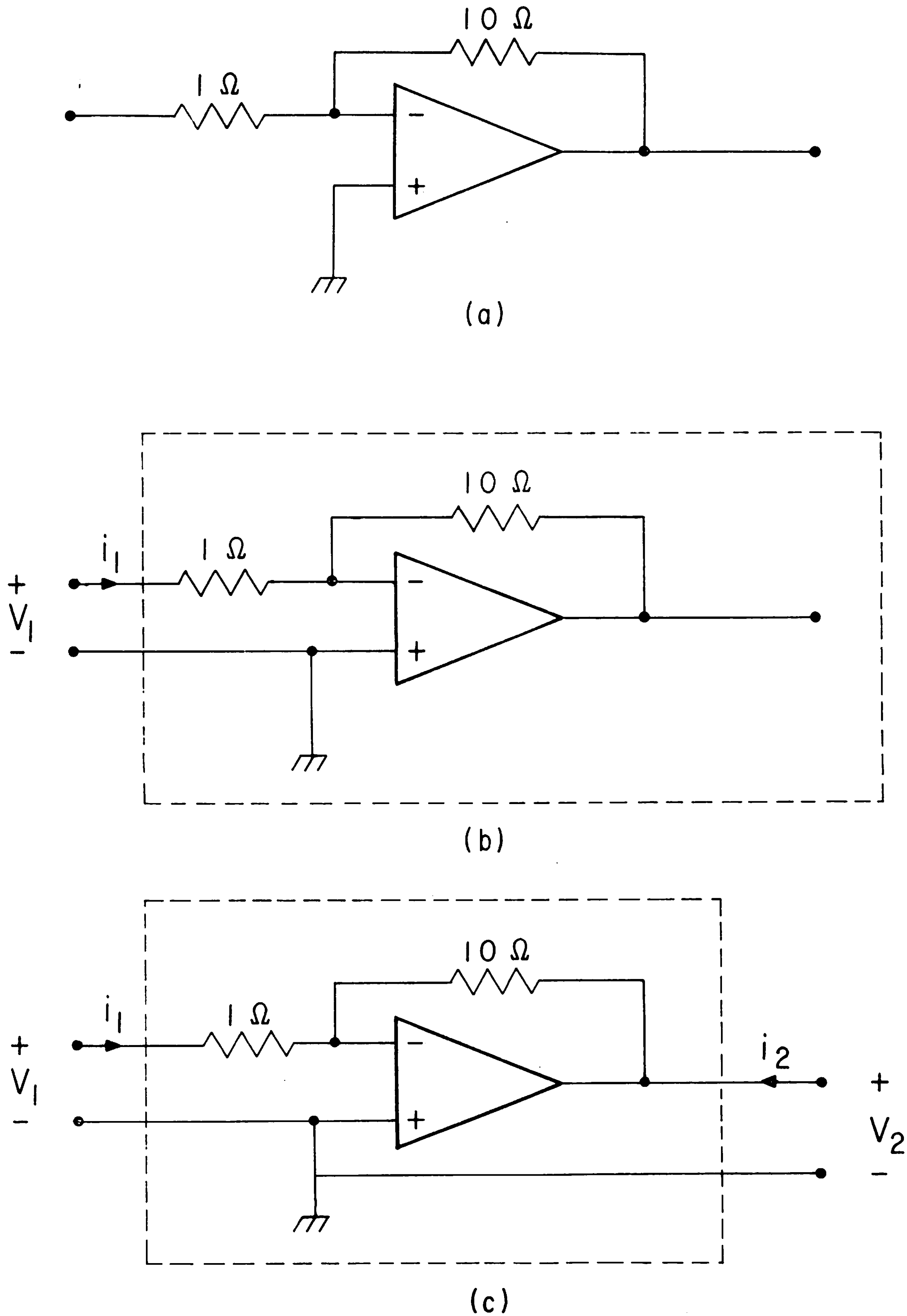
