SENSITIVITY INVARIANTS OF CONTINUOUSLY EQUIVALENT NETWORKS

by

E. S. Kuh

C. G. Lau

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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E. S. Kuh and C. G. Lau

ABSTRACT -- Two simple sensitivity invariants are derived for continuously equivalent networks. The first one states that the sum of sensitivities with respect to all elements in a network is invariant under continuously equivalent transformation. The second one states that the individual sensitivity for capacitances and inductances is invariant if there are no capacitance loops and inductance cut-sets in a network.

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The theory of continuously equivalent networks which was first introduced by Schoeffler [1] has generated considerable amount of interest recently [2-5]. The theory is an extension of the Howitt transformation [6] for deriving equivalent networks; however, it allows all elements in a network to vary continuously as functions of a single "dummy" parameter. The method is quite general, flexible and easily programmed. Schoeffler and others have employed the method to generate equivalent networks and meantime minimize sensitivity. The purpose of this paper is to derive two simple sensitivity invariants of continuously equivalent networks. The first invariant deals with the sum of sensitivities with respect to all elements in a network. The second invariant concerns the sensitivity of individual reactive elements in a network.

Consider an arbitrary linear time-invariant RLC network having n elements with values x_i, $i = 1, 2, \ldots, n$. Let us call the n-vector $\underset{\sim}{x}$ the element vector whose components are x_i. Let H be the particular network function of interest. Under continuously equivalent transformation $\underset{\sim}{x}$ is varied as a specific function of a dummy parameter z while H remains invariant. Let the conventional sensitivity for H with respect to x_i be denoted by $S_{x_i}^{H}$, that is

$$S_{x_i}^{H} \triangleq \frac{x_i}{H} \frac{\partial H}{\partial x_i} \tag{1}$$

Our first sensitivity invariant is stated below:

Theorem 1 : The sum of the sensitivities for the network function H with respect to all elements in a network is invariant under continuously equivalent transformation, that is,

$$\sum_{i=1}^{n} S_{x_i}^{H} \quad \text{is invariant} \tag{2}$$

This fact was observed in a special example in [5]; however, no proof was given. The proof for this general invariant is simple. Schoeffler has shown that under continuously equivalent transformation, the element vector $\underset{\sim}{x}$ satisfies a linear differential equation:

$$\frac{d\underset{\sim}{x}}{dz} = \underset{\sim}{F}\,\underset{\sim}{x} \tag{3}$$

where $\underset{\sim}{F}$ is an $n \times n$ constant matrix, and z is the dummy parameter. That is, the solution of Eq. (3) for any z gives the element value that maintains the same network function H. The gradient of H with respect to the element vector $\underset{\sim}{x}$ is

$$\underset{\sim}{q} \triangleq \text{grad}_{\underset{\sim}{x}} H$$

$$= \left[\frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2}, \ldots, \frac{\partial H}{\partial x_n} \right]^t \tag{4}$$

The n-vector $\underset{\sim}{q}$ is simply related to the n sensitivity functions $S_{x_i}^{H}$, $i = 1, 2, \ldots, n$. From Eq. (4) the ith element is

$$q_i = \frac{\partial H}{\partial x_i} \tag{5}$$

Thus

$$S_{x_i}^{H} = \frac{x_i q_i}{H} \tag{6}$$

and the sum of the sensitivities becomes

$$\sum_{i=1}^{n} S_{x_i}^{H} = \sum_{i=1}^{n} \frac{x_i q_i}{H} = \frac{\langle \underset{\sim}{x}, \underset{\sim}{q} \rangle}{H} \tag{7}$$

where $\langle \underset{\sim}{x}, \underset{\sim}{q} \rangle$ denotes the scalar product of the two vectors. Schoeffler has also shown that the $\underset{\sim}{q}$ vector satisfies the differential equation

$$\frac{d\underset{\sim}{q}}{dz} = -\underset{\sim}{F}^{t}\,\underset{\sim}{q} \tag{8}$$

which is the adjoint to the system of Eq. (3). From well-known property of the adjoint systems [7] we have

$$\langle \underset{\sim}{x}, \underset{\sim}{q} \rangle = \text{constant} \tag{9}$$

Substituting Eq. (9) in (7), we conclude that the sum of all sensitivities is invariant under continuously equivalent transformation.

It is also of interest to point out that Eqs. (3) and (8) represent a special case of a more general result obtained by Blostein [4]. Since Blostein's Theorem contains a minor error, the corrected version is stated below:

It is conceivable that under a more general continuous transformation, the differential equation for the element vector is nonlinear:

$$\frac{d\underset{\sim}{x}}{dz} = \underset{\sim}{f}(\underset{\sim}{x}) \tag{10}$$

Then, the vector $\underset{\sim}{q}$ satisfies the following linear differential equation:

$$\frac{d\underset{\sim}{q}}{dz} = - \left[\frac{d\underset{\sim}{f}}{\partial \underset{\sim}{x}}\right]^t \underset{\sim}{q} \tag{11}$$

where $\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}}$ is the $n \times n$ Jacobian matrix.

