Confidence Bands for A Distribution Function Using the Bootstrap[1]

By

P.J. Bickel
University of California, Berkeley

and

A.M. Krieger
University of Pennsylvania

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Department of Statistics
University of California
Berkeley, California

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Abstract: We first discuss the construction of bootstrap confidence bands for the distribution function F of a population for simple random sampling but do not which to to assume that F is continuous. The known alternative approach is to use the quantiles from the tabled Kolmogorov distribution. This approach is known to be conservative. It has already been shown in Bickel and Freedman (1981) that the bootstrap confidence band has the correct coverage probability asymptotically. We show by simulation that the bootstrap works well for small samples and outperforms the conservative approach particularly for distributions which have small carriers.

We also investigate the analogous problem of finding bootstrap confidence bands for the distribution function F of a population for a more complicated situation, stratified random sampling. The conservative approach in the previous situation is extended to this case when sampling is with replacement (we expect that it holds for sampling without replacement) and the supports of the conditional distributions in each stratum are not overlapping. If the strata overlap there seems to be little alternative to the bootstrap. Use of the bootstrap in setting confidence bands or curves in this way should prove widely applicable particularly when we leave the simple random sampling context as we have done. Asymptotic theory for the bootstrap confidence band is presented and the conservative and bootstrap approaches are compared for small samples by simulation.

Key Words: Confidence bands, bootstrap, stratified sampling.

1. INTRODUCTION AND THEORY

Perhaps the most interesting applications of Efron's (1979) bootstrap ideas are to situations where there is no clear alternative approach or known alternative approaches are unsatisfactory.

One example of this type, discussed in Bickel, Freedman (1981) is placing a confidence band on a distribution function (d.f.) F of a population when we observe a sample $X_1, \ldots, X_n$ from F but do not wish to assume that F is continuous, i.e. we observe some ties.

Let $\hat{F}_n$ be the empirical distribution of the sample,

$$D_n = \sqrt{n} \sup_x |\hat{F}_n(x) - F(x)|$$

be the Kolmogorov statistic, and $K_n(\cdot)$ the d.f. of D_n if F is continuous, the (tabled) Kolmogorov distribution. It is well known that, for all d,

$$P[D_n \le d] \ge K_n(d)$$

so that we can construct a level $(1-\alpha)$ confidence band for F, viz.,

$$\hat{F}_n \pm \frac{K_n^{-1}(1-\alpha)}{\sqrt{n}}$$

If F is discrete this band is conservative. The bootstrap alternative goes as follows. Given $X_1, \ldots, X_n$, let $X_1^*, \ldots, X_n^*$ conditionally be a sample from $\hat{F}_n$. Let F_n^* be the corresponding empirical d.f., and

$$D_n^* = \sup_x \sqrt{n}|F_n^*(x) - \hat{F}_n(x)| .$$

Form the bootstrap distribution of D_n,

$$K_n^*(d) = P[D_n^* \le d | X_1, \ldots, X_n]$$

which depends on $X_1, \ldots, X_n$ only.

If we let,

$$q_n^* = \inf \{d: \quad K_n^*(d) \ge 1 - \alpha\}$$

the bootstrap confidence band is just,

$$\hat{F}_n \pm \frac{q_n^*}{\sqrt{n}}$$

Bickel and Freedman show that, unless F is a point mass,

$$P_F[D_n \le q_n^*] \to 1 - \alpha$$

so that the band is asymptotically correct. In the next section, we present a small Monte Carlo study of the behavior of the bootstrap for n = 20, 40 and a variety of discrete distributions. This study seems to indicate,

(i) The bootstrap cofidence bands are reasonably accurate although they err on the side of liberality.

(ii) The bootstrap confidence bands are narrower on the average, substantially so for discrete distributions which have small carriers. In fact, the order of the effect is about 20% even for the uniform distribution on ten points.

These results, especially the second, are not too surprising. When F is a point mass the bootstrap band has 0 width as it should.

Encouraged by this result we investigate theoretically and with a small simulation the analogous problem for a more complicated situation, stratified sampling as discussed in Bickel, Freedman (1984), (B-F) for short. For $i = 1, \ldots, p$ observe a sample $\{X_{ij}\}$, $j = 1, \ldots, n_i$ from a stratum χ_i of a finite population $\chi = \chi_1 \cup \ldots \cup \chi_p$. The stratum size N_i is assumed known, and $n_i \geq 2$. The X_{ij} are drawn from χ_i either

a) With replacement

b) Without replacement.