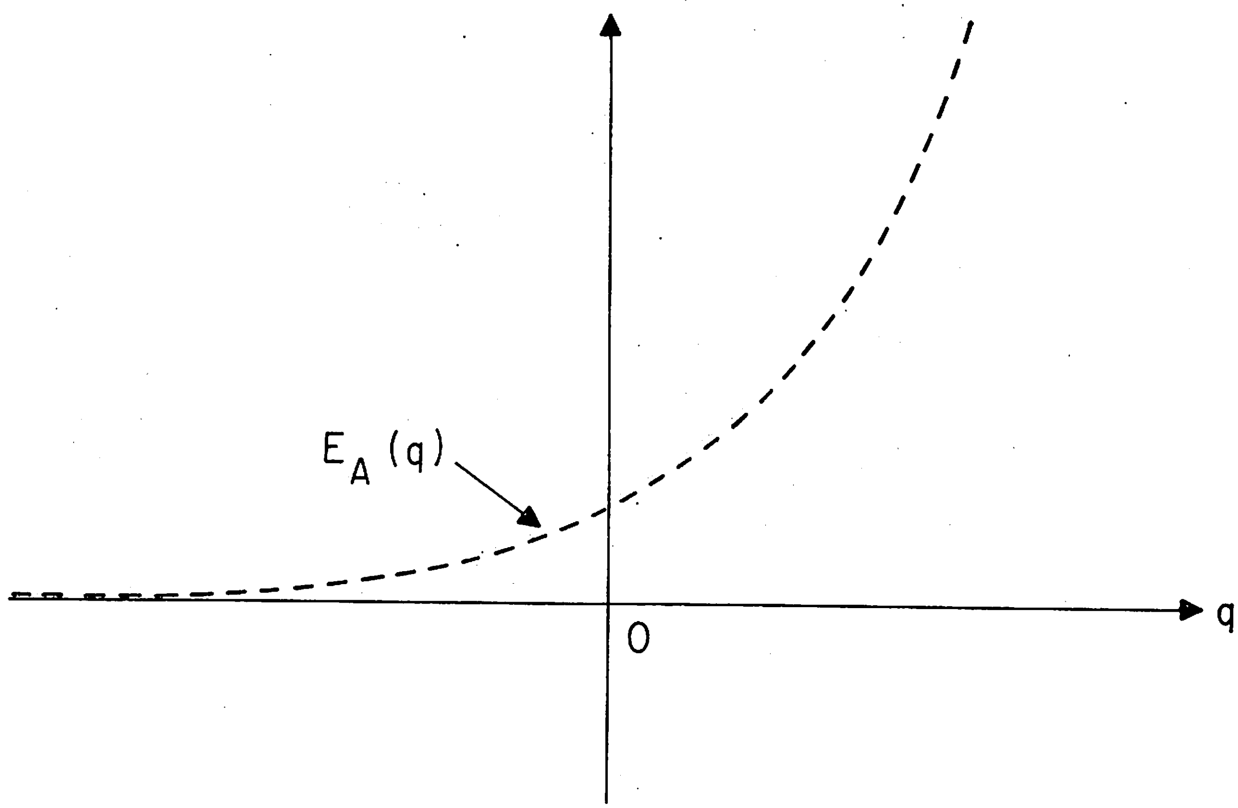


