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WITH INEQUALITY CONSTRAINTS

by

E. Polak and D.Q. Mayne

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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E. Polak and D. Q. Mayne

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

This paper presents a secant method, based on R. B. Wilson's formula, for the solution of optimization problems with inequality constraints. Global convergence properties are ensured by grafting the secant method onto a phase I-phase II feasible directions method, using a rate of convergence test for crossover control.

Key Words: Nonlinear programming, secant, quasi-Newton, Algorithm stabilization

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1. Introduction

When solving engineering design problems with distributed constraints, by means of outer approximation algorithms such as those described in [1], [2], one has to solve a large number of simpler optimization subproblems with a finite number of inequality constraints. In the context of engineering design, these subproblems are characterized by the fact that function and derivative evaluations are very costly, sometimes requiring as much as one minute of computer time (on a CDC 6400) for a simple function evaluation. In solving such subproblems by means of phase I-phase II feasible direction algorithms [3], one invariably finds that the time to solve a quadratic program, which yields Kuhn-Tucker type multipliers, is less than one tenth of the time required for a single function evaluation. This observation leads to the conclusion that the cost of solving quadratic programs at each iteration can be neglected in any scheme for solving such subproblems. While phase I-phase II feasible direction algorithms are quite dependable, they are inherently slow, which has led a number of researchers to look for better alternatives. The most successful alternatives so far have consisted of adaptations of Newton's and quasi-Newton methods. The development of these methods can be traced through the progression of papers by Wilson [4], Robinson [5], Han [6,7,8], and Powell [9]. So far, they have developed a number of important local convergence and rate of convergence results as well as some proposals for global stabilization based on exact penalty functions. The open questions in the current stabilization schemes [8,9] are those of (a) how to select the required exact penalty function constant and (b) how to ensure that the locally convergent superlinear method does in fact take over in the end.

Han [6,7] and Powell [9] have explored the use of quasi-Newton methods based on symmetric rank two updating formulas, such as BFGS. The main difficulty with such formulas in a global stabilization scheme is that they need to be supplied with a _sufficiently_ good initialization when one is _sufficiently_ close to a solution point. So far, no constructive tests have been proposed that can be used to determine when one is sufficiently close to a solution for the local method to converge.

On the other hand, there are some examples in the literature of effective global stabilization of secant like algorithms [10,11,12]. Although secant methods require one more gradient evaluations per iteration than BFGS type updates, they are considerably more robust, since the required precision of approximation can be enforced, and they have a higher rate of convergence because their updates use fewer past points.

In this paper, we present a superlinearly convergent, globally stabilized secant method, approximating Wilson's formulas [4], for optimization problems with inequality constants. The stabilization is accomplished by grafting the secant method onto a phase I-phase II method of feasible directions. The phase I-phase II method makes good progress in the initial iterations, and then, on the basis of a special test, it turns over the computation to the secant method when the superlinear rate of convergence of the secant method begins to manifest itself. As a result, we obtain a robust algorithm with excellent efficiency which should prove most useful in solving optimization problems arising in engineering design.

2. Building Blocks for an Algorithm

Consider the problem

$$\min\{f(x) \mid g^j(x) \leq 0,\ j \in \underline{m}\} \tag{2.1}$$

where $\underline{m} \triangleq \{1,2,\ldots,m\}$, and $f\colon \mathbb{R}^n \to \mathbb{R}$, $g^j\colon \mathbb{R}^n \to \mathbb{R}$, $j \in \underline{m}$, are <u>three times continuously differentiable</u>. To ensure that Kuhn-Tucker conditions [13] hold at solutions to (2.1) and that phase I-phase II feasible directions methods apply, we make the following hypothesis:

<u>Assumption 1</u>: For every $x \in \mathbb{R}^n$, $0 \notin \mathrm{co}\{\nabla g^j(x),\ j \in I(x)\}$, where co denotes the convex hull of the set and

$$I(x) \triangleq \{j \in \underline{m} \mid g^j(x) = 0\}\ . \tag{2.2}$$

□

When Assumption 1 is satisfied, if $\hat{x}$ is a local minimum for (2.1), then, according to the Kuhn-Tucker theorem [13], there exists a multiplier $\hat{y}$ such that (with $g = (g^1, g^2, \ldots, g^m)^T$)

$$\nabla f(\hat{x}) + \frac{\partial g(x)}{\partial x}^T \hat{y} = 0 \tag{2.3}$$

$$g(\hat{x}) \leqq 0 \tag{2.4}$$

$$\hat{y} \geqq 0 \tag{2.5}$$

$$\langle \hat{y}, g(\hat{x}) \rangle = 0 \tag{2.6}$$

We note that, unlike the case of equality constrained problems, the necessary optimality conditions for a local minimum of (2.1), (2.3)-(2.6), are different from those of a local maximum (in which $\hat{y} \leqq 0$). Hence a solution of (2.3)-(2.6) can only be a local minimum or saddle point.