Enumerate the i^{th} stratum as $\{x_{i1}, \ldots, x_{iN_i}\}$ and let F_i be the d.f. of the stratum, attaching mass N_i^{-1} to each x_{ij}, $j = 1, \ldots, N_i$. Write

$$N = \Sigma_{i=1}^{p} N_i$$

$$\pi_i = \frac{N_i}{N}, \text{ the stratum fraction}$$

Let

$$F = \Sigma_{i=1}^{p} \pi_i F_i$$

denote the d.f. of the population. Similarly let F_i denote the empirical d.f. of the sample $\{X_{ij}\}$, $j = 1, \ldots, n_i$ and,

$$n = \Sigma_{i=1}^{p} n_i \text{ --the total sample size}$$
$$\lambda_i = \frac{n_i}{n} \text{ --the sample fraction from the } i^{th} \text{ stratum}$$

and

$$\hat{F} = \Sigma_{i=1}^{p} \pi_i \hat{F}_i \text{ -- the usual estimate of } F.$$

As in the one sample case we can obtain a fixed width level $(1-\alpha)$ confidence band if we know the distribution Q_n of

$$D_n = \sup_x \sqrt{n} \; |\hat{F}_n(x) - F(x)| \; .$$

If the strata are contained in disjoint intervals it is easy to see that,

$$D_n = \max_{1 \leq i \leq p} \frac{\pi_i}{\sqrt{\lambda_i}} \sup_x \sqrt{n_i} \; |\hat{F}_i - F_i|$$

and hence

$$Q_n(t) = \Pi_{i=1}^{p} Q_{ni}(t\sqrt{\lambda_i}/\pi_i) \tag{1}$$

where Q_{ni} is the distribution of $\sup_x \sqrt{n_i} \; |\hat{F}_i(x) - F_i(x)|$.

If sampling is with replacement we can approximate quantiles conservatively using

$$Q_n(t) \geq \Pi_{i=1}^{p} \tilde{Q}_{n_i}\left(t\sqrt{\lambda_i}/\pi_i\right) \tag{2}$$

where $\tilde{Q}_m$ is the distribution of D_m when F is uniform on (0,1).

We conjecture that (2) also holds if sampling is without replacement. We expect that unless the number of distinct elements in each stratum is large the approximation is unduly conservative as we already noted in the one sample case.

An alternative is to approximate each of the Q_{ni} by the corresponding bootstrap distribution which is the same thing as approximating Q_n by the bootstrap distribution of D_n. We go into the details of a modification of this process which makes a necessary correction when some of the n_i are small, below.

If the strata are not disjoint formula (1) does not hold and, at first sight, bootstrap approximations seem to be the natural way to go.

The paper is organized as follows. We present and discuss the bootstrap bands and their asymptotic theory as well as extensions to variable width bands and other related situations in this section. Numerical studies and Monte Carlo simulations are presented in section 2 and the proofs of the asymptotic approximations are given in an appendix.

The bootstrap bands

a) Sampling with replacement: Given $\{X_{ij}\}$ $i = 1, \ldots, p$, $j = 1, \ldots, n_i$ let X^*_{ij} $j = 1, \ldots, n_i$ be a sample from the distribution $\hat{F}_i$ $i = 1, \ldots, p$. Define $\hat{F}^*_i$ as the empirical distribution of the $\{X^*_{ij}\}$ $j = 1, \ldots, n_i$. We want to estimate Q_n.

The naive bootstrap approximation is to use the conditional distribution of $\sup_x \sqrt{n}\, |\hat{F}_n(x) - \hat{F}(x)|$. Unfortunately this cannot be expected to work in general even asymptotically, if the number of strata is large unless all of the n_i are large. The problem comes from a variance mismatch. The conditional variance of $(\hat{F}^*_n(x) - \hat{F}_n(x))$ given the sample is an underestimate (biased down) of the variance of $(\hat{F}_n(x) - F(x))$.

$$E(\hat{F}^*_n(x) - \hat{F}(x))^2 = E\{\Sigma^p_{i=1}\, \pi_i^2\, \frac{\hat{F}_i(x)(1-\hat{F}_i(x))}{n_i}\}$$

$$= \Sigma^p_{i=1}\, \pi_i^2\, \frac{F_i(x)(1-F_i(x))}{n_i}\, \frac{n_i-1}{n_i}$$

So, if all the $n_i = 2$, we will be off by a factor of $\frac{1}{2}$. The solution we adopt, mentioned ("rescaling the c_i") in B-F (see also Rao and Wu (1984)), is to estimate the distribution of the stochastic process,

(3) $$Z_n(\cdot) = \sqrt{n}\, (\hat{F}_n(\cdot) - F(\cdot))$$

by the conditional (bootstrap) distribution of,

$$Z_n^*(\cdot) = n^{1/2}\left(\Sigma_{i=1}^{p} \pi_i\left(\frac{n_i}{n_i-1}\right)^{1/2}(\hat{F}_i^*(\cdot) - \hat{F}_i(\cdot))\right)$$

Specifically let,

$$D_n^* = \sup_x |Z_n^*(x)|$$
$$H_n^*(x, \{X_{ij}\}) = P[D_n^* \le x | \{X_{ij}\}]$$

and v_n be the $(1-\alpha)$ quantile of H_n^*

We propose a level $(1-\alpha)$ confidence band for F

(4) $$\hat{F}_n \pm v_n/\sqrt{n}$$

If the strata are disjoint (2) shows that this is a more conservative procedure than the naive bootstrap. Unfortunately, as we discuss below, in cases where there are a large number of small strata there is reason to believe that even this modification of the bootstrap doesn't work.

b) Sampling without replacement

Let

(5) $$\tilde{Z}_n(x) = \sqrt{n}\,(\hat{F}_n(x) - F(x))$$
$$\tilde{D}_n = \sup_x |\tilde{Z}_n(x)|,$$

in this case. We use the tilde to indicate that calculations are to be carried out under sampling without replacement in each stratum.

