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Simulation Techniques for Noise in Non-Autonomous Radio Frequency Circuits

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Abstract

In this paper we consider the problem of noise analysis for non-autonomous nonlinear RF circuits in presence of input signal phase noise. We formulate this problem as a stochastic differential equation and solve it in the presence of white-noise sources. We then relate this solution to results of the existing nonlinear time-domain and frequency-domain methods of noise analysis and point out the modifications required for the present techniques. We illustrate our technique using an example.

1 Introduction

In high speed communication, instrumentation and signal processing applications, random electrical noise that emanates from devices has a direct impact on critical high level specifications, for instance, bit error rate (BER) or signal to noise ratio (SNR), blocking performance, spectral leakage. Hence predicting noise in such systems at the design stage is extremely important. RF circuits are usually analyzed for their steady state behaviour under one or more periodic excitations. In a typical RF system, some amplifiers and other analog circuits such as mixers, filters and oscillators do not operate in small signal condition. These circuits usually have one or more large signal time-varying inputs which cause the statistics of the circuit noise sources to be time varying. Hence stationary noise analysis techniques, a la SPICE, are inadequate for analyzing the noise behaviour of such circuits since they do not capture important non-linear aspects such as frequency translation of noise spectra.

Several techniques have been proposed, both in the time domain [Hul92, DLSV96, OTIS93] and in the frequency domain [RLF98, RMM94], for predicting noise performance for nonlinear circuits. Both these classes of techniques take advantage of the fact that for most RF applications, the circuit is driven by periodic (or *quasi*-periodic) signals. Hence only the steady state performance of the circuit over a small time interval, usually over one period of the input signal is sufficient to describe its behaviour. Harmonic balance techniques assume that the periodic circuit response can be expressed in terms of a small number of harmonics and solve the nonlinear algebraic equations for each harmonic. Time domain techniques use finite difference Newton or shooting methods to obtain the steady state response of the circuit. Noise analysis is then performed by linearizing the circuit around this time varying response. The underlying assumption is that small perturbations, deterministic or stochastic, result in small deviations in the response of the circuit, leading to additive noise in the case of stochastic perturbations. This assumption is rigorously justified for stable non-autonomous systems in the presence of large deterministic signals. If the noisy input signal can be represented as an additive noise over and above the deterministic periodic signal, the small deviation assumption is again justified. The input signal noise can be assumed to be a circuit noise source with equivalent statistics, at the input node. These techniques conclude that the circuit noise statistics are periodically time-varying in presence of periodic inputs. However, [TKW96] conclude that only the stationary component of this noise is important without giving a mathematically rigorous justification. The popular use of noise figure of the individual blocks in a receiver path to compute its overall noise performance also assumes that the output noise is stationary. On the other hand, [RLF98] conclude that this notion of noise figure is not sufficient to characterize the output noise and resort to computing the full cyclostationary statistics at the output of a block.

However, it has been shown [DMR98] that it is not mathematically rigorous to view the oscillator output as a deterministic signal with additive phase and amplitude noise. This is due to the fact that linear perturbation analysis is not valid for autonomous systems. A mathematically consistent representation of the oscillator output is a sum two wide-sense stationary stochastic processes: a large signal output process with phase deviation which has the statistics of a Wiener process (Brownian motion) and a "small" amplitude noise process. Hence approaches that attempt to perform stationary/cyclostationary noise analysis of non-linear RF driven systems (e.g. mixers etc.) need to be carefully re-examined.

This paper addresses the problem of formulating and solving the circuit equations in presence of noisy oscillator input signal, which is assumed to the sum of a "small" amplitude deviation process and a large stochastic process, derived from the noiseless oscillator output by introducing Brownian motion phase variation. Our main results are summarized below:

- It is shown that the output of nonlinear non-autonomous systems in the presence of period input with Brownian motion phase deviation, is asymptotically wide-sense stationary.
- The Lorentzian spectrum of the input signal and the characteristics of the Brownian motion input phase deviation process are preserved at the output.
- Noisy input is shown to contribute a wide-band amplitude noise term at the output of the nonlinear circuit. This appears as a white noise source modulated by the time derivative of the steady state response of the system.
- Associated modifications to the existing cyclostationary noise analysis algorithms, both in the time and frequency domain are suggested.