As has been pointed out by S. Robinson, the system (2.3)-(2.6) cannot be solved by the extensions of Newton's method which he and Pshenichnyi have proposed [14,15], for the following reason. Let

$$\ell(x,y) \triangleq f(x) + \langle y, g(x) \rangle \ . \tag{2.7}$$

Then (2.3) becomes

$$\nabla_x \ell(\hat{x},\hat{y}) = 0 \ . \tag{2.8}$$

To solve, (2.3)-(2.6), given $(x_i, y_i) \in \mathbb{R}^n \times \mathbb{R}^m$, the extended Newton method [14] computes (x_{i+1}, y_{i+1}) according to the rule

$$\begin{aligned} x_{i+1} &= x_i + v_i \\ y_{i+1} &= y_i + w_i \end{aligned} \tag{2.9}$$

where (v_i, w_i) is the solution of

$$\text{minimize} \quad \tfrac{1}{2}\{\|v\|^2 + \|w\|^2\} \tag{2.10a}$$

subject to

$$\nabla_x \ell(x_i,y_i) + \frac{\partial^2 \ell(x_i,y_i)}{\partial x^2} v + \frac{\partial^2 \ell(x_i,y_i)}{\partial y \partial x} w = 0 \tag{2.10b}$$

$$g(x_i) + \frac{\partial g(x_i)}{\partial x} v \leqq 0 \tag{2.10c}$$

$$y + w \geqq 0 \tag{2.10d}$$

$$\langle y_i, g(x_i) + \frac{\partial g(x_i)}{\partial x} v \rangle + \langle g(x_i), w \rangle = 0 \tag{2.10e}$$

For a suitably bounded solution of (2.10) to exist, Robinson requires that the system (2.10b)-(2.10e) satisfy the LI condition in [14] in a neighborhood of a Kuhn-Tucker point. Unfortunately, it has been shown by Robinson that this system never satisfies his LI condition and hence may not be solvable by the extended Newton's method. Robinson proposed to salvage the situation by making use of an idea proposed but not analyzed, by Wilson [4]. Robinson had to strengthen his assumptions as follows.

<u>Assumption 2</u>: Let $(\hat{x},\hat{y})$ be any Kuhn-Tucker point for (2.1), i.e. $(\hat{x},\hat{y})$ satisfies (2.3)-(2.6). Then (i) $\hat{y}^j > 0$ for all $j \in I(\hat{x})$ (strict complementary slackness), (ii) $\frac{\partial^2 \ell(\hat{x},\hat{y})}{\partial x^2}$ is positive definite on the subspace $\{h \,|\, \langle \nabla g^j(\hat{x}), h \rangle = 0,\ j \in I(\hat{x})\}$ (second order sufficiency condition); and (iii) the vectors $\nabla g^j(\hat{x})$, $j \in I(\hat{x})$ are linearly independent. $\square$

Instead of solving (2.10), for (v_i, w_i), Wilson proposed to solve, instead, the smaller quadratic program below, for (v_i, y_{i+1}):†

$$\text{minimize} \quad \langle \nabla f(x_i), v \rangle + \frac{1}{2}\langle v, \frac{\partial^2 \ell(x_i,y_i)}{\partial x^2} v \rangle \tag{2.11}$$

subject to

$$g(x_i) + \frac{\partial g(x_i)}{\partial x} v \leqq 0 \tag{2.12}$$

Suppose v_i is a solution of (2.11) and y_{i+1} is the corresponding multiplier. Then, we find that (v_i, y_{i+1}) satisfy (2.10b)-(2.10d) and, in addition, y_{i+1} satisfies the nonlinear version of (2.10e), namely

$$\langle y_{i+1}, g(x_i) + \frac{\partial g(x_i)}{\partial x} v_i \rangle = 0 \tag{2.13}$$

Hence, when $v_i = 0$ solves (2.11), (x_i, y_{i+1}) satisfy (2.3)-(2.6).

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†From [5], (2.11)-(2.12) converges locally if $\frac{\partial^2 \ell}{\partial x^2}(\cdot,\cdot)$ is continuously differentiable, i.e. if $f(\cdot)$ and $g(\cdot)$ are only twice continuously differentiable.