To bootstrap the distribution of $\tilde{D}_n$ we can form samples $\{X_{ij}^*\}$ $j = 1, \ldots, n_i$ from estimated populations $\hat{\chi}_i$ obtained by copying X_{ij}, $j = 1, \ldots, n_i$ either $[\frac{N_i}{n_i}]$ times or $[\frac{N_i}{n_i}] + 1$ times with suitable probabilities as described in (B-F) and then form

$$\tilde{Z}_n^* = n^{1/2} \Sigma_{i=1}^p \pi_i(\hat{F}_i^* - \hat{F}_i)(\frac{n_i}{n_i-1} \frac{N_i-1}{N_i})^{1/2}$$

and proceed as for (4).

An alternative more practicable scheme suggested by Rao and Wu is to sample X_{ij}^{**} <u>with replacement</u> from $\hat{F}_i$, form the corresponding $\hat{F}_i^{**}$ let,

$$\tilde{Z}_n^{**}(\cdot) = n^{1/2} \Sigma_{i=1}^p \pi_i(\hat{F}_i^{**} - \hat{F}_i)(\frac{n_i}{n_i-1} \cdot \frac{N_i-n_i}{N_i-1})^{1/2}$$

The confidence band is again,

$$\hat{F}_n \pm \tilde{V}_n/\sqrt{n} \tag{6}$$

where $\tilde{v}_n$ is the $(1-\alpha)$ quantile of the conditional distribution of $D_n^{**} = \sup_x |\tilde{Z}_n^{**}(x)|$.

<u>Asymptotic theory for the bootstrap</u>

As in B-F (1984) we consider a sequence of problems indexed by n and give conditions on the strata and sampling fractions under which the bootstrap performs well asymptotically. Note that the F_i, π_i, λ_i, p now all depend on n. We make the dependence explicit when necessary by writing n as a (second) subscript. Let,

$$K_n(s,t) = E(Z_n(s)Z_n(t)) = \Sigma_{i=1}^{p} \frac{\pi_{in}^2}{\lambda_{in}} F_{in}(s)(1-F_{in}(t)) .$$

$$\tilde{K}_n(s,t) = E(\tilde{Z}_n(s)\tilde{Z}_n(t)) = \Sigma_{i=1}^{p} \frac{\pi_{in}^2}{\lambda_{in}} \frac{\rho_{in}}{n_{in}} F_{in}(s)(1-F_{in}(t))$$

where

$$\rho_{in} = n_{in}(N_{in} - n_{in})(N_{in} - 1)^{-1}$$

is the "effective sample size" for the i^{th} sample.

Assumptions:

A1: $\pi_{in}/\lambda_{in} \leq M < \infty$ all i,n

A2: $\sup_x |F_n(x) - F_\infty(x)| \to 0$

for some proper distribution F_∞.

A3: For all $s \leq t$

a) $K_n(s,t) \to K(s,t)$

b) $\tilde{K}_n(s,t) \to \tilde{K}(s,t)$

The assumptions are reasonable. The first requires the sampling fraction not be arbitrarily smaller than the stratum fraction. Often, they are equal.

The second specifies that the populations stabilize as $n \to \infty$. The last assumption of stability of the covariance structure of the empirical process is also reasonable but is included only for mathematical convenience.

Theorem 1. If assumptions A1, 2, 3a) (respectively b)) hold then there exists a Gaussian process $Z(\cdot)$ (respectively $\tilde{Z}(\cdot)$) with $D[-\infty, \infty]$ sample paths such that Z_n (respectively $\tilde{Z}_n$) converge weakly to Z, (respectively $\tilde{Z}$).

The functions $K(s,t)$, $\tilde{K}(s,t)$ are themselves necessarily covariance functions of Gaussian processes which in some cases can be identified. In particular, if p and the λ_i, π_i are fixed or stabilize to positive values, the strata are disjoint and the F_{in} converge to limits F_i then $K(s,t) = \tilde{K}(s,t)$ is the covariance of

$$Z = \Sigma_{i=1}^{p} \frac{\pi_i}{\sqrt{\lambda_i}} W_i^o (F_i(\cdot))$$

where the W_i^o are independent Brownian Bridges.

If the strata overlap the process can be represented as a weighted sum of dependent time transformed Brownian Bridges. In any case these representations do not suggest (save for (2)) an analytic approximation to the distribution of D_n. Of course we can obtain the distribution of $\sup_t |Z(t)|$ by simulation but that cannot be expected to improve on the bootstrap either in efficiency or simplicity.

Remark. An examination of the proof of Theorem 1 shows that assumption A1 sufficies to establish tightnes of the processes Z_n, $\hat{Z}_n$. Tightness is also not affected if the π_{in} are replaced by c_{in} with $|c_{in}| \leq K\, \pi_{in}$ for some $K < \infty$ all i, n.

