AN EIGHT-STATE SOLUTION TO THE FIRING SQUAD SYNCHRONIZATION PROBLEM

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ABSTRACT

A solution to the "Firing Squad Synchronization Problem" is presented which requires only eight states per module. The problem of synchronization for a finite but arbitrarily long row of synchronized modules is to find the state transition function of the modules under the following conditions: module inputs are from immediate neighbors only; for a "start" signal at either end of the row, all modules of the row will simultaneously go to the firing state. The solution is based on a procedure of consecutively bisecting the array down to a length of only two modules at which point the logic carries the modules to the firing state. Time to fire is about 3n where n is the number of modules in the row. An exact expression is presented. Several extensions of the firing-squad problem are suggested. As a check of the solution, the firing-squad procedure was simulated on a computer.

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I. INTRODUCTION

The firing squad synchronization problem devised by J. Myhill first arose in connection with machine self-reproduction in J. von Neumann's cellular automaton.[1] The firing squad synchronization problem was specifically concerned with simultaneously activating all the cells of that portion of a modular lattice which contained the dormant copy of the parent machine. Myhill stated the problem for the one-dimensional case, but the solution carries over to the two- and three-dimensional cases by using the first firing squad to activate more firing squads.

Consider a finite, but <u>arbitrarily</u> long, one-dimensional array of finite-state machines, all of which are called soldiers. The machines are synchronous and identical, and the state of each machine at time $t + 1$ is a function of its state and the state of its two neighbors at time t. The problem is to specify the states and the transitions of the soldiers in such a way that for a starting signal introduced at either end of the array, all the soldiers will be able to go eventually but simultaneously to the firing state. Initially, i.e., at t = 0, all the soldiers are in the quiescent state. The two soldiers at the ends of the array are allowed to be different from the other soldiers since they have only one neighbor. The structure of all the soldiers is fixed, while the length of the array is arbitrary.

A. Waksman presented a 16-state machine as an optimum solution.[2] The solution presented here requires only eight states. However, the 16-state solution reaches the firing state at time 2n-2--n being the number of machines in the array, while the eight-state solution needs about 3n. The eight-state solution presented here has been checked on a computer by simulation for arrays of length 2 to 210.

Since several solutions of the firing squad problem have already been found, several extensions of the problem are suggested in order to make the solution reasonable for such physical systems as signal repeaters between stars and for such conceptual models as cellular differentiation in living organisms where it may be desirable to bring a modular array into a single state with some degree of simultaneity.

II. DESCRIPTION OF SOLUTION

The solution of the firing-squad problem is based upon the idea of propagating two signals down the linear array that will bisect the array into two new, equal-length rows. At the point of bisection, new signals are injected into each new row, and the process continues and successively generates smaller, equal-length rows. When the rows are reduced to a length of two modules, the logic carries the modules to the firing state in three time units.

The bisecting process is realized by two signals, one of which travels down the row three times faster than the other, reflects off the opposite end of the row, and meets the slower signal in the middle. Collision of the fast and slow signals indicates bisection, and four new signals are initiated -- a fast and a slow signal to the right and a similar set to the left. (Fig. 1). When the rows are of length two, collision of the fast signals occurs immediately and forces the modules to the firing state. The solution requires $n \geq 2$ where n is the length of the array.

All modules are initially in the quiescent state, state q. A start signal to an end module sends it to state A. State A initiates the bisecting signals. To get one signal to propagate across the row three times faster than the other, the fast signal, state f, moves one module per time unit, while the slow signal moves one module every three time units. For the slow signal, a module in state a goes to state b and then to state c. State c causes a neighboring module to go to state a, and the slow signal process continues. Which neighbor the fast signal or the slow signal moves to is determined by inhibition signals that propagate along directly behind both the slow and fast signals. The inhibition signals prevent fan-out of the fast and slow signals. The use of inhibition reduced the number of states required in my earlier 13-state solution and simplified the problem of signal reflection at row boundaries.