The intuition behind these results is the fact that the non-autonomous system in conjunction with the driving oscillator can be viewed as a composite large oscillator. Hence the observations made in [DMR98] about the output of the noisy oscillator carry over to this composite system which is also autonomous. However, for the nonlinear circuits the frequencies of interest typically are far away from the input signal frequency (or any harmonics of that) and hence the noise due to the phase deviation process in the frequency range of interest is small, compared to the wide band amplitude noise process. Hence, in contrast to the oscillator phase noise analysis, we concentrate on the amplitude noise process here.

The rest of the paper is organized as follows. In Section 2 we introduce some basic mathematical notation about non-autonomous systems. We begin our analysis (Section 3) by briefly reviewing the system noise equations in presence of a deterministic large periodic input signal. We then analyze the noiseless system with input signal phase noise only and show that the general noise analysis is an extension of this case. Finally (Section 4) we demonstrate our technique with an example.

2 Mathematical Preliminaries

The dynamics of a unperturbed non-autonomous system can be described by the following system of differential equations

$$\dot{x} = f(x) + b_0(t) \tag{2.1}$$

where $x \in \mathbb{R}^n$ is a vector of state variables, $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ and $b_0(t) : \mathbb{R} \to \mathbb{R}^n$ is deterministic T-periodic input. We assume that this equation satisfies the Cauchy-Peano existence and uniqueness theorem for the initial value problem [Gri90]. We further assume that the system is stable in the sense that in the absence of $b_0(t)$, the steady state solution of this equation is 0. We assume that the steady state solution of this system (in presence of $b_0(t)$) is given by $x_s(t)$, which is also periodic with period T. This assumption is justified for almost all non-autonomous RF components except frequency dividers where the output is periodic with a larger period T'. The analysis we present here is therefore not valid for frequency dividers.

We are interested in the response of this system in the presence of noise, both in the form of circuit intrinsic noise $D(x)\xi(t)$ and phase noise in the input signal of the form $b_0(t+\alpha(t))$ where $D(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ describes the connectivity and modulation of the noise sources, $\xi(\cdot) : \mathbb{R} \to \mathbb{R}^p$ are white noise sources and $\alpha(\cdot) : \mathbb{R} \to \mathbb{R}$ is the phase deviation process of the input signal which is a scaled Brownian motion process, i.e., of the form $\sqrt{c}B(t)$ where c is the rate of increase of the variance. Hence the modified system is governed by the following differential equation.

$$\dot{x} = f(x) + b_0(t + \alpha(t)) + D(x)\xi(t)$$

or equivalently, in stochastic differential equation form as

$$\mathrm{d}x = f(x)\mathrm{d}t + b_0(t + \alpha(t))\mathrm{d}t + D(x)\mathrm{d}B_p(t) \tag{2.2}$$

where $B_p(t)$ is a p-dimensional Brownian motion. For sake of simplicity, we use the state equation formulation to describe the system. These results and techniques can be extended to the mixed differential-algebraic equation formulation (for instance, as in modified nodal analysis (MNA)) of the form $\frac{\mathrm{d}q(x)}{\mathrm{d}t} + I(x) = 0$ in a straightforward manner.

3 Noise Analysis of Non-Autonomous Systems

We begin with a brief review of classical cyclostationary noise analysis.

3.1 Cyclostationary Approach

Consider the above system of equations (2.2) but with ideal input source signal $b_0(t)$, i.e.,

$$\mathrm{d}x_s = f(x)\mathrm{d}t + b_0(t)\mathrm{d}t + D(x)\mathrm{d}B_p(t) \tag{3.1}$$

Assume that the perturbed response of this system is $x_s(t)+y(t)$ where $y(t)$ is the small stochastic deviation of the response of the system. Substituting this in (3.1) we have

$$\begin{aligned}\mathrm{d}x_s(t) + \mathrm{d}y(t) = f(x_s(t)+y(t))\mathrm{d}t + b_0(t)\mathrm{d}t \\ + D\left(x_s(t)+y(t)\right)\mathrm{d}B_p(t)\end{aligned}$$

