FREQUENCY DOMAIN PASSIVITY CONDITIONS
FOR LINEAR TIME-INVARIANT LUMPED NETWORKS

by

J. W. Gannett and L. O. Chua

Memorandum No. UCB/ERL M78/21

11 May 1978

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

ADDENDUM

After this memo was completed, the authors discovered that the sufficient condition for passivity given by Theorem (3.18), page 22, could be easily generalized so as to include the scattering representation. Replace condition (d) with the following weaker condition:

(d) $\underset{\sim}{a}^T \underset{\sim}{c} + \underset{\sim}{c}^T \underset{\sim}{a} \leq 0$,

that is, $\mathrm{Re}(\underset{\sim}{w}^H \underset{\sim}{a}^T \underset{\sim}{c}\underset{\sim}{w}) \leq 0$ for all $\underset{\sim}{w} \in \mathbb{C}^n$. The proof of Theorem (3.18) requires only two modifications. In (4), we can only conclude that the real part of the left-hand side is nonpositive for $t > t_1$, and the first equality in (6) becomes an inequality. Since this more general version of Theorem (3.18) applies to the scattering representation, Theorem (5.4) on pages 27-28 should be replaced by the following:

<u>(5.4) Theorem (Necessary and Sufficient Passivity Conditions for the Scattering Representation)</u>.

(i) If network $\mathcal{N}$ is passive, the scattering matrix $\underset{\sim}{G}(\cdot)$ is a bounded real matrix.

(ii) If network $\mathcal{N}$ is controllable and its scattering matrix $\underset{\sim}{G}(\cdot)$ is bounded real, then network $\mathcal{N}$ is passive.

FREQUENCY DOMAIN PASSIVITY CONDITIONS

FOR LINEAR TIME-INVARIANT LUMPED NETWORKS

J. W. Gannett and L. O. Chua

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

A rigorous yet elementary derivation of frequency domain passivity conditions for linear time-invariant lumped networks is presented. The conditions are derived in a coordinate-independent form and then applied to the hybrid and scattering representations.

Research sponsored by the Office of Naval Research Contract N00014-76-C-0572 and the National Science Foundation Grant ENG77-22745.

I. INTRODUCTION

Frequency domain passivity conditions for linear time-invariant networks are so basic that many derivations have been presented in the literature. These derivations usually assume an input-output representation for the network, as opposed to the state space representation. Some of these derivations are non-rigorous, while the rigorous derivations often involve sophisticated mathematics beyond the reach of the average engineer. By adopting the state space representation in this report and considering only lumped networks, we are able to present a rigorous derivation for frequency domain passivity conditions which does not involve overly sophisticated mathematics.

The state space representation has another advantage. Consider the problem of finding sufficient conditions which guarantee that a given network $\mathcal{N}$ is passive. The input-output description is unable to handle this problem because it deals only with the zero-state response of the network. The one-port $\mathcal{N}$ shown in the figure illustrates this problem. For zero initial conditions (i.e., zero initial voltage across the capacitor), $\mathcal{N}$ looks like an ideal one-port short circuit -- a passive network. However, if $\mathcal{N}$ is constructed at time 0 with an initial voltage v_{co} across the capacitor, then $\mathcal{N}$ looks like an independent voltage source with waveform $v(t) = v_{co}e^{-t}$ for $t \geq 0$. Arbitrarily large amounts of energy can be extracted from $\mathcal{N}$ by connecting a sufficiently small positive resistor across its terminals. Thus $\mathcal{N}$ should certainly be classified as an active (i.e., non-passive) network. The definition of passivity given in this report, based on the state space representation, classifies this network correctly. Moreover, this definition allows us

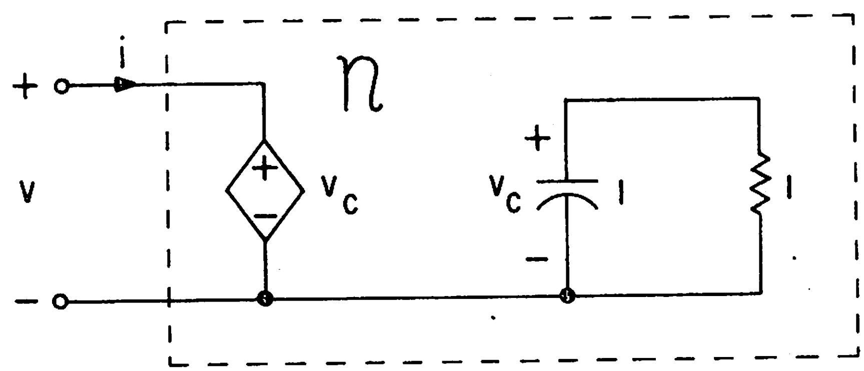

Figure. An example of a one-port electrical network $\mathcal{N}$ which is active and uncontrollable.

to obtain sufficient conditions which guarantee that a network is passive.