How the row is bisected depends upon whether the length of the row is an odd or even integer. If its length is odd, only the one central module of the row goes to the bisection state, state A. If the length is even, the two central modules go to state A. Each module goes to the firing state if and only if it and its neighbors are in states c or f.

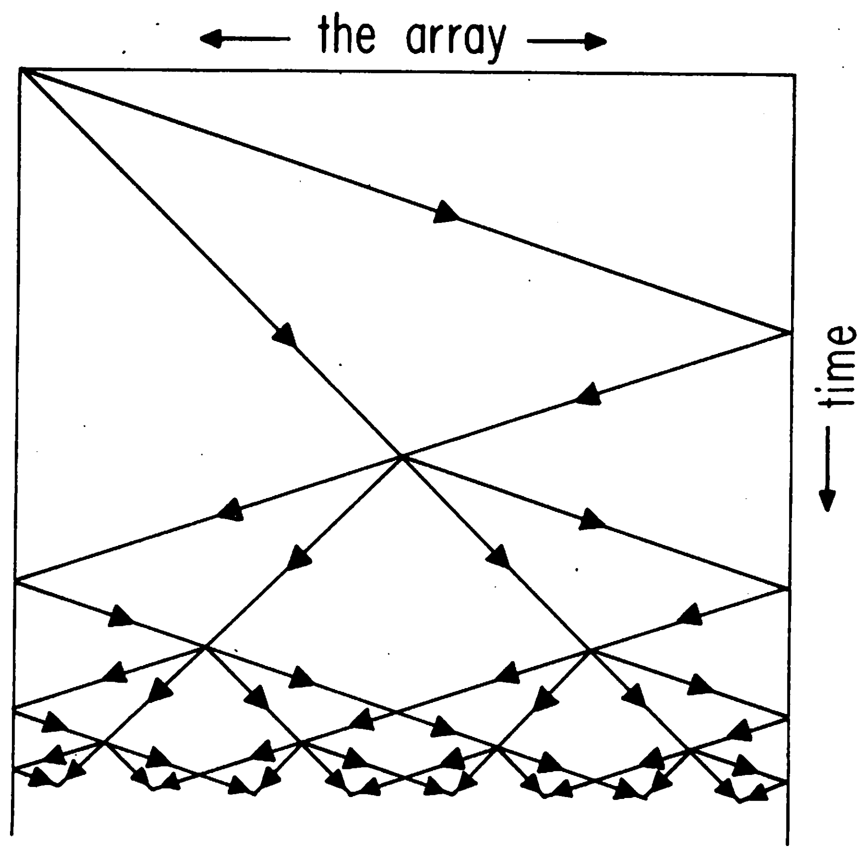

Fig. 1. The pattern of the bisecting signals in the array.

Eight States of Each Module.

q	Quiescent state
a, b, c	Three states that comprise the slow bisecting signal. State b also inhibits fan-out of the fast bisecting signal.
d	Inhibition state preventing fan-out of the slow signal.
f	State generating the fast bisecting signal.
A	Bisection state produced by intersection of the slow and fast bisecting signals. State A generates the bisecting signals.
F	Firing state

Examples of Transition Scheme

The module states as a function of time and position in the array are shown for arrays of lengths 12 and 13 (Fig. 2). Quiescent states are shown by blanks.

III. THEOREM CONCERNING BISECTION

Denote module i of the array by m_i .

Definition: A row at time t is any sequence of modules $\{m_i, m_{i+1}, \ldots, m_j\}$, $j - i \geq 2$, such that for $k = i$ or j, m_k is a module at the end of the array or it has been in state A at or before time t . For $p = i + 1, i + 2, \ldots, j-1$, m_p has not been in state A at or before time t .