From Theorem 1 we can immediately conclude that under the appropriate conditions D_n converges weakly to $\sup_x |Z(x)|$ and $\tilde{D}_n$ converges weakly to $\sup_x |\tilde{Z}(x)|$. Under the same conditions, Z_n^* converges weakly to Z in probability, $\tilde{Z}_n^*$ and $\tilde{Z}_n^{**}$ converge to $\tilde{Z}$ in probability.

A4: The distribution of

a) $\sup_x |Z(x)|$

b) $\sup_x |\tilde{Z}(x)|$

is continuous. By a theorem of Tsirelson (1975) these distributions are continuous expect possibly at the lower endpoint of their support.

Theorem 2.

a) If A1, 2, 3a), 4a) hold and the X_{ij} are obtained by sampling with replacement then

$$P[F_n \in \hat{F}_n \pm \frac{v_n}{\sqrt{n}}] \to 1 - \alpha .$$

b) If A1, 2, 3b), 4b) hold and the X_{ij} are obtained by sampling without replacement then,

$$P[F_n \in \hat{F}_n \pm \frac{\bar{v}_n}{\sqrt{n}}] \to 1 - \alpha .$$

This result does not say much about the adequacy of the bootstrap approximation if we have many small strata since typically then $K = \bar{K} = 0$ so that $Z = 0$ and D_n tends in law to 0. Condition A4 is then violated and we have no information on how the quantiles of the bootstrap distribution of D_n compare to those of Q_n. In fact we expect the bootstrap approximation to fail in those cases.

For instance, consider the situation where the strata are disjoint $\pi_i = \lambda_i = \frac{1}{p}$ for all i and $n_i = 2$ for all i. Suppose that $N_i = \infty$ for all i and F_∞ is continuous. Without loss of generality we can take F_i to be uniform on $[(i-1)/p, i/p)$. From (1)

$$Q_n(t) = \tilde{Q}_n^p\ (t\sqrt{p})$$

It is easy to see that the density Q_2' is continuous on $[0, \sqrt{2}]$ and $\tilde{Q}(t) \sim 2(1 - t/\sqrt{2})$ as $t \uparrow \sqrt{2}$. Then, by standard arguments, the law of $\sqrt{p}\ (1 - D_n \sqrt{p/2})$ tends in distribution to the law with density $2te^{-t^2}$, $t > 0$ and hence $D_n\sqrt{p/2} \xrightarrow{P} 1$. By a similar argument, the bootstrap distribution of $D_n \sqrt{p/2}$ is that of the maximum of p independent random variables U_i with common distribution

$$P[U_i = 0] = \frac{1}{2} = 1 - P[U_i = \frac{1}{2}]$$

so that $D_n^* \sqrt{p/2} = \frac{1}{2}$ with probabity tending to 1 which is not right. Even scaling D_n^* up by $\sqrt{2}$ does not help. In fact, no matter what the scaling the bootstrap doesn't work since even the distribution of $\sqrt{p}\ (1 - \sqrt{2}\ D_n^* \sqrt{p})$ is asymptotically point mass at 0.

This limit theorem can be used to derive further limit theorems for related processes such as the quantile process. Applications include the Lorenz curve discussed by Gastwirth (1972) as well as weighted versions of the Z_n and Z_n^* processes which can in principle be used to give variable width confidence bands. We do not pursue these here.

2. MONTE CARLO STUDIES

i) <u>The bootstrap in the discrete one sample problem</u>. The parameters of our simulations were chosen as follows:

<u>Distributions</u> F: a) Uniform on integers 1 to 10

b) Poisson (1)

c) Binomial (4,.5)

d) Binomial (9,.5)

These situations were chosen to provide some possibilities of exploration of the effects of

i) Size of carrier: small finite c); large finite a), d); infinite b)

ii) Modality: multimodal a); unimodal b)-d)

<u>Sample sizes</u>: $n = 20, 40$

<u>Number of samples</u>: 1,000

<u>Number of bootstrap replications per sample</u>: 1,000

<u>Confidence levels</u>: $\alpha = .10, .05, .01$.

For each sample, α we calculated

1) The bootstrap confidence limits for F i.e., $q_n^*/\sqrt{n}$

2) D_n.

For each F, n, α we estimated by averaging across samples,

a) $P[D_n \le q_n^*]$ --the probability of coverage of the bootstrap band

b) $P[D_n \le K_n^{-1}(1-\alpha)]$ --the probability of coverage of the conservative K-S band

We also estimated

c) q_n --the actual $(1-\alpha)$ quantile of the d.f. of D_n

d) $E(q_n^*)$ and the percent differences $(E(q_n^*) - q_n)/q_n$, $(K_n^{-1} - q_n)/q_n$.

Our results show a consistent pattern for all three levels so that below we only table results for the 95% level. Given our number of replications we expect the probabilities tabled to be accurate within about ±1% and the width $E(q_n^*/\sqrt{n})$ to be accurate within about ±2%. We can summarize our conclusions as follows:

1) The coverage probability of the bootstrap behaved satisfactorily. It never differed from its nominal value by more than 1% and was not conservative in only three instances, binomial (4,.5) for n = 20, 40 and uniform for n = 20.

2) The percent differences between the expected bootstrap quantile and the actual is negligible. The conservative band is consistently between 20% (for n = 20) and 30% (for n = 40) too long.