Outlining the report, Section II presents definitions, lemmas, and theorems which will be needed in later sections. In Section II we have tried to present standard electrical engineering concepts in a rigorous manner. The main purpose of Section II is to introduce our notation. Section III describes the mathematical representation of the network and derives passivity conditions in a coordinate-independent form. It is assumed that the reader has some familiarity with the state space representation for linear systems. In Section IV the results of Section III are applied to the hybrid representation. The results in Section IV can be summarized as follows: If a network with a hybrid representation is passive, then its hybrid matrix is positive real. Conversely, if the hybrid matrix for a controllable network is positive real, then the network is passive. In Section V the results of Section III are applied to the scattering representation. Only a necessary condition is derived, namely, if a network with a scattering representation is passive, then its scattering matrix is bounded real.

All integrals in this report are Lebesgue integrals. This allows a very general class of input functions. The reader who is unfamiliar with the Lebesgue integral may assume that the class of allowable inputs is the set of all piecewise continuous functions, then the corresponding integrals reduce to the ordinary Riemann integral.

II. PRELIMINARY RESULTS

This section contains definitions, lemmas, and theorems which will be needed in later sections.

(2.1) Definitions. If $\underset{\sim}{w} \in \mathbb{C}^{nxm}$ (i.e., $\underset{\sim}{w}$ is an nxm matrix with complex elements) and if $\underset{\sim}{w} = \underset{\sim}{u} + j\underset{\sim}{v}$, where $\underset{\sim}{u},\underset{\sim}{v} \in \mathbb{R}^{nxm}$, then, by definition, Re $\underset{\sim}{w} \triangleq \underset{\sim}{u}$ and Im $\underset{\sim}{w} \triangleq \underset{\sim}{v}$. The transpose of $\underset{\sim}{w}$ is denoted by $\underset{\sim}{w}^T$, the complex conjugate of $\underset{\sim}{w}$ is denoted by $\bar{\underset{\sim}{w}} \triangleq \underset{\sim}{u} - j\underset{\sim}{v}$, and $\underset{\sim}{w}^H \triangleq \bar{\underset{\sim}{w}}^T$. An nxn Hermitian matrix $\underset{\sim}{w}$ is one for which $\underset{\sim}{w}^H = \underset{\sim}{w}$. The term real matrix will denote a matrix with real elements, and the term complex matrix will denote a matrix with complex elements.

(2.2) Lemma. If $\underset{\sim}{M} \in \mathbb{C}^{nxn}$ and $\underset{\sim}{w} \in \mathbb{C}^n$, then $\mathrm{Re}(\underset{\sim}{w}^H\underset{\sim}{M}\underset{\sim}{w}) = \frac{1}{2}\underset{\sim}{w}^H(\underset{\sim}{M}+\underset{\sim}{M}^H)\underset{\sim}{w}$ and $j\mathrm{Im}(\underset{\sim}{w}^H\underset{\sim}{M}\underset{\sim}{w}) = \frac{1}{2}\underset{\sim}{w}^H(\underset{\sim}{M}-\underset{\sim}{M}^H)\underset{\sim}{w}$.

Proof. Note that $\underset{\sim}{M} = \frac{1}{2}(\underset{\sim}{M}+\underset{\sim}{M}^H) + \frac{1}{2}(\underset{\sim}{M}-\underset{\sim}{M}^H)$. Thus $\underset{\sim}{w}^H\underset{\sim}{M}\underset{\sim}{w} = \frac{1}{2}\underset{\sim}{w}^H(\underset{\sim}{M}+\underset{\sim}{M}^H)\underset{\sim}{w} + \frac{1}{2}\underset{\sim}{w}^H(\underset{\sim}{M}-\underset{\sim}{M}^H)\underset{\sim}{w}$. Now

$$[\frac{1}{2}\underset{\sim}{w}^H(\underset{\sim}{M}+\underset{\sim}{M}^H)\underset{\sim}{w}]^H = \frac{1}{2}\underset{\sim}{w}^H(\underset{\sim}{M}+\underset{\sim}{M}^H)\underset{\sim}{w}$$

$$[\frac{1}{2}\underset{\sim}{w}^H(\underset{\sim}{M}-\underset{\sim}{M}^H)\underset{\sim}{w}]^H = -\frac{1}{2}\underset{\sim}{w}^H(\underset{\sim}{M}-\underset{\sim}{M}^H)\underset{\sim}{w}.$$

Thus $\frac{1}{2}\underset{\sim}{w}^H(\underset{\sim}{M}+\underset{\sim}{M}^H)\underset{\sim}{w}$ is a real number, and $\frac{1}{2}\underset{\sim}{w}^H(\underset{\sim}{M}-\underset{\sim}{M}^H)\underset{\sim}{w}$ is an imaginary number.