As is well known (see [10,11,16]), secant methods are considerably more efficient than Newton's method. To construct a secant version of (2.11)-(2.12), we only need to substitute a finite difference approximation for the matrix $\frac{\partial^2 \ell(x,y)}{\partial x^2}$, in (2.12), defined as follows. Let $(x_{i-n},y_{i-n}),(x_{i-n+1},y_{i-n+1}),(x_{i-n+2},y_{i-n+2}),\ldots,(x_i,y_i)$ be given and let H_i be a matrix whose j^{th} column $h_{j,i}$ $(j=1,2,\ldots,n)$ is given by

$$h_{j,i}=\frac{1}{\Delta_k}\{\nabla_x \ell(x_{i-n+k}+\Delta_k e_j, y_{i-n+k}) - \nabla_x \ell(x_{i-n+k},y_{i-n+k})\} \tag{2.14}$$

with $k \in \{1,2,\ldots,n\}$, e_j the j^{th} unit vector in $\mathbb{R}^n$ and

$$\Delta_k \triangleq \|x_{i-n+k}-x_{i-n+k-1}\| + \|y_{i-n+k}-y_{i-n+k-1}\| \tag{2.15}$$

The specific order in which k and j appear in (2.14) is immaterial, but assumed to result from a cyclic column replacement scheme. We now state a secant method.

<u>Algorithm 2.1</u> (Secant)

<u>Data</u>: $(x_0,y_0) \in \mathbb{R}^n \times \mathbb{R}^m$, $H_0 \in \mathbb{R}^{n\times n}$ (symmetric), $\varepsilon > 0$, $\alpha_1 \in (0,1)$.

<u>Step 0</u>: Set $i = 0$.

<u>Step 1</u>: Solve

$$\min\{\langle\nabla f(x_i),v\rangle + \tfrac{1}{2}\langle v,H_i v\rangle \,|\, g(x_i) + \frac{\partial g(x_i)}{\partial x} v \leqq 0\} \tag{2.16}$$

for v_i and the corresponding multiplier $y_{i+1} \triangleq y_i + w_i$. Set $x_{i+1} = x_i + v_i$.

<u>Step 2</u>: Compute

$$h_{j,i+1} = \frac{1}{\Delta_i}[\nabla_x \ell(x_{i+1}+\Delta_i e_i, y_{i+1}) - \nabla_x \ell(x_{i+1},y_{i+1})] \tag{2.17}$$

with $j = i \bmod n$, and

$$\Delta_i = \min\{(\|v_i\|+\|w_i\|),\varepsilon\} \ . \tag{2.18}$$

Set

$$H_{i+1} = [h_{1,i}, h_{2,i}, \ldots, h_{j-1,i}, h_{j,i+1}, h_{j+1,i}, \ldots, h_{n,i}] \tag{2.19}$$

Step 3: Set $i = i+1$ and go to step 1. □

The following theorem follows directly from Theorem 3.1 in [5], Lemma 3 in [12] and Theorem 2 in [11].

Theorem 2.1: Let $(\hat{x},\hat{y})$ be a Kuhn-Tucker pair for problem (2.1), i.e. it satisfies (2.3)-(2.6), and suppose that Assumption 2 holds. Then there exists a $\hat{\rho} > 0$ such that if Algorithm 2.1 has constructed an infinite sequence $\{(x_i,y_i)\}_{i=1}^{\infty}$ in which for some $i_0 \geq 0$, $(x_{i_0-n-1}, y_{i_0-n-1}), \ldots, (x_{i_0}, y_{i_0})$ are in the ball

$$B((\hat{x},\hat{y}),\hat{\rho}) \triangleq \{(x,y) \mid \|x-\hat{x}\| \leq \hat{\rho},\ \|y-\hat{y}\| \leq \hat{\rho}\}, \tag{2.20}$$

then the following hold.

(a) The sequence $\{(x_i,y_i)\}_{i=i_0}^{\infty}$ constructed by Algorithm 2.1 converges, superlinearly, to $(\hat{x},\hat{y})$, with root rate τ_n, i.e., see [17, (9.2.5)], $0 < \overline{\lim}(\|x_i-\hat{x}\|+\|y_i-\hat{y}\|)^{1/\tau_n^i} < 1$, where τ_n is the unique positive root of $t^{n+1} - t^n - 1 = 0$ ($\tau_n \in (1,2)$ and $\tau_n \searrow 1$ as $n \to \infty$).