Figure 5



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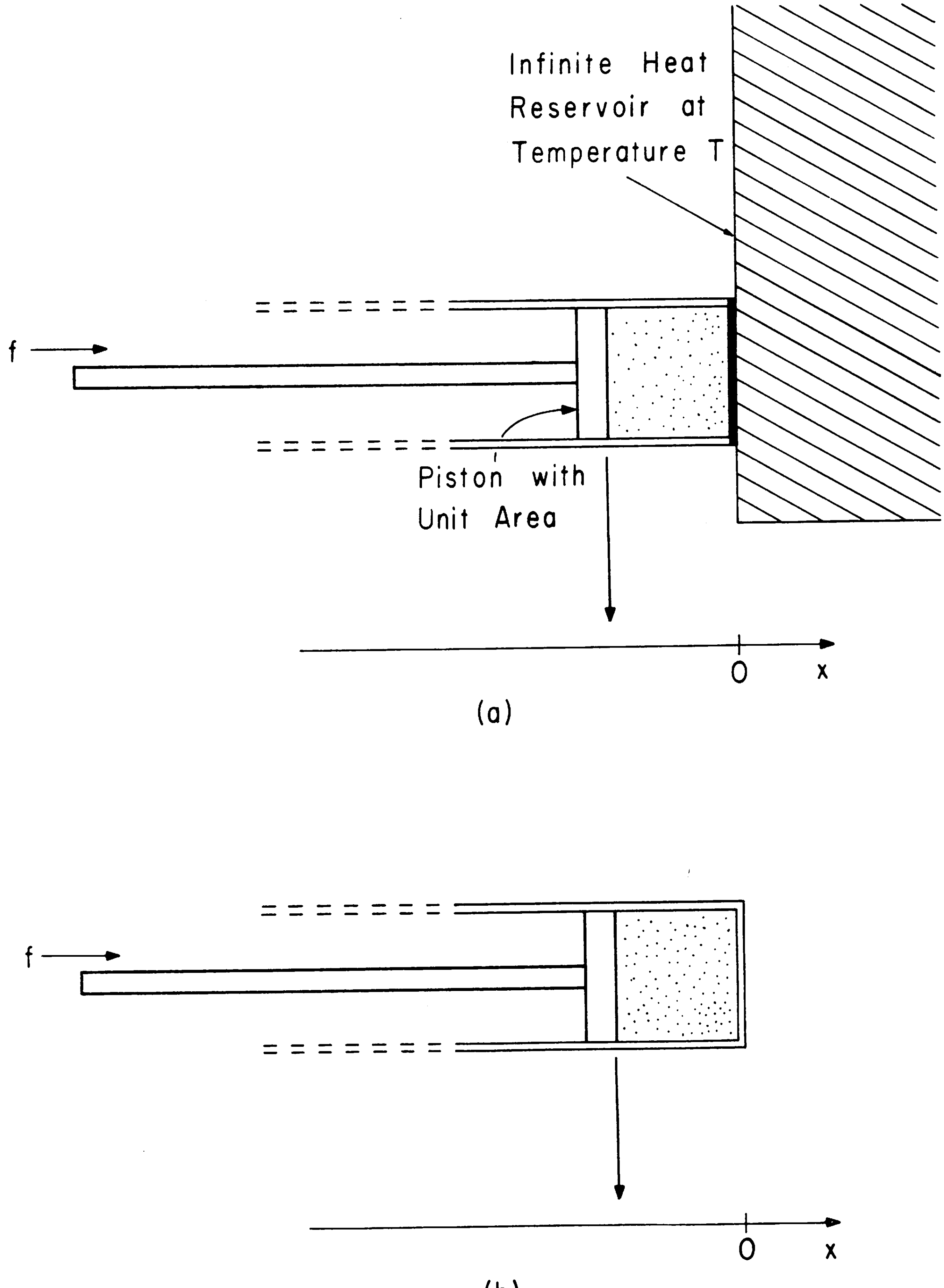
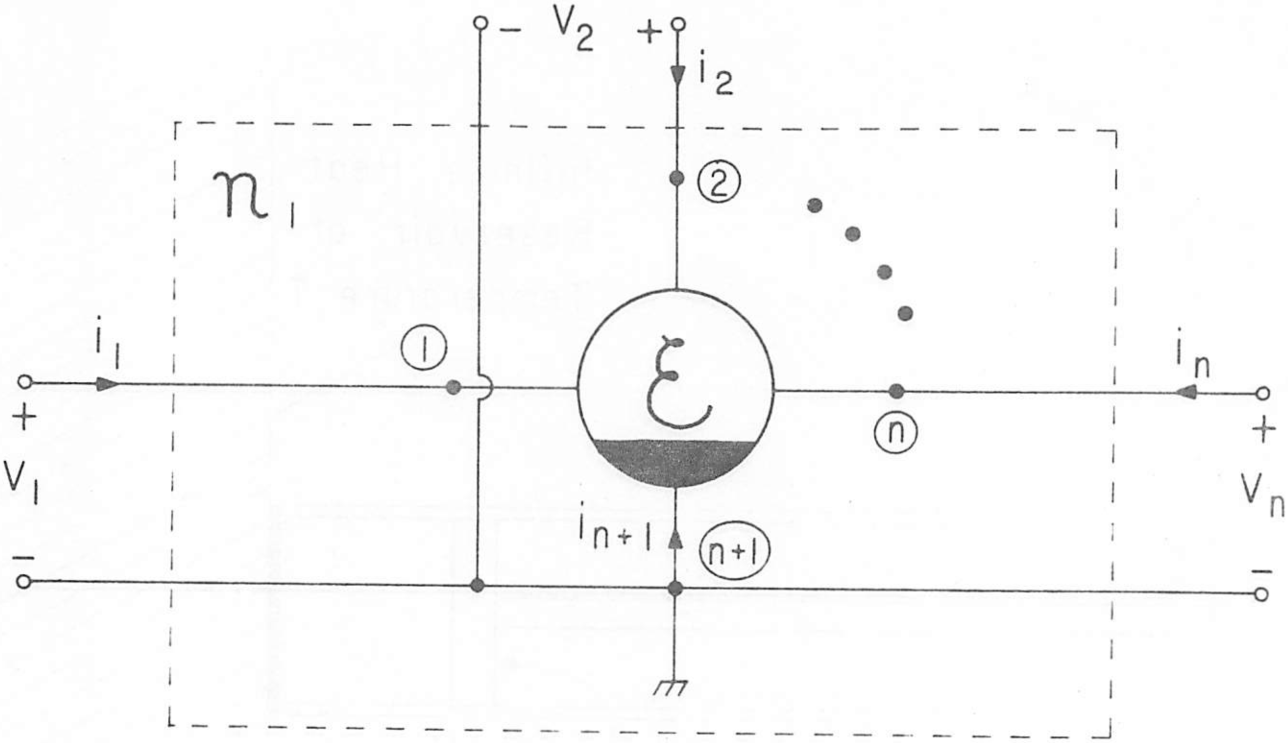