3) The risk of the bootstrap in terms of coverage probability decreases with n and its advantage over the K-S band increases with n.

We also made these comparisons for an unrealistic situation in which the bootstrap band can be calculated exactly viz binomial (1,p). In this case, $\sqrt{n}\, q_n^*$ is just the (1-α) quantile for $|B(n,\hat{p}) - n\hat{p}|$ where B(n,p) is a binomial (n,p) variable and $\hat{p}$ is the observed fraction of successes. This is a situation where we expect the conservatism of the K-S band to be most extreme and the bootstrap bands to be shortest. This is indeed the case. For instance, if $p = \frac{1}{3}$, the conservative band is 30-50% too long. However, the actual bootstrap coverage probabilities are seriously low for n = 20 (90% for the nominal 95% interval). For n = 40 the bootstrap is satisfactory and highly preferable to the conservative band.

TABLE 1: Coverage probabilities for the 95% bands.

	B		C	
a)	.96	.96	.99	.98
b)	.94	.95	.99	.99
c)	.94	.94	.98	.99
d)	.95	.99	.96	.99

TABLE 2: Quantiles for the 95% bands.

	A		B		C		(B−A)/A		(C−A)/A	
a)	.25	.18	.25	.18	.29	.21	0	0	.18	.20
b)	.23	.16	.22	.16	-	-	0	0	.27	.33
c)	.24	.16	.22	.16	-	-	-.06	0	.24	.29
d)	.25	.18	.24	.17	-	-	-.03	-.02	.18	.20

Code: a)-d) Distributions

A - Actual

B - Bootstrap

C - Conservative

Column 1: n = 20, Column 2: n = 40

ii) <u>The bootstrap in stratified sampling</u>. The parameters of our simulations were chosen as follows:

<u>Number of strata</u>: $p = 3, 6$

<u>Distributions within strata</u>: In all cases the ith stratum population is a sample of size N_i from an appropriate uniform distribution U_i. The same populations were, of course, used throughout the simulation. The appropriate uniforms were as follows:

a) $U_i = U((i-1)/p)$, $i \le i \le p$; we refer to this case as disjoint.

b) If $p = 3$; $U_1 = U(0, .50)$, $U_2 = U(.25, .75)$ and $U_3 = U(.50, 1.0)$.
If $p = 6$; $U_1 = U(0, .25)$, $U_2 = U(.15, .40)$, $U_3 = U(.30, .55)$, $U_4 = U(.45, .70)$, $U_5 = U(.60, .85)$ and $U_6 = U(.75, 1.0)$; we refer to this case as overlapping.

Sample sizes: If $p = 3$; a) (10,10,10)
b) (20,20,20)
c) (40,10,10)
d) (10,40,10)

If p = 6; a) (5,5,5,5,5,5)

b) (10,10,10,10,10,10)

c) (15,15,15,5,5,5)

d) (5,5,15,15,15,5)

We refer to these cases as a) Small = b) Large = c) First * d) Middle * .

Population sizes:

1. Sampling with replacement (SWR)

   If p = 3; 20 per stratum

   If p = 6; 10 per stratum

2. Sampling without replacement (SWOR)

   If p = 3; 20 per stratum in Small =

   40 per stratum in Large =

   80 per stratum in First * and Middle *

   If p = 6; 10 per stratum in Small =

   20 per stratum in Large =

   30 per stratum in First * and Middle *

Note that in the population case, the population sizes were chosen to be twice the largest sample size. Note also that in the overlapping case the actual amount and type of overlap depends on the populations (strata) which we created. There are, of course, no ties but the interlacing of the strata makes (1) invalid.

Number of samples: 1,000

Number of bootstrap replications per sample: 1,000

Confidence levels: α = .10, .05, .01.

For each sample, α we calculated

1) The bootstrap confidence limits of D_n^* (D_n^{**}) for process (population)

2) D_n($\bar{D}_n$) for process (population).

We estimated by averaging across samples

a) $P[D_n \leq D_n^*$ (D_n^{**})] for process (population) -- the probability of coverage of the bootstrap band

b) $P[D_n \leq K_n^{-1}(1-\alpha)]$ -- where $K_n^{-1}(1-\alpha)$ is computed according to the right-hand side of equation (2). This only applies in the process in the disjoint case.

We also estimated

c) q_n -- the actual $(1-\alpha)$ quantile of the d.f. of D_n

d) $E(q_n^*)$ ($E(q_n^{**})$) for process (population) and the percent differences $(E(q_n^*) - q_n)/q_n$, $(K_n^{-1} - q_n)/q_n$.

Our results show a consistent pattern for α = .01, .05 and .10 so that below we only table results for the 95% level. We can summarize our conclusions as follows:

1) The coverage probabilities for the bootstrap behaved satisfactorily. The tendency was for it to be better when the supports were disjoint than when the supports were overlapping. The overlapping cases tend to be less conservative than the disjoint cases. The coverage probabilities were conservative when there were six strata and the supports were disjoint.

2) The bootstrap band tended to be conservative in the SWR case with disjoint supports. The bootstrap band was not necessarily conservative when the supports were overlapping. The bootstrap quantiles for the SWOR case were comparable to the actual quantiles for the SWOR case but the band was not necessarily conservative.