(b) There exist $M > 0$, $\delta \in (0,1)$ such that $(\|x_{i+1}-x_i\|+\|y_{i+1}-y_i\|) \leq M\delta^{\tau_n^i}$.

(c) For

$$p(x_i,y_i) \triangleq \frac{1}{2}\{\|\nabla_x \ell(x_i,y_i)\|^2 + \|g(x_i)_+\|^2\} \tag{2.21}$$

$p(x_{i+1},y_{i+1}) \leq p(x_i,y_i)$ holds for all $i \geq i_0$, where $g(x_i)_+$ is a vector with components $\max\{0, g^j(x_i)\}$, $j \in \underline{m}$. □

We see from Theorem 2.1 that the secant method (2.1) has excellent local convergence properties (local in the sense that it depends on a sufficiently good initial guess x_0, H_0). However, it, as well as other secant methods, has rather poor global convergence. In [11,12,18] we find stabilization schemes for secant and Newton algorithms based on the descent function $p(\cdot,\cdot)$ in (2.21). Note that $p(\cdot,\cdot)$ achieves its minimum value of zero at any Kuhn-Tucker pair $(\hat{x},\hat{y})$.

The stabilization schemes in [12,18] fall back on a gradient method for minimizing $p(x,y)$ when the secant method fails or when (x_i,y_i) is unacceptably far from a solution. Unfortunately, the computation of $\nabla_x p(x,y)$ involves the computation of $\frac{\partial^2 \ell(x,y)}{\partial x^2}$, which nullifies the savings achieved by the use of the secant matrices H_i. Hence we find it necessary to resort to an alternative approach. Specifically, we propose to stabilize the secant methods by grafting it onto a dual phase I-phase II method of feasible directions, stated below (see also [3]). This method has the desirable feature that it produces multipliers at each iteration that can be used for updating the H_i. Furthermore, it does not require the Hessian $\frac{\partial^2 \ell(x,y)}{\partial x^2}$. This method will be used to obtain a sufficiently good approximation to a Kuhn-Tucker point, and to the required Hessian $\frac{\partial^2 \ell(x,y)}{\partial x^2}$, for the secant method to converge. The details of meshing the two algorithms together will be given in the next section.

We shall need the following notation. Let

$$\psi(x) \triangleq \max_{j\in\underline{m}}\{g^j(x)\} \tag{2.22}$$

$$\psi(x)_+ \triangleq \max\{0,\psi(x)\} \tag{2.23}$$

Algorithm 2.2 (Phase I-Phase II Method of Feasible Directions)

Data: $x_0 \in \mathbb{R}^n$, $\alpha \in (0,1)$, $\beta \in (0,1)$, $\delta > 1$.

Step 0: Set $i = 0$.

Step 1: Compute

$$\theta(x_i) \triangleq \max_{y}\{ \sum_{j\in\underline{m}} y^j (g^j(x_i)-\psi(x_i)_+)-y^0\delta\psi(x_i)_+ -\frac{1}{2}\| \sum_{j\in\underline{m}} y^j\nabla g^j(x_i)+y^0\nabla f(x_i)\|^2 \Big| \; y \geqq 0, \; \sum_{i=0}^{m} y^j = 1\} \tag{2.24}$$

and set

$$h_i \triangleq -[\bar{y}_i^0\nabla f(x_i) + \sum_{j\in\underline{m}} \bar{y}_i^j \nabla g^j(x_i)] \tag{2.25}$$

where $(\bar{y}_i^0, \bar{y}_i^1, \ldots, \bar{y}_i^m)$ is the solution of (2.24).

Step 2: If $\theta(x_i) = 0$, stop. Else compute the smallest integer $k_i \geq 0$ such that

$$\left.\begin{aligned} f(x_i+\beta^{k_i}h_i) - f(x_i) &\leq \beta^{k_i}\alpha\theta(x_i) \\ g^j(x_i+\beta^{k_i}h_i) &\leq 0 \quad \forall j \in \underline{m} \end{aligned}\right\} \text{ if } \psi(x_i)_+ = 0 \tag{2.26a}$$

and

$$\psi(x_i+\beta^{k_i}h_i) - \psi(x_i) \leq \beta^{k_i}\alpha\theta(x_i) \text{ if } \psi(x_i)_+ > 0. \tag{2.26b}$$

Step 3: Set $x_{i+1} = x_i + \beta^{k_i}h_i$, set $i = i+1$ and go to step 1. □

The following result is proved in [3], with part (e) following from Theorem 1.3.66 in [13].