Infinite Heat
Reservoir at
Temperature T
f
Piston with
Unit Area
0
x
(a)
f
0
x
(b)

Figure 7


- V_2 +
i_2
$\mathcal{N}_1$
2
i_1
1
$\mathcal{E}$
i_n
+
V_1
−
n
+
V_n
−
i_{n+1}
n+1

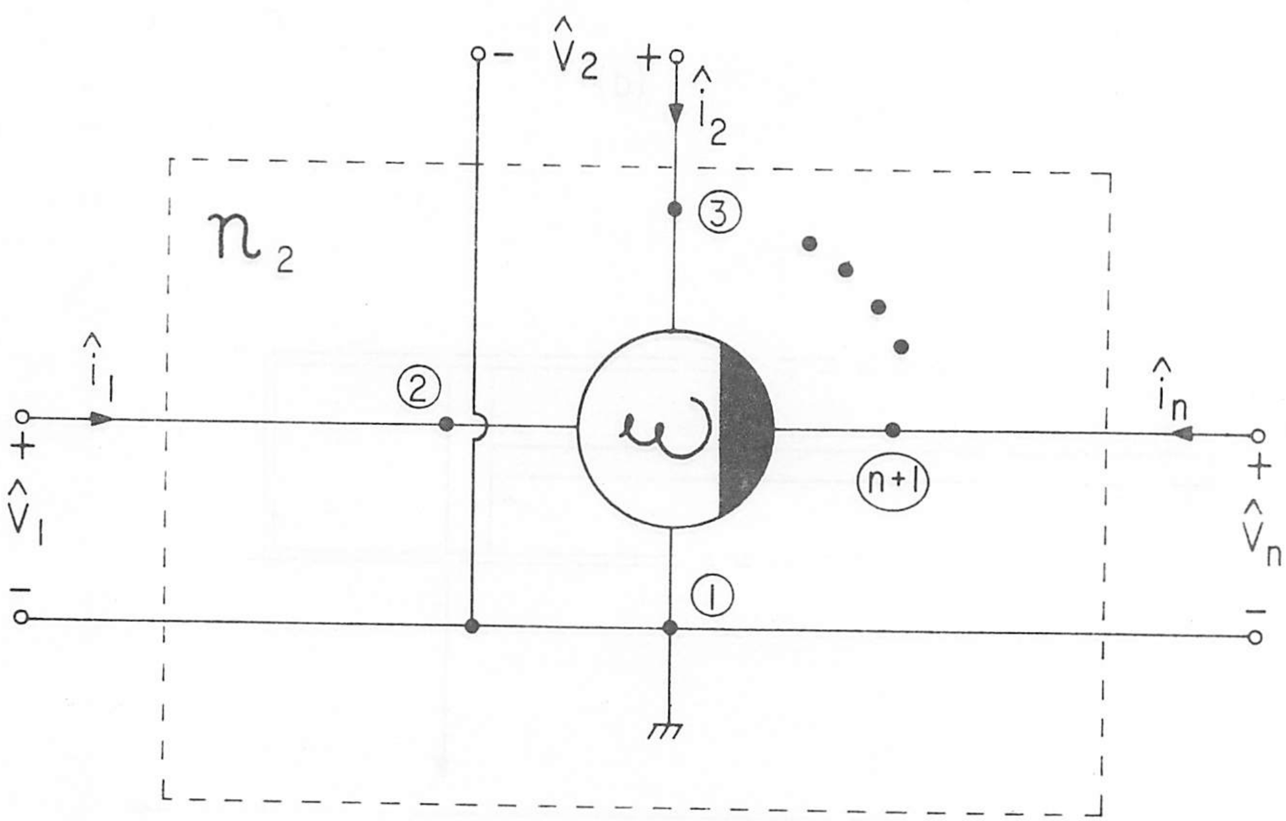

- $\hat{V}_2$ +
$\hat{i}_2$
$\mathcal{N}_2$
3
$\hat{i}_1$
2
ω
$\hat{i}_n$
+
$\hat{V}_1$
−
n+1
+
$\hat{V}_n$
−
1