Linearizing $f(x_s(t)+y(t))$ around $x_s(t)$, ignoring $y(t)$ in the argument of $B(\cdot)$ and using that fact that $x_s(t)$ satisfies (2.1), the above equation reduces to

$$\mathrm{d}y(t) \approx \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_s(t)} y(t)\mathrm{d}t + D(x_s(t))\mathrm{d}B_p(t) \tag{3.2}$$

where $\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_s(t)} = J(t)$ is the Jacobian of $f(x)$ evaluated at $x_s(t)$. Since $x_s(t)$ is T-periodic, it follows that $J(t)$ is also T-periodic. Since $D(x_s(t))$ is also T-periodic, and the system of equations is linear in $y(t)$, the above system of equations describes a linear periodic time-varying system of equations governing the deviation of the circuit response and $y(t)$ is also cyclostationary. The time-varying statistics of $y(t)$ are usually computed by considering the periodic time-varying noise as an input to a linear periodically time-varying system corresponding to (3.2) which is computed directly from the steady state response of the circuit.

3.2 Response to Input Signal Phase Noise

We now introduce our approach to solving (2.2). To illustrate the basic principles we will assume that the nonlinear circuit itself is noiseless, i.e., $D(x) = 0$. We will relax this assumption later. As indicated earlier, the additive amplitude noise component of the input signal can also be absorbed in the circuit equations so we will only consider an input signal which has phase deviation but no amplitude noise, i.e., of the form $b_0(t+\alpha(t))$. Hence (2.2) reduces to

$$\dot{x} = f(x) + b_0(t+\alpha(t))$$

or equivalently

$$\mathrm{d}x = f(x)\mathrm{d}t + b_0(t+\alpha(t))\mathrm{d}t \tag{3.3}$$

where as before $\alpha(t) = \sqrt{c}B(t)$. Assuming that c is small, i.e., the input signal phase noise is small and the system is stable, the response of the system is of the form

$$x_s(t+\alpha(t)) + y_1(t)$$

where $y_1(t)$ is assumed to be small. By choosing the response to be of this form, we are assuming that the circuit is able to follow any variations in instantaneous input frequency. This is a valid assumption if input signal phase noise (i.e., c) is assumed to be small and the nonlinear circuit is stable (non-oscillatory).

Definition 3.1 *Define* $s(t) = t + \alpha(t)$.

Also let

$$\dot{x}_s(t) = \frac{\mathrm{d}x_s}{\mathrm{d}t} \text{ and } \ddot{x}_s(t) = \frac{\mathrm{d}^2x_s}{\mathrm{d}t^2}$$

We note that

$$\begin{aligned}\mathrm{d}x_s(s(t)) &= x_s(s(t+\mathrm{d}t)) - x_s(s(t)) \\ &= \dot{x}_s(s(t))\mathrm{d}s(t) + \frac{1}{2}\ddot{x}_s(s(t))[\mathrm{d}s(t)]^2 \\ &= \dot{x}_s(s(t))(\mathrm{d}t + \sqrt{c}\mathrm{d}B(t)) + \frac{c}{2}\ddot{x}_s(s(t))\mathrm{d}t\end{aligned}$$

where we have used the fact that $(\mathrm{d}t)^2 = \mathrm{d}t\mathrm{d}B(t) = 0$ and $(\mathrm{d}B(t))^2 = \mathrm{d}t$ [Øk98]. Notice that the second term in the above expansion is due to the fact that $\mathrm{d}B(t)$ is of the order of $\sqrt{\mathrm{d}t}$. Substituting this expression in (3.3) and linearizing $f(x)$ around $x_s(s(t))$ we obtain

$$\begin{aligned}&\mathrm{d}y_1(t) + \dot{x}_s(s(t))(\mathrm{d}t + \sqrt{c}\mathrm{d}B(t)) + \frac{c}{2}\ddot{x}_s(s(t))\mathrm{d}t \\ &\approx f(x_s(s(t)))\mathrm{d}t + \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_s(s(t))} y_1(t)\mathrm{d}t + b_0(s(t))\mathrm{d}t\end{aligned}$$

Since $x_s(t)$ is the steady state solution of (2.1),

$$\frac{\mathrm{d}x_s}{\mathrm{d}t}(s(t)) = f(x_s(s(t))) + b_0(s(t))$$

and hence

$$\begin{aligned}\mathrm{d}y_1(t) = J(s(t))y_1(t)\mathrm{d}t + M_1(s(t))\mathrm{d}B(t) \\ + M_2(s(t))\mathrm{d}t\end{aligned} \tag{3.4}$$

where $M_1(t) = -\sqrt{c}\dot{x}_s(t)$ and $M_2(t) = -0.5c\ddot{x}_s(t)$ are also T-periodic.