Theorem 2.2: Suppose that Assumption 2.1 is satisfied. Then

(a) $\theta(\cdot)$ is continuous.

(b) $\theta(x) = 0$ if and only if x is a Kuhn-Tucker point for the problem (2.1), i.e. for some $y \in \mathbb{R}^m$, (x,y) satisfies (2.3)-(2.6). $\theta(x) < 0$ otherwise.

(c) If the sequence $\{x_i, \bar{y}_i\}$ constructed by Algorithm 2.2. is finite, then its last element $(x_s, \bar{y}_s)$ defines a Kuhn-Tucker pair (x_s, y_s), with $y_s^j \triangleq \frac{1}{\bar{y}_s^0} \bar{y}_s^j$, $j = 1,2,\ldots,m$.

(d) If the sequence $\{x_i,\bar{y}_i\}$ constructed by Algorithm 2.2 is infinite, then any accumulation point $(\hat{x},\hat{\bar{y}})$ of $\{x_i,\bar{y}_i\}$ defines a Kuhn-Tucker pair $(\hat{x},\hat{y})$, with $\hat{y}$ defined by $\hat{y}^j = \frac{1}{\hat{\bar{y}}^0}\hat{\bar{y}}^j$, $j = 1,2,\ldots,m$.

(e) If the sequence $\{x_i\}$ constructed by Algorithm 2.2 is infinite and bounded and Problem 2.1 has only isolated Kuhn-Tucker points, then $x_i \to \hat{x}$ as $i \to \infty$, with $\hat{x}$ a Kuhn-Tucker point. □

3. The Robust Algorithm

We now state our algorithm which uses the phase I-phase II method of feasible directions (Algorithm 2.2) for bringing the iterative process into a region where the secant method (Algorithm 2.1) converges. Recall that $p(\cdot)$ was defined in (2.21).

Algorithm 3.1 (Globally Stabilized Secant Method)

Parameters: $\alpha \in (0,1)$, $\beta \in (0,1)$, $\gamma \in (0,1)$, $\delta \geq 1$, $\bar{\varepsilon} > 0$, $\varepsilon_0 > 0$.

Data: $x_0 \in \mathbb{R}^n$, $H_0 \in \mathbb{R}^{n\times n}$, symmetric.

Step 0: Set $i = 0$, $\ell = 0$, $\varepsilon = \varepsilon_0$. Compute $\bar{\theta} = \max_y\{ \sum_{j=1} y^j(g^j(x_0) - y^0\delta\psi(x_0)_+) - y^0\delta\psi(x_0)_+ - \frac{1}{2}\|y^0\nabla f(x_0) + \sum_{j=1}^m y^j\nabla g^j(x_0)\|^2 \,|\, y \geqq 0, \sum_{j=0}^m y^j = 1\}$, and corresponding multipliers $\bar{y}_0^j$, $j = 0,1,\ldots,m$. Set $y_0^j = \frac{1}{\bar{y}_0^0}\bar{y}_0^j$, $j \in \underline{m}$ if $\bar{y}_0^0 \neq 0$, and set $y_0^j = \bar{y}_0^j$, $j \in \underline{m}$, otherwise.

Step 1: Solve for v_i and the corresponding multiplier η_{i+1} the QP:

$$\min\{\langle\nabla f(x_i),v\rangle + \tfrac{1}{2}\langle v,H_i v\rangle \,|\, g(x_i) + \frac{\partial g(x_i)}{\partial x}v \leqq 0\} \tag{3.1}$$

Step 2: If (3.1) has a solution (v_i,η_{i+1}) and $v_i = 0$, stop. If (3.1) has a solution (v_i,η_{i+1}), with $v_i \neq 0$ and

$$\|v_i\| + \|\eta_{i+1} - y_i\| \leq \gamma^\ell \tag{3.2}$$

set

$$x_{i+1} = x_i + v_i \tag{3.3}$$

$$y_{i+1} = \eta_{i+1}\ ; \tag{3.4}$$

compute H_{i+1} by replacing the k^{th} column, $h_{k,i}$, of H_i, with $k = i \bmod n$, by the vector

$$h_{k,i+1} \frac{1}{\Delta_i}[\nabla_x \ell(x_{i+1}+\Delta_i e_k, y_{i+1}) - \nabla_x \ell(x_i, y_{i+1})] \tag{3.5a}$$

with

$$\Delta_i = \min\{\bar{\varepsilon}, \|x_{i+1}-x_i\|+\|y_{i+1}-y_i\|\}. \tag{3.5b}$$

Set $i = i+1$, $\ell = \ell+1$ and go to step 1.