Figure 8

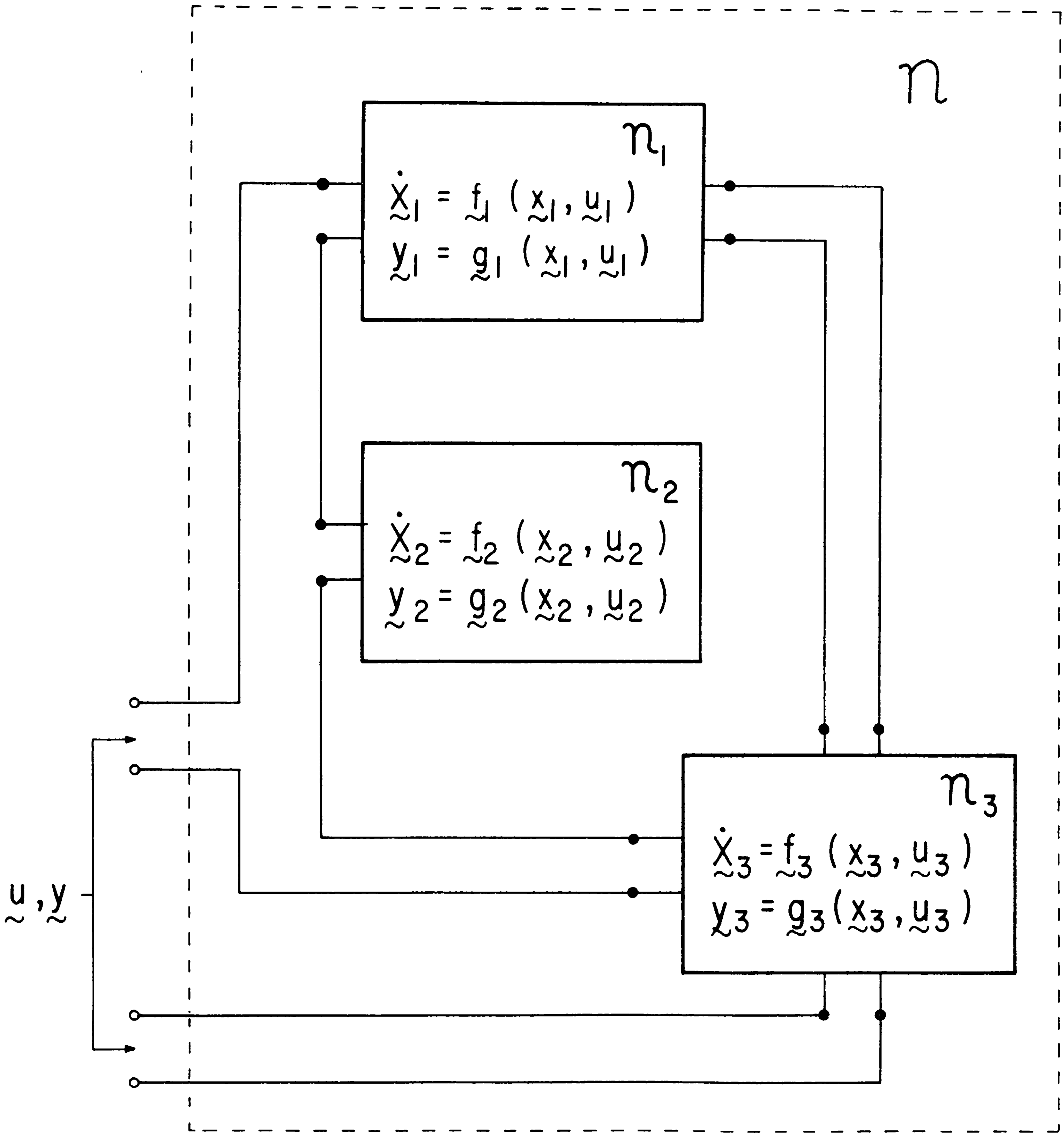

𝔑
𝔑₁
Ẋ₁ = f₁ (x₁, u₁)
y₁ = g₁ (x₁, u₁)
𝔑₂
Ẋ₂ = f₂ (x₂, u₂)
y₂ = g₂ (x₂, u₂)
𝔑₃
Ẋ₃ = f₃ (x₃, u₃)
y₃ = g₃ (x₃, u₃)
u, y