Theorem: For a row of length $n > 2$ with the bisection signals introduced at the left (right), if the length of the row is an even integer, intersection of the fast and slow signals, i.e., bisection, occurs with states c and f (f and c) occupying the respective two central modules. If the length of the row is odd, intersection occurs with the central module in state a, the right (left) adjacent module in state f.

n=12

A											
b	f										
c	b	f									
d	a	b	f								
d	b		b	f							
d	c			b	f						
	d	a			b	f					
	d	b				b	f				
	d	c					b	f			
		d	a					b	f		
		d	b						b	f	
		d	c							b	f
			d	a							f
			d	b						f	b
			d	c					f	b	
				d	a			f	b		
				d	b		f	b			
				d	c	f	b				
					A	A					
				f	b	b	f				
			f	b	c	c	b	f			
		f	b	a	d	d	a	b	f		
	f	b		b	d	d	b		b	f	
f	b			c	d	d	c			b	f
f			a	d			d	a			f
b	f		b	d			d	b		f	b
	b	f	c	d			d	c	f	b	
		A	A					A	A		
	f	b	b	f			f	b	b	f	
f	b	c	c	b	f	f	b	c	c	b	f
f	a	d	d	a	f	f	a	d	d	a	f
b	A	d	d	A			A	d	d	A	b
f	b	f	f	b	f	f	b	f	f	b	f
f	c	f	f	c	f	f	c	f	f	c	f
F	F	F	F	F	F	F	F	F	F	F	F

n=13

A												
b	f											
c	b	f										
d	a	b	f									
d	b		b	f								
d	c			b	f							
	d	a			b	f						
	d	b				b	f					
	d	c					b	f				
		d	a					b	f			
		d	b						b	f		
		d	c							b	f	
			d	a							b	f
			d	b								f
			d	c							f	b
				d	a					f	b	
				d	b				f	b		
				d	c			f	b			
					d	a	f	b				
					d	A	b					
					f	b	f					
				f	b	c	b	f				
			f	b	a	d	a	b	f			
		f	b		b	d	b		b	f		
	f	b			c	d	c			b	f	
f	b			a	d		d	a			b	f
f				b	d		d	b				f
b	f			c	d		d	c			f	b
	b	f	a	d				d	a	f	b	
		b	A	d				d	A	b		
		f	b	f				f	b	f		
	f	b	c	b	f		f	b	c	b	f	
f	b	a	d	a	b	f	b	a	d	a	b	f
f		b	d	b		f		b	d	b		f
b	f	c	d	c	f	d	f	c	d	c	f	b
	A	A		A	A		A	A		A	A	
f	b	b	f	b	b	f	b	b	f	b	b	f
f	c	c	f	c	c	f	c	c	f	c	c	f
F	F	F	F	F	F	F	F	F	F	F	F	F

Fig. 2. Module states with time.

Proof: Let T_f denote the amount of time required for the fast signal to move to the indicated point of intersection. Time = 1 when one module of the row is in state A. Let T_s denote the amount of time required for the slow signal to move to the indicated point of intersection.

For a row of length n, n even, $n = 2m$,
then $T_f = n + m = 3m$,

$T_s = 3(\frac{n}{2}) = 3m$.

for $n = 6$: $T_f = T_s = 3m = 9$

t = 1	A
t = 2	bf
t = 3	cbf
t = 4	a f
t = 5	b f
t = 6	c f
t = 7	a f
t = 8	b f
t = 9	cf

For n odd, $n = 2m + 1$
then $T_f = n + m = (2m + 1) + m = 3m + 1$,
$T_s = 3m + 1$.

Consequently, $T_f = T_s$ for both n even and n odd, and the type of collision indicates which module(s) are in the center of the row. Q.E.D.

The Transition Tables for Interior and End Modules

The transition tables for the interior modules are in the form

$s_i(t)$			
T.N.	$s_{i-1}(t)$	$s_{i+1}(t)$	$s_i(t+1)$

where the $s_i(t)$ is the present state of module i ; the neighbors are in states $s_{i-1}(t)$ and $s_{i+1}(t)$; and the transition number T.N. indicates which branch from state $s_i(t)$ in the state transition diagram is to be taken. The transition relationships for the end modules are of the form

Numbers refer to conditions for transition; see Table 1.