Else, proceed.

Step 3: Compute $\theta(x_i)$, $\bar{y}_i^j$, $j = 1,2,\ldots,m$ as solutions of

$$\theta(x_i) = \max_y\{\sum_{j=1}^m y^j(g^j(x_i)-\psi(x_i)_+) - y^0\delta\psi(x_i)_+ - \frac{1}{2}\|y^0\nabla f(x_i)+\sum_{j=1}^m y^j\nabla g^j(x_i)\|^2 \Big| \sum_{j=0}^m y^j = 1,\ y \geqq 0\}. \tag{3.6}$$

Step 4: Set

$$h_i = -[\bar{y}_i^0\nabla f(x_i) + \sum_{j\in \underline{m}} \bar{y}_i^j \nabla g^j(x_i)]. \tag{3.7}$$

Step 5: Compute the smallest integer $k_i \geq 0$ such that

$$\left.\begin{aligned} f(x_i+\beta^{k_i}h_i) - f(x_i) &\leq \beta^{k_i}\alpha\theta(x_i) \\ \psi(x_i+\beta^{k_i}h_i) &\leq 0 \end{aligned}\right\} \text{ if } \psi(x_i)_+ = 0 \tag{3.8a}$$

and

$$\psi(x_i+\beta^{k_i}h_i) - \psi(x_i) \leq \beta^{k_i}\alpha\theta(x_i) \quad \text{if} \quad \psi(x_i)_+ > 0 \tag{3.8b}$$

Set

$$x_{i+1} = x_i + \beta^{k_i}h_i \tag{3.9}$$

Step 6: Compute $\theta(x_{i+1})$, $\bar{y}_{i+1}^j$, $j = 0,1,\ldots,m$ by solving (3.6), with x_{i+1} replacing x_i. Set $y_{i+1} = (y_{i+1}^1,\ldots,y_{i+1}^m)^T$, with $y_{i+1}^j = \frac{1}{\bar{y}_{i+1}^0}\bar{y}_{i+1}^j$, if $y_{i+1}^0 \neq 0$, and set $y_{i+1}^j = \bar{y}_{i+1}^j$, $j = 1,\ldots,m$ otherwise.

Step 7: Set $i = i+1$.

Step 8: If $\theta(x_i) \geq \gamma\bar{\theta}$, set $\bar{\theta} = \theta(x_i)$. Compute H_i to be the matrix with columns $h_{k,i}$, $k = 1,2,\ldots,n$,

$$h_{k,i} = \frac{1}{\varepsilon}[\nabla_x \ell(x_i+\varepsilon e_k, y_i) - \nabla_x \ell(x_i, y_i)]$$

Set $\varepsilon = \gamma\varepsilon$ and go to step 1. Else go to step 4. □

We now proceed to examine the properties of Algorithm 3.1.

Lemma 3.1. Suppose that Assumption 2.2 holds and that Algorithm 3.1 constructs an infinite sequence $\{(x_i,y_i)\}$ such that for $i \geq i_0 \geq 0$, (x_i,y_i) is constructed in step 2. Then $\{(x_i,y_i)\}$ converges superlinearly, with root rate τ_n, to a Kuhn-Tucker pair.

Proof: By construction, in step 2, $\|x_{i+1}-x_i\| + \|y_{i+1}-y_i\| \leq K\gamma^i$ for all $i \geq i_0$ and some $K > 0$. Hence, setting $z \triangleq (x,y)$, $\|z\| \triangleq \|x\| + \|y\|$, we have, for any $k \geq 1$ and $i \geq i_0$,

$$\|z_{i+k}-z_i\| \leq K \sum_{\ell=i}^{i+k-1} \gamma^\ell < K \sum_{\ell=i}^{\infty} \gamma^\ell , \tag{3.10}$$

which shows that $\{z_i\}$ is Cauchy and hence converges to a point $\hat{z} = (\hat{x},\hat{y})$. As a result, the matrices $H_i \to \frac{\partial^2 \ell(\hat{x},\hat{y})}{\partial x^2}$ as $i \to \infty$. Now suppose that $(\hat{x},\hat{y})$ is not a Kuhn-Tucker pair. Then, the corresponding solution $(\hat{v},\hat{w})$ of

$$\min\{\langle\nabla f(\hat{x}),v\rangle + \frac{1}{2}\langle v,\frac{\partial^2 \ell(\hat{x},\hat{y})}{\partial x^2}v\rangle \,|\, g(\hat{x}) + \frac{\partial g(\hat{x})}{\partial x}v \leqq 0\} > 0 \tag{3.11}$$