Figure 9

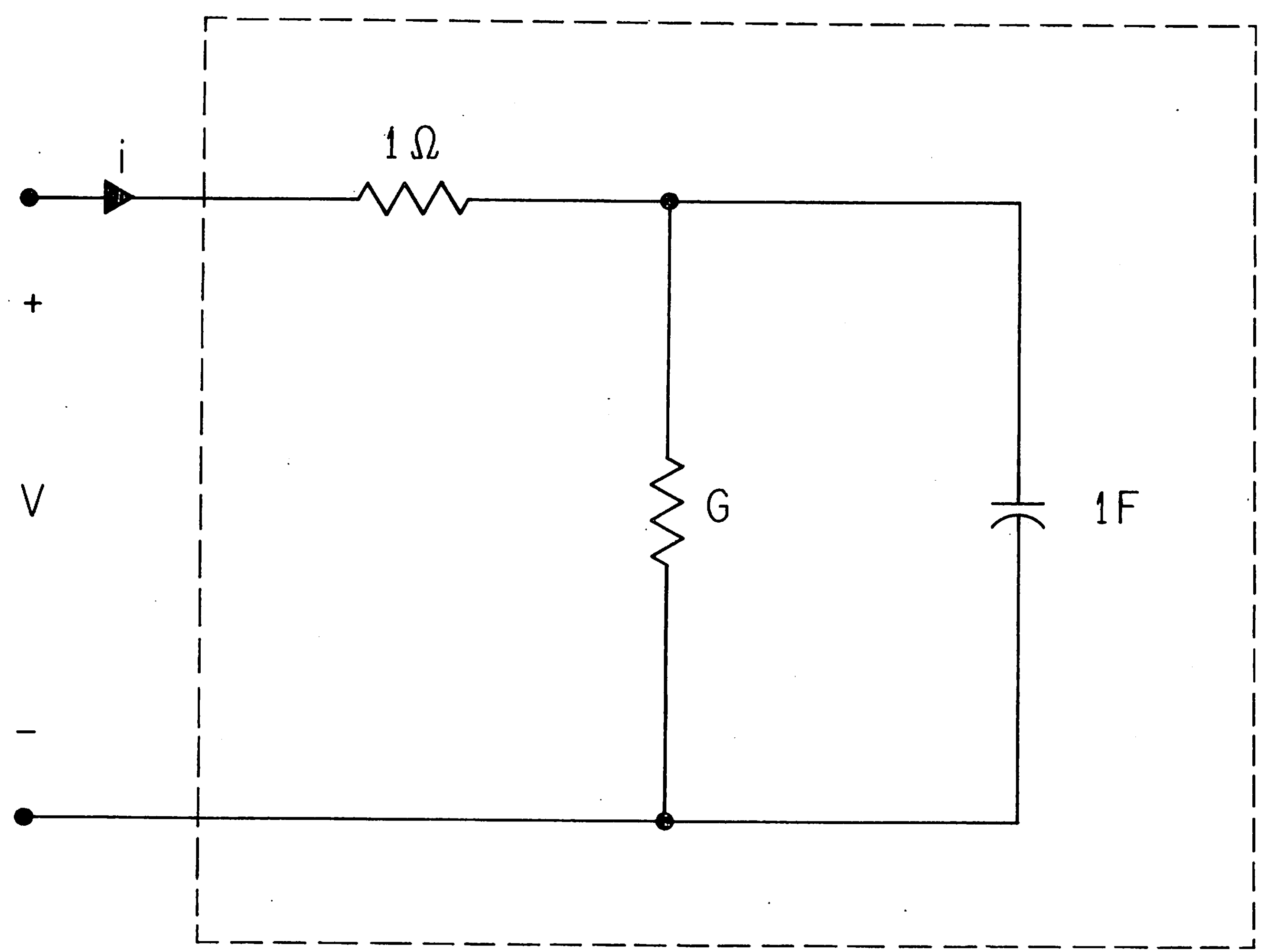