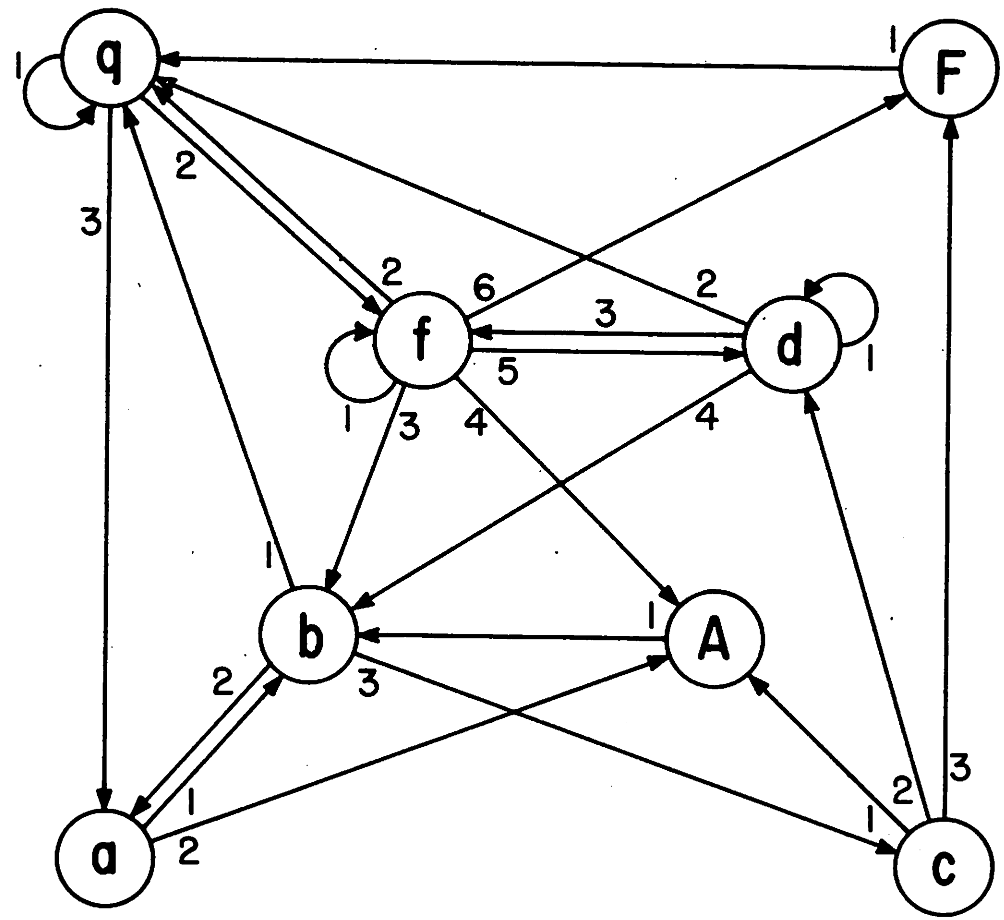

Fig. 3. State transition diagram for an interior module.

Numbers refer to conditions for transition
(see Table 2).