Remark:

- The term $M_1(s(t))\mathrm{d}B(t)$ represents a white noise source modulated by the time derivative of the steady state response. This means that phase noise in the input signal results in a time-varying wide-band noise at the output of the nonlinear circuit.
- The periodic coefficients J, M_1 and M_2 are evaluated at $s(t) = t + \alpha(t)$ and *not* at t.

- (3.4) is a stochastic differential equation which is linear in $y_1(t)$ and the terms $M_1(s(t))\mathrm{d}B(t)$ and $M_2(s(t))\mathrm{d}t$ represent two inputs to this linear system. Hence $y_1(t)$ can be represented as $y_{11}(t) + y_{12}(t)$ where $y_{11}(t)$ satisfies

$$\mathrm{d}y_{11}(t) = J(s(t))y_{11}(t)\mathrm{d}t + M_1(s(t))\mathrm{d}B(t) \quad (3.5)$$

and $y_{12}(t)$ satisfies

$$\mathrm{d}y_{12}(t) = J(s(t))y_{12}(t)\mathrm{d}t + M_2(s(t))\mathrm{d}t \quad (3.6)$$

To solve (3.5) we make the following useful observations:

Definition 3.2 *Define $U(t)$ as*

$$U(t) = \sqrt{c}B(t) \mod T$$

Lemma 3.3 *The solution of (3.5) is the same as the solution of*

$$\mathrm{d}y_{11}(t) = J(t+U(t))y_{11}(t)\mathrm{d}t + M_1(t+U(t))\mathrm{d}B(t)$$

Proof: Follows from the fact that $J(t)$ and $M_1(t)$ are T-periodic. ■

Lemma 3.4 *Asymptotically $U(t)$ is a random process which is uniformly distributed between 0 and T for every t.*

Proof: Obvious. ■

Definition 3.5 *Define $r = t + U(t)$ and $z_{11}(r) = y_{11}(t)$.*

Then using the fact that c is small, it follows that (3.5) is equivalent to the following equation

$$\mathrm{d}z_{11}(r) = J(r)z_{11}(r)\mathrm{d}r + M_1(r)\mathrm{d}B(r)$$

Note that this equation is in the exact same form as (3.2). This means that $z_{11}(r)$ is a cyclostationary process. Moreover, since $J(\cdot)$ is the Jacobian of a stable system, if $M_1(r)w(r)$ is small, $z_{11}(r)$ is small for all r. Hence the above analysis is consistent.

Using the fact that $y_{11}(t) = z_{11}(r) = z_{11}(t+U(t))$ and $U(t)$ is uniformly distributed between 0 and T for all t,

Theorem 3.6

- *$y_{11}(t)$ is stationary*
- *The autocorrelation $\mathbb{E}\left[y_{11}(t)y_{11}^T(t+\tau)\right]$, where $y_{11}(t)$ is the solution of*

$$\mathrm{d}y_{11}(t) = J(t+\alpha(t))y_{11}(t)\mathrm{d}t + M_1(t+\alpha(t))\mathrm{d}B(t)$$

is the stationary component of $\mathbb{E}\left[z_{11}(t)z_{11}^T(t+\tau)\right]$ where $z_{11}(t)$ is the solution of

$$\mathrm{d}z_{11} = J(t)z_{11}(t)\mathrm{d}t + M_1(t)\mathrm{d}B(t)$$

Proof: [Pap91] ■

Now we consider (3.6). Defining $z_{12}(s) = y_{12}(t)$ as before we conclude that $z_{12}(s)$ satisfies the following differential equation

$$\mathrm{d}z_{12} = J(s)z_{12}(s)\mathrm{d}s + M_2(s)\mathrm{d}s$$

Using the same arguments we can conclude that the steady state solution $z_{12}(s)$ of the above equation remains small and bounded for small c's. $y_{12}(t) = z_{12}(t+\alpha(t))$ is therefore a wide-sense stationary stochastic process with a noise spectrum which is very similar to the spectrum of $x_s(t+\alpha(t))$ except that it is much smaller in magnitude. Hence as indicated in Section 1 this typically contributes to noise power outside the frequency band of interest.