satisfies

$$\|\hat{v}\| + \|\hat{w}\| > 0 \tag{3.12}$$

where $\hat{y} + \hat{w}$ is the corresponding Kuhn-Tucker multiplier for (3.11). Now, with $v_i = x_{i+1} - x_i$, $w_i = y_{i+1} - y_i$, we have, by construction, that

$$\|v_i\| + \|w_i\| \geq \min\{\| v \|+\| w \| \,\Big|\, \nabla_x \ell(x_i,y_i) + H_i v + \frac{\partial g(x_i)}{\partial x}^T w = 0, \qquad (3.13)$$
$$g(x_i) + \frac{\partial g(x_i)}{\partial x} v \leqq 0;\ y_i + w \geqq 0;\ \langle y_i + w, g(x_i) + \frac{\partial g(x_i)}{\partial x} v\rangle = 0\}$$

Now, $x_i \to \hat{x}$ and $y_i \to \hat{y}$. Hence, $H_i \to \frac{\partial^2 \ell(\hat{x},\hat{y})}{\partial x^2}$ and therefore, we must have, since $\|v_i\| + \|w_i\| \to 0$, and since $(\hat{v},\hat{w})$ is optimal for (3.13) with (x_i,y_i) and H_i replaced by $(\hat{x},\hat{y})$ and $\frac{\partial^2 \ell(\hat{x},\hat{y})}{\partial x^2}$, respectively, that

$$0 = \lim(\|v_i\|+\|w_i\|) \geq \|\hat{v}\| + \|\hat{w}\| \qquad (3.14)$$

But (3.14) contradicts (3.12) and hence $(\hat{x},\hat{y})$ must be a Kuhn-Tucker pair. The rest of the lemma now follows from the fact that $H_i \to \frac{\partial^2 \ell(\hat{x},\hat{y})}{\partial x^2}$ and Theorem 2.1. □

Lemma 3.2: Suppose Algorithm 3.1 constructs an infinite sequence $\{(x_i,y_i)\}$ containing two infinite subsequences, one constructed in steps 5 and 6 and the other in step 2. Let $K \subset \{0,1,2,...\}$ be such that if $i \in K$, then (x_i,y_i) is constructed in steps 5, 6, and (x_{i+1},y_{i+1}) is constructed in step 2. Then any accumulation point of $\{(x_i,y_i)\}_{i\in K}$ is a Kuhn-Tucker pair.

Proof: Let $K' \subset K$ be infinite and such that $(x_i,y_i) \xrightarrow{K'} (\hat{x},\hat{y})$. Then, by construction in step 8 of Algorithm 3.1, for any i, $i+k \in K'$, $k \geq 1$, $\theta(x_{i+k}) \geq \gamma\theta(x_i)$ and hence $\theta(x_i) \xrightarrow{K'} 0$. The desired result now follows from Theorem 2.2a,b. □

Theorem 3.1: Suppose that Assumption 2.2 holds and that Algorithm 3.1 constructs an infinite bounded sequence $\{(x_i,y_i)\}$. Then $\{(x_i,y_i)\}$ converges superlinearly, with root rate τ_n, to $(\hat{x},\hat{y})$ a Kuhn-Tucker pair for Problem 2.1.

Proof: If there is an i_0 such that (x_i, y_i) is constructed in step 2 for all $i \geq i_0$, then we are done by Lemma 3.1. Next, it follows from Theorem 2.2 that if (x_i, y_i) is constructed in steps 5 and 6 of Algorithm 3.1 for $i = i_0, i_0+1, \ldots,$ then there exists an $i_1 \geq i_0$ such that for all $i \geq i_1$, $\theta(x_i) \geq \gamma\bar{\theta}$ for any $\bar{\theta} < 0$. Hence the algorithm will attempt to construct x_{i_1+1}, y_{i_1+1} in step 2. Now, by Lemma 3.2, if $\{(x_{i_k}, y_{i_k})\}$ is the subsequence of $\{(x_i, y_i)\}$ such that (x_{i_k}, y_{i_k}) is constructed in steps 5 and 6 and (x_{i_k+1}, y_{i_k+1}) is constructed in step 2, then any accumulation point $(\hat{x}, \hat{y})$ of (x_{i_k}, y_{i_k}) is a Kuhn-Tucker pair. Hence, step 2 is entered with $(x_{i_k}, y_{i_k}) \xrightarrow{K} (\hat{x}, \hat{y})$ and $H_{i_k} \xrightarrow{K} \frac{\partial^2 \ell(\hat{x}, \hat{y})}{\partial x^2}$. It now follows from Theorem 2.1b that there exists an i_2 such that (x_i, y_i) is constructed in step 2 for all $i \geq i_2$ and hence we are done. □