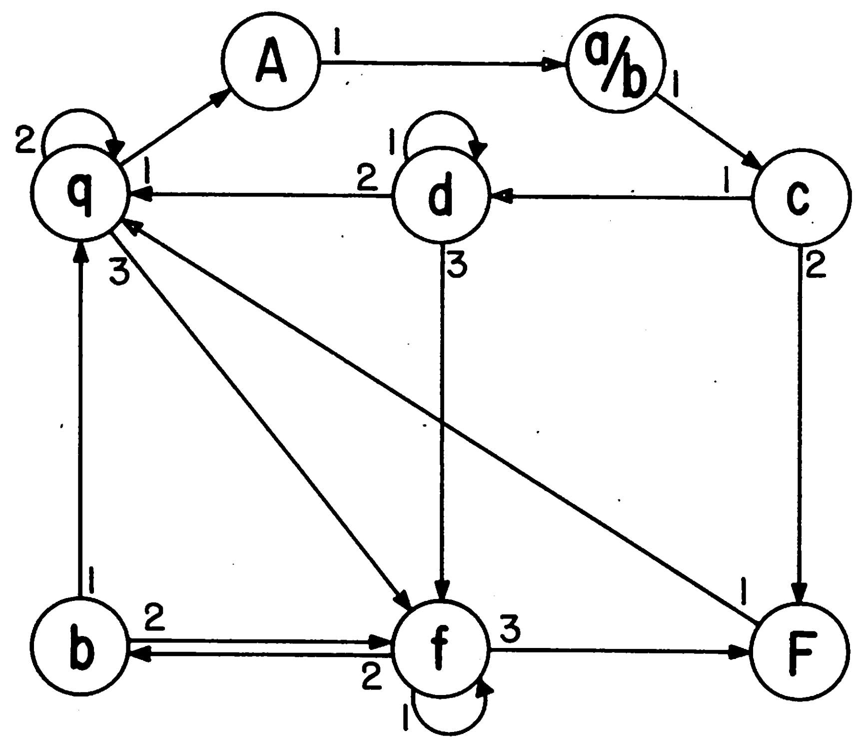

Fig. 4. State transition diagram for the end modules.

Table 1. Transitions for interior modules.

q

1	q q	q
2	f f b f f b f q q f A q q A A A	f
3	c φ φ c	a

A

1	φ φ	b

F

1	φ φ	q

f

1	b b b f f b	f
2	a f f a a a a b b a a d d a	q
3	q b b q q d d q	b
4	b c c b c d d c	A
5	q q q f f q	d
6	c c c f f c	F

d

1	a φ φ a b φ φ b	d
2	q c c q c c d f f d f f	q
3	q A A q d A A d	f
4	A A	b

a

1	d f̸ f̸ d	b
2	d f f d	A

b

1	a f f a q f f q	q
2	c f f c	a
3	f f b f f b q d d q b b	c

c

1	d f f d	A
2	d f̸ f̸ d φ b b φ	d
3	c c c f f c f f	F

Table 2. Transitions for the end modules.

q		
1	Start	A
2	q b d	q
3	f A	f

f		
1	b	f
2	q a	b
3	c	F

d		
1	a b	d
2	c	q
3	A	f

b		
1	f	q
2	A	f

F		
1	ϕ	q

A		
1	ϕ	a

a		
1	ϕ	c

c		
1	b	d
2	f	F

$s_i(t)$		
T. N.	$s_a(t)$	$s_i(t+1)$

where s_a denotes the state of the module adjacent to the end module.

The "don't care" condition is indicated by "ϕ", and "not f" by "$\not f$". In almost all instances we have been able to ignore the difference between the state of the module and its output (input to neighbors) by equating output with state, but for the end modules the (a/b) symbol of the state transition diagram (Fig. 4) denotes that the module is in state a with an output of "state b".

IV. TIME REQUIRED TO REACH FIRING STATE

Theorem: With each module of the row initially in the quiescent state, the amount of time required to reach the firing state is

$$T = 3n - 2 + \sum_{i=1}^{L} [n_i] \, ,$$

where n is the number of modules in the entire array ($n > 2$), and $[n_i]$ is defined as follows:

$$n_1 = n,$$

$$\langle n_i \rangle = \begin{cases} n_i \text{ if } n_i \text{ is even,} \\ n_i + 1 \text{ if } n_i \text{ is odd,} \end{cases}$$

$$n_{i+1} = \frac{\langle n_i \rangle}{2} \, ,$$

$$[n_i] = \begin{cases} 0 \text{ if } n_i \text{ is even,} \\ 1 \text{ if } n_i \text{ is odd.} \end{cases}$$

The condition $n_L = 2$ determines L. Trivially, $n_L = 2$ always exists for some finite L, $n > 1$. Since $n_p = 1$ for $p > L$, the condition is unique.