3) In the disjoint SWR case, the conservative were more conservative on the whole than the bootstrap. This shows up in the quantiles as expected. It also shows up in the coverage probabilities. In the overlapping SWR case, where there is no justification for the "conservative" approximation it does poorly and substantially worse than the bootstrap.

4) One difficulty in both coverage probabilities and quantile estimation stems from the discreteness of the distribution of q_n. For example, the .95 quantile of q_n in the disjoint Small =, SWOR case with six strata is estimated

to be .3651 which is considerably below the bootstrap estimate of .4564. However, the distribution across the 1000 samples is: .1826 (61 times), .2739 (555 times), .3651 (343 times) and .4564(41 times). This means that if we had gotten .4564 at least nine more times then the bootstrap would be excellent.

Table 3 Bootstrap (Conservative) Coverage Probabilities

SWR

	Three Strata		Six Strata	
	Disjoint	Overlapping	Disjoint	Overlapping
Small *	.938 (.980)	.924 (.876)	.967 (.967)	.894 (.827)
Large *	.959 (.959)	.966 (.885)	.957 (.999)	.970 (.970)
First *	.964 (.964)	.917 (.886)	.956 (.983)	.922 (.913)
Middle *	.960 (.960)	.932 (.932)	.965 (.987)	.948 (.951)

SWOR

	Three Strata		Six Strata	
	Disjoint	Overlapping	Disjoint	Overlapping
Small *	.971	.956	.978	.932
Large *	.910	.939	.950	.929
First *	.923	.922	.979	.930
Middle *	.935	.925	.978	.935

Table 4a) Quantiles in the SWR Case

Disjoint

p	Sample	A	B	C	$\frac{B-A}{A}$	$\frac{C-A}{A}$
3	Small =	.8216	.7698	.8418	-.063	.025
3	Large =	.7746	.7947	.8594	.026	.109
3	First *	1.0328	1.0887	1.1216	.054	.086
3	Middle *	1.0328	1.0887	1.1216	.054	.086
6	Small =	.5477	.6124	.5972	.118	.090
6	Large =	.5164	.5447	.6655	.055	.289
6	First *	.7746	.8529	.8184	.101	.056
6	Middle *	.7746	.8546	.8184	.103	.056

Overlapping

p	Sample	A	B	$\frac{B-A}{A}$
3	Small =	1.0042	.9575	-.047
3	Large =	.9037	.9434	.044
3	First *	1.2910	1.2207	-.054
3	Middle *	1.1619	1.1424	-.017
6	Small =	.7746	.8546	.103
6	Large =	.6455	.6818	.056
6	First *	.9037	.8715	-.036
6	Middle *	.8176	.8643	.057

Table 4b) Quantile in the SWOR Case

Disjoint

p	Sample	A	B	$\frac{B-A}{A}$
3	Small =	.5477	.5585	.020
3	Large =	.5809	.5693	-.020
3	First *	1.0651	1.0248	.038
3	Middle *	1.0328	1.0248	-.008
6	Small =	.3651	.4564	.250
6	Large =	.4518	.4181	-.075
6	First *	.7316	.8041	.099
6	Middle *	.7316	.8041	.099

Overlapping

p	Sample	A	B	$\frac{B-A}{A}$
3	Small =	.6390	.6766	.059
3	Large =	.7101	.7261	.023
3	First *	1.2587	1.2298	-.023
3	Middle *	1.0973	1.0555	-.038
6	Small =	.4564	.5223	.144
6	Large =	.5164	.5292	.025
6	First *	.8176	.8375	.021
6	Middle *	.8176	.8153	-.029

APPENDIX

The proof of Theorems 1 and 2 are technical. The important special case of part a) of Theorem 1, $\pi_{in} = \lambda_{in}$, F_∞ uniform appears in Rechtschaffen (1975). Here we sketch the argument for part b) which essentially includes that for part a).

Proof of Theorem 1.

1) Finite dimensional distributions of $\tilde{Z}_n$ converge to finite dimensional distributions of $\tilde{Z}$.

$\tilde{Z}_n(s)$ is of the form $\Sigma_{i=1}^{p} c_i(Y_{i.} - \mu_i)$ where $\{Y_{ij}\}$, $j = 1, \ldots, n_i$, is a sample without replacement from a population of N_i 0's and 1's mean $\mu_i = F_i(s)$ and the $Y_{i.}$ are independent. Moreover,

$$E(\Sigma_{i=1}^{p} c_i(Y_{i.} - \mu_i))^2 = \tilde{K}_n(s,s) \ .$$

If $\tilde{K}(s,s) = 0$, $\tilde{Z}_n(s) \to 0$. Otherwise, we can show $\tilde{Z}_n(s)$ converges in law to $N(0,\tilde{K}(s,s))$ by verifying the conditions of Theorem 3 of (B-F). These can easily be seen to be implied by

$$\inf_i \{\frac{\lambda_{in}^2}{\pi_{in}^2} n\} \to \infty$$

which follows from A1. The same argument establishes asymptotic normality for linear combinations of $\tilde{Z}_n(s_1), \ldots, \tilde{Z}_n(s_k)$ for arbitrary $s_1, \ldots, s_k$.