The proof of this result is again simple. Since by definition the network function H is invariant with respect to the parameter z, we have $\frac{dH}{dz} = 0$. Using Eq. (10), we obtain

$$\frac{dH}{dz} = \sum_{i=1}^{n} \frac{\partial H}{\partial x_i} \frac{dx_i}{dz} = \left\langle \underset{\sim}{q}, \frac{d\underset{\sim}{x}}{dz} \right\rangle$$

$$= \left\langle \underset{\sim}{q}, \underset{\sim}{f} \right\rangle = 0 \tag{12}$$

Differentiating the above with respect to z, we have

$$\left\langle \frac{d\underset{\sim}{q}}{dz}, \underset{\sim}{f} \right\rangle + \left\langle \underset{\sim}{q}, \frac{d\underset{\sim}{f}}{dz} \right\rangle = 0 \tag{13}$$

But

$$\frac{d\underset{\sim}{f}}{dz} = \sum_{i=1}^{n} \frac{d\underset{\sim}{f}}{\partial x_i} \frac{dx_i}{dz} = \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} \underset{\sim}{f} \tag{14}$$

Equation (13) can be written as

$$\left\langle \frac{d\underset{\sim}{q}}{dz}, \underset{\sim}{f} \right\rangle = - \left\langle \underset{\sim}{q}, \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} \underset{\sim}{f} \right\rangle$$

$$= - \left\langle \left[\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} \right]^t \underset{\sim}{q}, \underset{\sim}{f} \right\rangle \tag{15}$$

Since the above is valid for all $\underset{\sim}{f}$, Eq. (11) follows.

Our second sensitivity invariant is based on the state space formulation of continuously equivalent transformation. Consider the state equations of any linear time-invariant RLC network:

$$\underset{\sim}{M} \frac{d\underset{\sim}{y}}{dt} = \underset{\sim}{N} \underset{\sim}{y} + \underset{\sim}{b} u$$

where $\underset{\sim}{y}$ is the state vector, $\underset{\sim}{b} u$ represents the input, $\underset{\sim}{M}$ is a square matrix with elements contributed by inductances and capacitances in a network, and $\underset{\sim}{N}$ is a square matrix with elements due to resistances. Calahan [2] has shown that under continuously equivalent transformation, $\underset{\sim}{M}$ and $\underset{\sim}{N}$ satisfy the following differential equations

$$\frac{d\underset{\sim}{M}}{dz} = \underset{\sim}{B}\underset{\sim}{M} + \underset{\sim}{M}\underset{\sim}{A} \tag{16}$$

$$\frac{d\underset{\sim}{N}}{dz} = \underset{\sim}{B}\underset{\sim}{N} + \underset{\sim}{N}\underset{\sim}{A} \tag{17}$$

where $\underset{\sim}{B}$ and $\underset{\sim}{A}$ are constant square matrices containing parameters to be chosen. Our second sensitivity invariant is given next.

Theorem 2: Under continuously equivalent transformation, the individual sensitivity for the network function H with respect to the capacitances and inductances is invariant if there are no capacitance loops and inductance cut-sets in the network.

The proof is given as follows: For a network without capacitance loop and inductance cut-set, the matrix $\underset{\sim}{M}$ in the state equation is diagonal and is of the form

$$\underset{\sim}{M} = \begin{bmatrix} C_1 & & & & & & \\ & \ddots & & & & 0 & \\ & & C_j & & & & \\ & & & \ddots & & & \\ & & & & L_1 & & \\ & 0 & & & & \ddots & \\ & & & & & & L_k \\ & & & & & & & \ddots \end{bmatrix} \tag{18}$$

The differential equation for the element vector corresponding to capacitances and inductances are of specially simple form. For example, combining (16) and (18), we obtain typical equations:

$$\frac{dC_j}{dz} = (b_{jj} + a_{jj})\, C_j \triangleq f_j C_j \tag{19}$$

and

$$\frac{dL_k}{dz} = (b_{kk} + a_{kk})\, L_k \triangleq f_k L_k \tag{20}$$

where f_j and f_k are constants. In other words the differential equation for the element vector which includes only the capacitances and inductances of the network is of the form

$$\frac{d\underset{\sim}{x}}{dz} = \underset{\sim}{F}\,\underset{\sim}{x} = \begin{bmatrix} f_1 & & & 0 \\ & f_2 & & \\ & & \ddots & \\ 0 & & & f_n \end{bmatrix} \underset{\sim}{x} \tag{21}$$

where $\underset{\sim}{F}$ is diagonal. The solution of Eq. (21) is simply

$$x_i(z) = e^{f_i z}\, x_i(0) \qquad i = 1, 2, \ldots, n \tag{22}$$

The differential equation for the vector $\underset{\sim}{q}$ is

$$\frac{d\underset{\sim}{q}}{dz} = -\underset{\sim}{F}^t \underset{\sim}{q} = \begin{bmatrix} -f_1 & & & 0 \\ & -f_2 & & \\ & & \ddots & \\ 0 & & & -f_n \end{bmatrix} \underset{\sim}{q} \tag{23}$$

and the solution is

$$q_i(z) = e^{-f_i z} q_i(0) \qquad i = 1, 2, \ldots, n \tag{24}$$

Thus the sensitivity for the $\underline{i}$th reactive element is

$$S_{x_i}^H = \frac{x_i(z)\, q_i(z)}{H} = \frac{x_i(0)\, q_i(0)}{H} \tag{25}$$

which is invariant, under continuously equivalent transformation.

In conclusion, we have derived two useful sensitivity invariants for continuously equivalent networks. These invariants can be used as simple guidelines for designers in generating equivalent networks when sensitivity minimization is desired.

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