Example

Consider the array of length 41. $n = 41 = n_1$

$n_1 = 41$	$\langle n_1 \rangle = 42$	$[n_1] = 1$
$n_2 = 21$	$\langle n_2 \rangle = 22$	$[n_2] = 1$
$n_3 = 11$	$\langle n_3 \rangle = 12$	$[n_3] = 1$
$n_4 = 6$	$\langle n_4 \rangle = 6$	$[n_4] = 0$
$n_5 = 3$	$\langle n_5 \rangle = 4$	$[n_5] = 1$
$n_6 = 2$	$\langle n_6 \rangle = 2$	$[n_6] = 0$

$$L = 6, \quad T = 3n - 2 + \sum_{i=1}^{6} [n_i] = 123 - 2 + 4 = 125$$

Proof:

The proof is by finite induction.

1a. For $n = 2$, $T = 4$ both by the formula and by the logic.

1b. For $n = 3$, $T = 8$ both by the formula and by the logic.

2a. Adopting the convention T_n for the time to fire, T, of an array of length n, we assume that n is even, $n = 2m$, $m > 1$.

To divide once (time from an intersection to the next intersection) takes a time $2m + m = 3m$ by the bisection theorem and the new rows are of length m, i.e.,

$n_1 = 2m$, $n_2 = m$.

$T_n = 3m + T_m$.

$$T_n = 3n - 2 + \sum_{i=1}^{L} [n_i]$$

$$= 3(2m) - 2 + ([n_1] + \sum_{i=2}^{L} [n_i]), \quad \text{but } [n_1] = [2m] = 0, \ n_2 = m,$$

(cont'd)

$$T_n = 3m + (3m - 2 + \sum_{i=2}^{L} [n_i]).$$

Since $m = n_2$, we may write $T_m = 3m - 2 + \sum_{i=2}^{L} [n_i]$.

Thus $T_n = 3m + T_m$ so that if T_m is correct, T_n is correct, n even.

2b. Assume that n is odd, $n = 2m - 1$, $m > 2$

Dividing the row once takes time $(2m - 1) + (m - 1) = 3m - 2$ by the bisection theorem. The new rows are of length m.

$n_1 = n$, $n_2 = m$, $[n_1] = 1$.

Thus it must be shown that the timing formula satisfies the expression

$$T_n = 3m - 2 + T_m.$$

$$T_n = 3n - 2 + \sum_{i=1}^{L} [n_i]$$

$$= 3(2m - 1) - 2 + (1 + \sum_{i=2}^{L} [n_i])$$

$$= (3m - 2) + (3m - 2 + \sum_{i=2}^{L} [n_i]).$$

Since $m = n_2$, we may write $T_m = 3m - 2 + \sum_{i=2}^{L} [n_i]$.

Thus $T_n = 3m - 2 + T_m$ and if T_m is correct then T_n is also, when n is odd.

3. Since it has been shown that the expression for the time to the firing state is correct for $n = 2$, $n = 3$, and that it satisfies the recursive relationships

$$T_n = 3m + T_m \text{ for } n = 2m, \quad n > 2 \text{ or}$$

$$T_n = 3m - 2 + T_m \text{ for } n = 2m - 1, \quad n > 3,$$

it follows by the principle of finite induction that the relationship is correct for any row of length n, $n > 1$. Q.E.D.

As observed previously, the time to fire is about 3n.

V. EXTENSIONS OF THE FIRING SQUAD SYNCHRONIZATION PROBLEM

If the solution to the firing squad problem were found to be relevant to physical systems, there are several extensions of the problem that would be of interest. Relevant systems might be repeater stations spaced between stars or biological cells in the process of differentiation or heart muscles in the process of contraction.