Conclusion:

The algorithm which we have presented consists of three segments. The first is the phase I-phase II method of feasible directions which carried the computation into the convergence region of the second segment: the superlinearly converging the secant method. The third segment consists of the tests which detect whether the computation has entered into the region of convergence of the secant method. The tests are based on rate of convergence detection, as defined by (3.2), and on improved nearness to a solution, as defined by step 8. Both of these tests depend on the parameter $\gamma \in (0,1)$. It appears that both tests are easiest satisfied, and hence the secant method is allowed to continue, if $\gamma \approx 1$. A good choice for the remaining parameters is difficult to indicate a priori and will have to be determined by the user on the basis of experience.

Given our past experience with secant methods we feel that the algorithm which we have described in this paper will be quite competitive with other algorithms for the problem in question and hence will prove to be a useful addition to the optimizer's arsenal.

References

[1] D.Q. Mayne, E. Polak and R. Trahan, "An outer approximation algorithm for computer-aided design problems," JOTA, in press. Also M77/10, Electronics Research Laboratory, University of California, Berkeley, 1977.

[2] C. Gonzaga and E. Polak, "On constraint dropping schemes and optimality functions for a class of outer approximation algorithms," M77/68, Electronics Research Laboratory, University of California, Berkeley, 1977, also SIAM J. Control and Optimization, in Press.

[3] E. Polak, R. Trahan and D.Q. Mayne, "Combined Phase I-Phase II methods of feasible directions," Mathematical Programming, in press. Also 77/39, Electronics Research Laboratory, University of California, Berkeley, 1977.

[4] R.B. Wilson, "A simplified algorithm for concave programming," Ph.D. dissertation, Graduate School of Business Administration, Harvard University, Cambridge, Mass., 1963.

[5] S.M. Robinson, "Perturbed Kuhn-Tucker points and rates of convergence for a class on nonlinear programming algorithms," Mathematical Programming, vol. 7 (1974), pp. 1-16.

[6] S.P. Han, "Superlinearly convergent variable metric algorithms for general nonlinear programming problems," Mathematical Programming vol. 11 (1976), pp. 263-282.

[7] ______, "Dual variable metric algorithms for constrained optimization," SIAM J. Control and Optimization, vol. 15, no. 4 (1977), pp. 546-566.

[8] ______, "A globally convergent method for nonlinear programming," JOTA, vol. 22 (1977), pp. 297-309.

[9] M.J.D. Powell, "A fast algorithm for nonlinearly constrained optimization calculations," presented at 1977 Dundee Conference on Numerical Analysis.

[10] E. Polak, "A modified secant method for unconstrained optimization," Mathematical Programming, vol. 6, no. 3 (1974), pp. 264-280.

[11] ______, "A globally convergent secant method with applications to boundary value problems," SIAM J. Numer. Anal., vol. 11, no. 3 (1974), pp. 529-537.

[12] E. Polak and I. Teodoru, "Newton derived methods for nonlinear equations and inequalities," in Nonlinear Programming, O.L. Mangasarian, R.R. Meyer and S.M. Robinson, eds., Academic Press, 1975, pp. 255-277.

[13] E. Polak, Computational Methods in Optimization, Academic Press, 1977.

[14] S.M. Robinson, "Extension of Newton's method to mixed systems of nonlinear equations and inequalities," Tech. Summary Report No. 1161, Mathematics Research Center, University of Wisconsin, Madison, Wis., 1971. Also Numerische Mathematik, vol. 19 (1972), pp. 341-347.

[15] B. N. Pshenichnigi and . M. Danilin, Numerical Methods in Extremal Problems (in Russian) (Chapter III,), Nauka, Moscow, 1975.

[16] R.P. Brent, "Some efficient algorithms for solving systems of nonlinear equations," SIAM J. Numer. Anal., vol. 10, no. 2 (1973), pp. 327-344.

[17] J.M. Ortega and W.C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York, 1970.

[18] H. Mukai and E. Polak, "On the use of approximations in algorithms for optimization problems with equality and inequality constraints," SIAM J. Numer. Anal., vol. 15, no. 4 (1978), pp. 674-693.