Figure 10

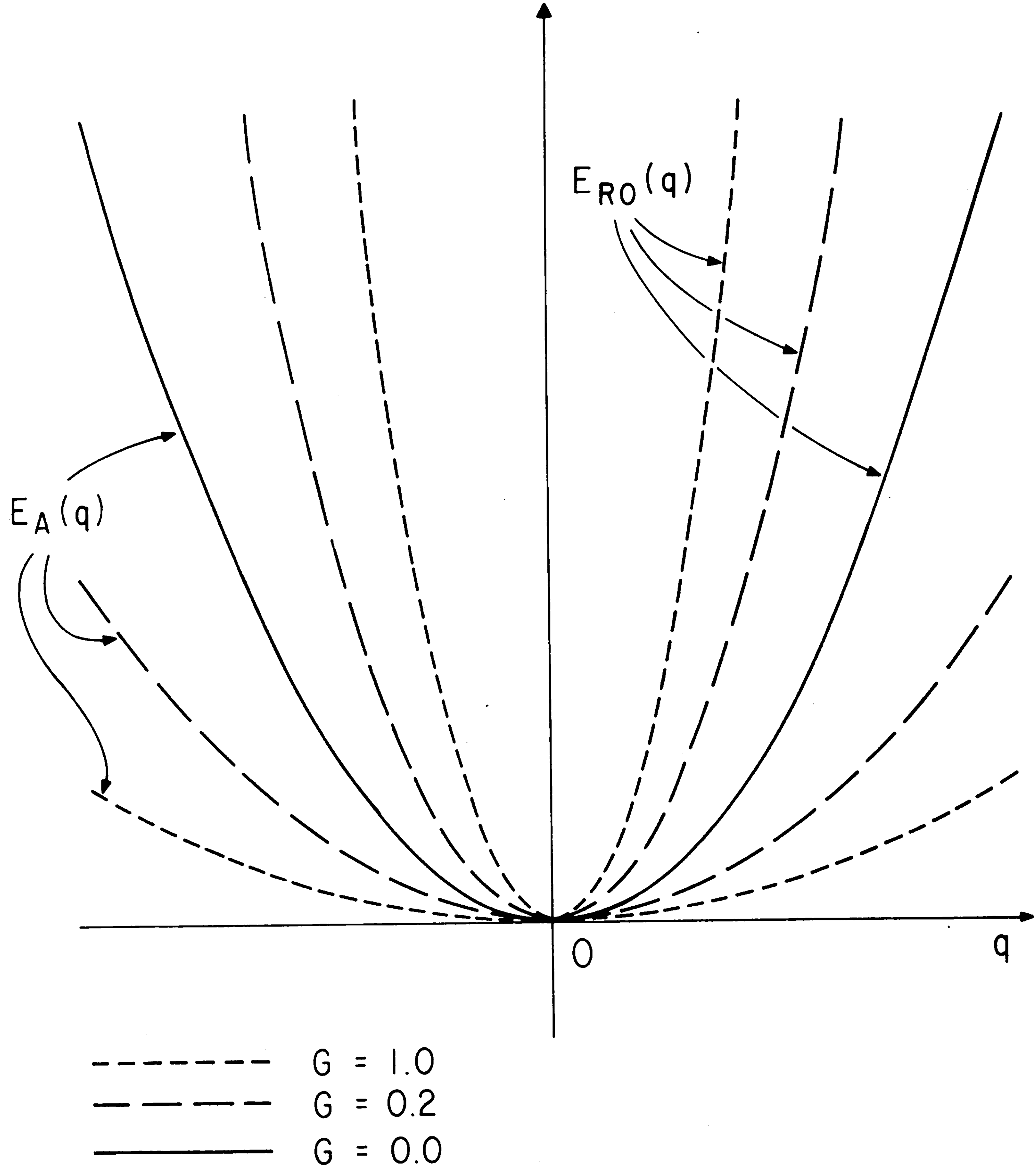


Figure 11