In one extension of the problem, the process could start from any module, not just from the end modules. This extension and an indicated solution was suggested by R. Moore of the Department of Mathematics, U. C., Berkeley. The solution is based upon finding the mirror image of the initially triggered module, using these two modules (the triggered and image modules) to find the center of the row by bisection, then using the previously described procedure to get to the firing state. There are several ways to find the image module. (See Fig. 5).

While Fig. 6 is only a sketched example of the solution, Professor Moore's observation reduces the remainder of the problem to details.

A second extension of the firing squad problem would be to find a reliable solution in the presence of logical errors in computation. That

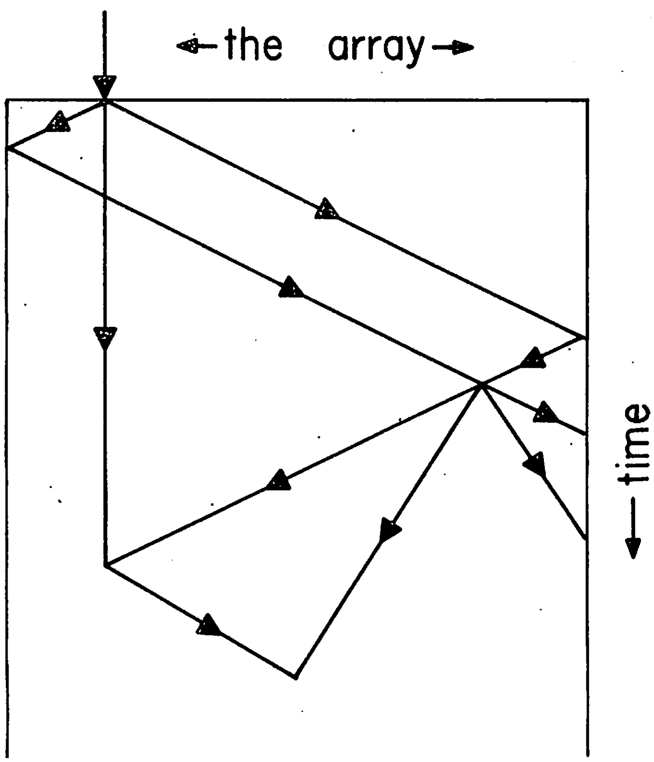

Fig. 5. The initial signal pattern for an arbitrary starting position.