3.3 General Noise Analysis

We now consider (2.2)

$$\mathrm{d}x = f(x)\mathrm{d}t + b_0(t+\alpha(t))\mathrm{d}t + D(x)\mathrm{d}B_p(t)$$

We assume that the response of the circuit is of the form

$$x_s(t+\alpha(t)) + y_0(t)$$

Proceeding exactly as in the previous subsection, we conclude that $y_0(t) = y_{01}(t) + y_{02}(t)$ where $y_{01}(t)$ satisfies

$$\begin{aligned}\mathrm{d}y_{01}(t) = J(s(t))y_{01}(t)\mathrm{d}t + M_1(s(t))\mathrm{d}B(t) \\ + M_0(s(t))\mathrm{d}B_p(t)\end{aligned} \quad (3.7)$$

where $B(t)$ and $B_p(t)$ are uncorrelated and $M_0(t) = B(x_s(t))$. $y_{02}(t)$ is still given by (3.6). (3.7) can be rewritten as

$$\mathrm{d}y_{01}(t) = J(s(t))y_{01}(t)\mathrm{d}t + M(s(t))\mathrm{d}B_{p+1}(t)$$

where

$$M(t) = \begin{bmatrix} M_1(t) & M_0(t) \end{bmatrix} \text{ and } B_{p+1}(t) = \begin{bmatrix} B(t) \\ B_p(t) \end{bmatrix}.$$

Corollary 3.7

- *$y_{01}(t)$ is stationary*
- *The autocorrelation $\mathbb{E}\left[y_{01}(t)y_{01}^T(t+\tau)\right]$, where $y_{01}(t)$ is the solution of*

$$\mathrm{d}y_{01}(t) = J(t+\alpha(t))y_{01}(t)\mathrm{d}t + M(t+\alpha(t))\mathrm{d}B_{p+1}(t)$$

is the stationary component of $\mathbb{E}\left[z_{01}(t)z_{01}^T(t+\tau)\right]$ where $z_{01}(t)$ is the solution of

$$\mathrm{d}z_{01} = J(t)z_{01}(t)\mathrm{d}t + M(t)\mathrm{d}B_{p+1}(t)$$

Hence we conclude that we can still use the existing nonlinear noise simulation algorithms for predicting noise in the non-autonomous nonlinear systems with a couple of modifications.

- We need to add another noise source to the noise equations corresponding to the phase to wide-band amplitude noise conversion of the input signal phase noise by the nonlinear system. For this we first need to perform noise analysis of the oscillator(s) to determine the phase noise performance of the input signal.
- We only need to consider the stationary component of the cyclostationary noise statistics computed by the algorithm.

Remark: The above analysis makes the assumption that the input signal phase noise are uncorrelated with the circuit noise sources and noise coming from any other input port. Consider the case when the LNA in the receiver path is driven by a small desired signal and a large blocker. The blocker acts as an LO for the nonlinearities present in the LNA. Hence the LNA output consists of a large in-band blocker along with LNA output noise which is correlated to the blocker. Noise analysis of subsequent blocks will have to take this correlation into account until the in-band power of the blocker drops below the noise floor. This can be problem can be finessed by analyzing the cascade of circuit blocks till the in-band power of the blocker is negligible. This does increase the circuit size but if efficient algorithms coupled with iterative linear solvers are used, the running time increases almost linearly (actually $O(n \log n)$).

4 Experimental Results

The noise simulation algorithm is implemented in MATLAB. We use the time domain technique presented in [DMR98] for performing noise simulation for oscillators and the harmonic balance based technique presented in [RLF98] to perform the noise analysis of the non-autonomous portions of the circuit. The steady state response of the circuit and the Jacobians are computed by performing transient simulations in SPICE3 and later handed over to MATLAB.