2) <u>The $\tilde{Z}_n$ processes are tight in $D[-\infty,\infty]$.</u>

<u>Case (i): F continuous.</u>

Then it's enough to show that $\tilde{Z}_n(F_\infty^{-1}(\cdot))$ are tight on $[0,1]$, or equivalently to establish tightness on $D[0,1]$ for $\tilde{Z}_n$ where all the F_i concentrate on $[0,1]$ and F_∞ is the uniform distribution on $(0,1)$.

By a theorem of Hoeffding (1963), for $0 \le s \le t \le 1$,

$$E(\tilde{Z}_n(t) - \tilde{Z}_n(s))^4 \le E(Z_n(s) - Z_n(t))^4 \tag{7}$$

where Z_n is obtained by sampling with replacement in each stratum as in (3). But

$$E(Z_n(s) - Z_n(t))^4 = n^2\{\Sigma_{i=1}^p \pi_{in}^4 E(\hat{F}_{in}(s,t] - F_{in}(s,t])^4$$

$$+ 6 \sum_{i \neq j} \pi_{in}^2 \pi_{jn}^2 E(\hat{F}_{in}(s,t] - F_{in}(s,t])^2 E(\hat{F}_{jn}(s,t] - F_{jn}(s,t])^2\}$$

$$\le n^2\{\Sigma_{i=1}^p \frac{\pi_{in}^4}{n_{in}^3} F_{in}(s,t] + 6(\sum_i \frac{\pi_{in}^2}{n_{in}} F_{in}(s,t])^2\}$$

$$\le n^{-1}M^3 F_n(s,t] + 6M^2 F_n^2(s,t]$$

Since $F_n(s,t] \to (t-s)$ uniformly we can now argue as in Shorack (1972), that the Z_n are tight.

The argument for the remaining cases relies on the follow lemma and corollary.

Let T be a fixed subset of R and

$$\hat{F}_i(x,T) = \hat{F}_i\{(-\infty,x] \quad T\}$$

and define $F_i(\cdot,T)$, $F(\cdot,T)$ similarly suppressing dependence on M. Suppose we sample <u>with replacement</u>. Let,

$$(8) \qquad Q(T) = \sup_x |n^{1/2} \Sigma_{i=1}^p \pi_i(\hat{F}_i(x,T) - F_i(x,T))| .$$

Lemma 1. If A1 holds then, for all $r \geq 1$ there exist $C_r < \infty$, and a function γ: $[0,1] \to R^+$ with $\lim_{a \to 0} \gamma(a) = \infty$, (all depending on M only) such that

$$(9) \qquad P[Q(T) \geq t] \leq C_r \gamma^{-r}(F(\bar{\mathcal{S}}))t^{-r}$$

Proof. Let $(\hat{F}_1'(\cdot,T), \ldots, \hat{F}_p'(\cdot,T))$ be a copy of $(\hat{F}_1(\cdot,T), \ldots, \hat{F}_p(\cdot,T))$. By arguing as in Le Cam (1982)

$$(10) \qquad P[\sup_x n^{1/2} |\Sigma_{i=1}^p \pi_i(\hat{F}_i(x,T) - \hat{F}_i'(x,T)| \geq t]$$

$$\leq CP[|n^{1/2} \Sigma_{i=1}^p \frac{\pi_i}{n_i} (P_i - P_i') \geq \frac{t}{4}]$$

where C is a natural constant and the P_i, P_i' are independent Poisson

$((\log 2)n_i F_i(\mathcal{S}))$. By Bernstein's inequality, the right hand side of (10) can be bounded by

$$C \inf_{u\geq 0} \exp\{-\frac{ut}{4} + 2 \log 2 \ \Sigma_{i=1}^{p} n_i F_i(T)[\cosh(n^{-1/2}\frac{\pi_i}{\lambda_i} u) - 1]$$

$$\leq C \inf_u \exp\{-\frac{ut}{4} + (\log 2)M^3 u^2 F(T) e^{Mu}\}$$

$$= C \exp\{-t \inf_u [(\log 2)M^3 F(T) e^{Mu} \frac{u^2}{t} - \frac{u}{4}]\}$$

Define $v\colon R^+ \to R^+$ by

$$v(w) \exp v(w) = w$$

Then v is increasing, $v(\infty) = \infty$, $v(t) \leq t$. We claim that for any $\varepsilon > 0$, $M < \infty$.

(11) $$\inf_{u\geq 0} (e^{Mu} \varepsilon \frac{u^2}{t} - \frac{u}{4}) \leq -v(tM/8\varepsilon)/8M .$$

To see (11) put $u = M^{-1}v(\frac{tM}{8\varepsilon})$. Put $\varepsilon = (\log 2)M^3 F(T)$, $\delta = 8M^2 \log 2$
$\gamma(a) = v(F^{-1/2}(T))/8M$ $v(a^{-\frac{1}{2}})$ and conclude that for $t \geq \delta F^{1/2}(\mathcal{S})$, (10) is bounded by

(12) $$C \exp(-t\gamma(F(\mathcal{S})))$$

Now, by symmetrization,

$$EQ^r(T) \le E\sup_x \left|n^{1/2}\Sigma_{i=1}^p \pi_i(\hat{F}_i(x,T) - \hat{F}_i'(x,T))\right|^r$$

$$\le \delta^r F^{\frac{r}{2}}(T) + C\Gamma(r)\gamma^{-r}(F(T))$$

by (12). The lemma follows.