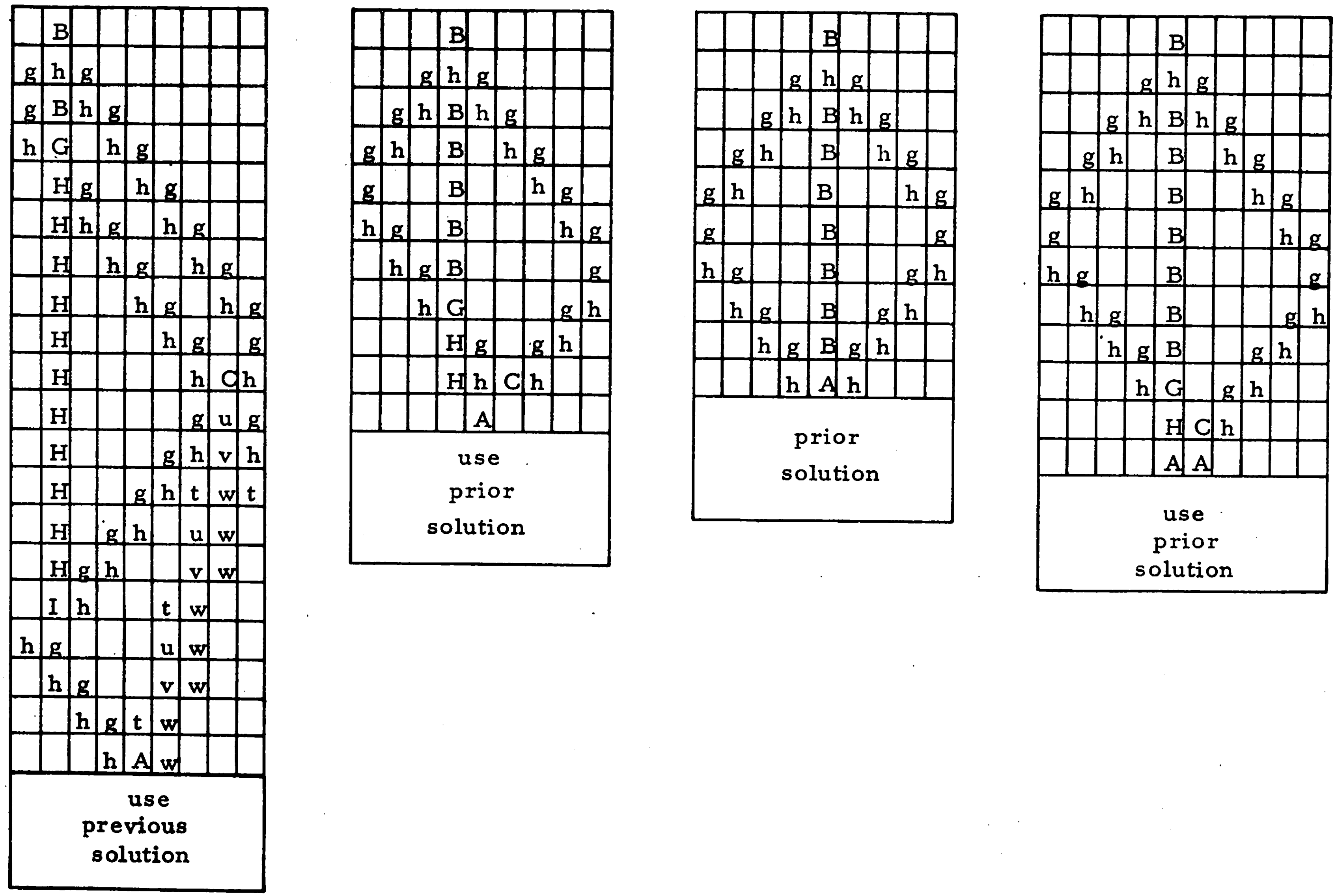

Fig. 6. Examples for n = 9, 10.

is, devise an error-correcting scheme that can reconstruct lost bisecting signals and eliminate extraneous states, when each module has a probability p of computing the next state incorrectly.

A third extension of the firing squad problem would be a solution for systems where the modules are not synchronous but have slightly different time intervals between state transitions. In this extension the bisecting signals would propagate erratically down their respective rows and might eventually cause some portions of the array to go to the firing state before others. To make the problem explicit, assume that the new state of a module depends upon the states of its neighbors at its time of transition. (If a neighbor is in the process of transition, use that neighbor's prior state.) It would be useful to say something about how extensive the error-correcting code needs to be as a function of the maximum difference in transition times of the modules. A modification of the problem would be to bound the maximum difference in transition times between any two <u>adjacent</u> modules.

VI. CONCLUSION

The Firing Squad Synchronization problem is concerned with <u>simultaneously</u> activating all modules of a synchronous, linear, modular array of <u>arbitrary</u> length. Information is transferred only between <u>immediate</u> neighbors. The "start the process" signal may be introduced to only one module. A solution to the Firing Squad Synchronization problem is presented which requires only eight states per module. Due to the general nature of the problem, the solution is expected to offer insight into the communication processes during mitosis and differentiation in living cells and have application for such devices as signal repeaters between stars. Several extensions of the firing squad problem are suggested which are of practical as well as theoretical interest. Their intent is to remove the requirements on the modules of synchrony and error-free computation of states.

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