We illustrate our technique using an example. Consider the oscillator shown in Figure 1. The basic configuration is a Colpitts oscillator. This circuit has 11 state variables and 8 noise sources. c was computed to be 3.19×10^{-19} sec which corresponds to relative noise power of 98.1 dBc/Hz below the carrier at an offset frequency of 100 kHz. This oscillator is used to generate the 2.2 GHz LO which drives the mixer shown

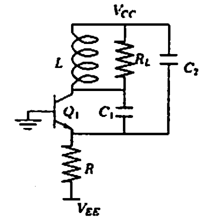

Figure 1: Colpitt's oscillator

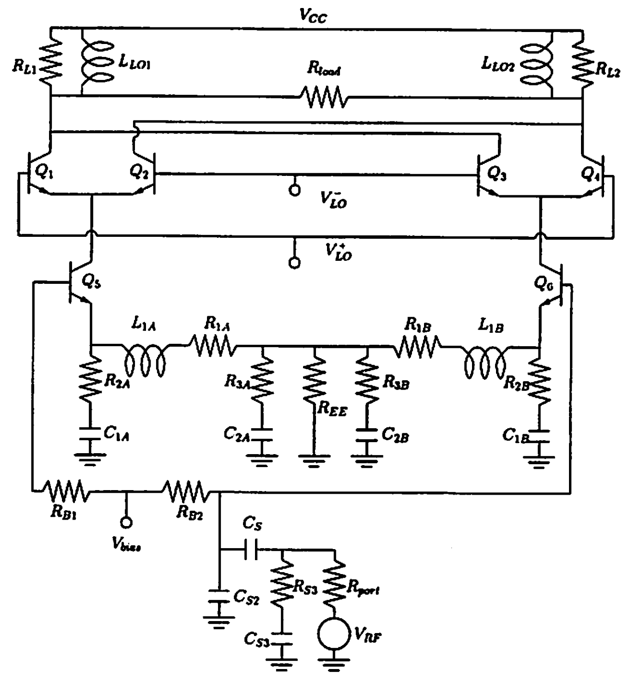

Figure 2: Gilbert cell based mixer

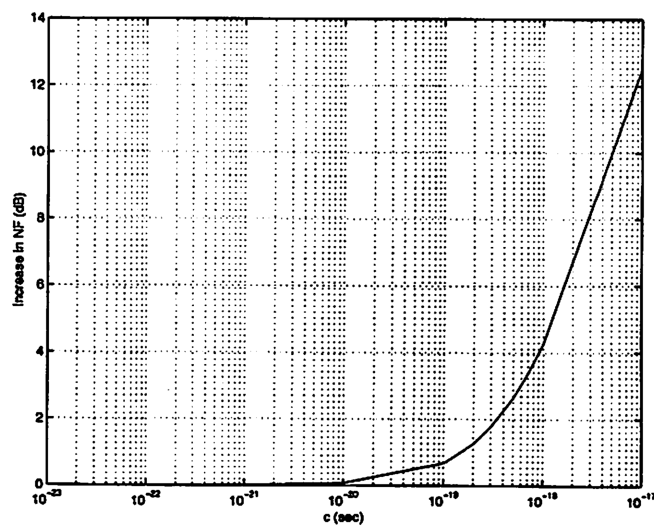

Figure 3: Increase of mixer NF with input signal phase noise

in Figure 2. The mixer circuit has 53 state variables along with 46 noise sources (excluding the one added for the oscillator noise contribution). The RF signal is assumed to come from a 50 Ω port at 2.4 GHz. The noise figure of this mixer at the IF port at 200MHz, without the contribution of the LO phase, noise was computed to be 9 dB. Including the effect of LO phase noise, the noise figure increased to 10.85 dB.

Figure 3 shows the increase in noise figure (from the noiseless oscillator case) as a function of c for this circuit. This increase is negligible for $c < 1 \times 10^{-20}$ sec but as c increases beyond this value, the noise figure degrades rapidly. This cross-over point is the value of c where the input signal phase noise starts dominating over the circuit noise. This also suggests that for this particular mixer, it is an overkill for the LO to have phase noise performance better than 113 dBc/Hz at 100 kHz offset.

5 Conclusions

This paper addresses the problem of performing noise simulation for non-autonomous nonlinear circuits driven by large periodic signals which are themselves generated by oscillators and therefore have phase noise. We showed that noise at the output of these systems is stationary and that we can use a modified version of existing nonlinear noise simulation techniques to evaluate noise performance. We illustrate this technique using a simple example.

This technique, as presented here can only handle white noise sources. However for noise with long-term correlations, i.e., flicker noise, the steps outlines above are not rigorously justified. [DLSV96] used the modulated stationary noise model to analyze flicker noise. However, the asymptotic arguments in this formulation need to be carefully examined before these results can be carried over to the flicker noise case as well.

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