Suppose we sample <u>without replacement</u>. Define $\tilde{Q}(T)$ by (8) .

Corollary 1. Under the conditions of Lemma 1, for all $t > 0$, $r \ge 1$

$$P[\tilde{Q}(T) \ge t] \le C_r \gamma^{-r}(F(S))t^{-r}$$

Proof. The map $f \to \sup_x |f(x)|$ from $D[-\infty,\infty]$ to R^+ is convex. By applying the vector version of Hoeffding's (1963) theorem we can deduce that for all $r \ge 1$,

$$E\tilde{Q}^r(T) \le EQ^r(T) .$$

The next case we deal with is,

ii) <u>F_∞ carried on a finite set</u> $S = \{s_1, \ldots, s_k\}$. Then,

$$\sup_x |F_n(x,S) - F_\infty(x)| \to 0$$

The process,

$$\sqrt{n}(\hat{F}_n(x,S) - F_n(x,S)) = \Sigma_{a=1}^k \sqrt{n}(\hat{F}_n(x,\{s_a\}) - F_n(x,\{s_a\}))$$

Tightness of the summands is equivalent to tightness of

$$\Sigma_{i=1}^{p} \pi_i \sqrt{n}(\hat{F}_{in}(\{s_a\}) - F_{in}(\{s_a\}))$$

which follows readily from (B-F) Theorem 3. Finally we can apply Corollary 1 to conclude that,

$$\sup_x |\tilde{Z}_n(x) - \sqrt{n}(\hat{F}_n(x,S) - F_n(x,S))| = \sup_x |\sqrt{n}(\hat{F}_n(x,S^c) - F_n(x,S^c)| \to 0$$

since

$$\sup_x |F_n(x,S^c)| \to 0$$

Tightness follows.

iii) <u>$F_\bullet$ discrete but carried on a denumerable set</u> $S = \{a_1, a_2, \ldots, \}$

By case (ii) $\sqrt{n}(\hat{F}_n(\cdot,S_K) - F_n(\cdot,S_K)$ is tight for every $S_K = \{a_1, \ldots, a_K\}$. Then use Corollary 1 to show that for every $\varepsilon > 0$ there exists K such that for all n,

$$P[\sup_x |\sqrt{n}(\hat{F}_n(x,S_K^c) - F_n(x,S_K^c))| \geq \varepsilon] \leq \varepsilon$$

Tightness follows.

iv) The general case. Write,

$$F_\infty(x) = F_\infty(x,S) + F_\infty(x,S^c)$$

where $F_\infty(\cdot,S)$, $F_\infty(\cdot,S^c)$ are respectively the substochastic discrete and continuous parts of F_∞. Note that $\sup_x |F_n(x,S) - F_\infty(x,S)| \to 0$ and apply cases (i)-(iii).

Theorem 1 is proven.

Proof of Theorem 2. Theorem 2 is equivalent to showing that

$$Z_n^*(\cdot) \to Z(\cdot)$$

weakly in probability and

$$\tilde{Z}_n^{**}(\cdot) \to \tilde{Z}(\cdot)$$

weakly in probability. Consider case a). We can apply Theorem 1 with π_{in} replaced by $\pi_{in}(\frac{n_i}{n_i-1})^{1/2}$ provided we can verify A2, A3 with the appropriate limits. That means we need only show that

$$\sup_x \left| \sum_{i=1}^{p} \pi_{in}\left(1 - \left(\frac{n_i}{n_i-1}\right)^{1/2}\right) (\hat{F}_{in}(x) - F_{in}(x)) \right| \to 0 \tag{13}$$

and

$$(14) \qquad \Sigma_{i=1}^{n} \pi_{in}^{2} \frac{n_i}{n_i-1} \hat{F}_{in}(x)(1-\hat{F}_{in}(y)) \to K(x,y) \quad \text{for all } x \le y$$

But, $\sqrt{n}\, \Sigma_{i=1}^{p} \pi_{in}\left(1-\left(\frac{n_i}{n_i-1}\right)^{1/2}\right)(\hat{F}_{in}(\cdot) - F_{in}(\cdot))$ are tight by the remark following Theorem 1. Also,

$$E\, \Sigma_{i=1}^{p} \pi_{in}\left(1-\left(\frac{n_{in}}{n_{in}-1}\right)^{1/2}\right)(\hat{F}_{in}(x) - F_{in}(x))^2$$

$$\le Mn^{-1}\, \Sigma_{i=1}^{p} \pi_{in}\left(1-\left(\frac{n_{in}}{n_{in}-1}\right)^{1/2}\right)^2 F_{in}(x) \to 0$$

since $n_{in} \ge 2$. Claim (13) follows. Claim (14) is a special case of Theorem 3(ii) of (B-F).

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