Q.E.D.

(2.3) Observation. If $\underset{\sim}{M}$ is an nxn Hermitian matrix, it follows from Lemma (2.2) that $\underset{\sim}{w}^H\underset{\sim}{M}\underset{\sim}{w}$ is a real number for all $\underset{\sim}{w} \in \mathbb{C}^n$.

(2.4) Definitions. If $\underset{\sim}{M}$ is an nxn Hermitian matrix, the notation $\underset{\sim}{M} \geq 0$ means that $\underset{\sim}{w}^H\underset{\sim}{M}\underset{\sim}{w} \geq 0$ for all $\underset{\sim}{w} \in \mathbb{C}^n$.

The symbol $\underset{\sim}{I}$ will denote the nxn identity matrix.

The right-half plane (abbreviated RHP) is the set $\{s \in \mathbb{C}: \mathrm{Re}(s) \geq 0\}$. The open RHP is the set $\{s \in \mathbb{C}: \mathrm{Re}(s) > 0\}$.

A function $f: \Psi \to \mathbb{C}$ is defined to be holomorphic in an open set

$\Psi \subset \mathbb{C}$ if its derivative exists (in the complex sense) at each point of of Ψ.[1] A complex-valued function $f(\cdot)$ is defined to be meromorphic in an open set $\Psi \subset \mathbb{C}$ if there exists a set P of isolated points in Ψ such that $f(\cdot)$ is holomorphic in $\Psi \backslash P$ and each point of P is a pole of $f(\cdot)$.

Now consider the matrix-valued map $\underset{\sim}{M}: \Psi \subset \mathbb{C} \to \mathbb{C}^{n\times n}$, i.e., $\underset{\sim}{M}(\cdot)$ is an nxn matrix of complex-valued functions of a complex variable. By definition, $\underset{\sim}{M}(\cdot)$ is holomorphic in an open set $\Psi \subset \mathbb{C}$ if each element of $\underset{\sim}{M}(\cdot)$ is holomorphic in Ψ.

(2.5) Definition. $\underset{\sim}{M}: \Psi \subset \mathbb{C} \to \mathbb{C}^{n\times n}$ is a positive real matrix if Ψ contains the open RHP and

(i) $\underset{\sim}{M}(\cdot)$ is holomorphic in the open RHP.

(ii) $\underset{\sim}{M}(\sigma)$ is a real matrix for real, positive σ.[2]

(iii) $\underset{\sim}{M}(s) + \underset{\sim}{M}^H(s) \geq 0$ for all s in the open RHP.

(2.6) Definition. $\underset{\sim}{M}: \Psi \subset \mathbb{C} \to \mathbb{C}^{n\times n}$ is a bounded real matrix if Ψ contains the open RHP and

(i) $\underset{\sim}{M}(\cdot)$ is holomorphic in the open RHP.

(ii) $\underset{\sim}{M}(\sigma)$ is a real matrix for real, positive σ.

(iii) $\underset{\sim}{I} - \underset{\sim}{M}^H(s)\underset{\sim}{M}(s) \geq 0$ for all s in the open RHP.

(2.7) Definitions. A point $s_0 \in \mathbb{C}$ is a k-th order pole of the matrix-valued map $\underset{\sim}{M}: \Psi \subset \mathbb{C} \to \mathbb{C}^{n\times n}$ if there exists an element of $\underset{\sim}{M}(\cdot)$ with

---

[1] Some authors use the terms analytic or regular in the same sense that holomorphic is used here.

[2] Condition (ii) can be replaced by the condition that $\underset{\sim}{M}(\bar{s}) = \overline{\underset{\sim}{M}(s)}$ for s in the open RHP. The reflection principle [1] shows that the resulting definition is equivalent. The same comment applies to Definition (2.6).

a k-th order pole at s_0 and every other element of $\underset{\sim}{M}(\cdot)$ is either holomorphic at s_0 or has a pole at s_0 of order not greater than k. A first-order pole of $\underset{\sim}{M}(\cdot)$ is called a <u>simple pole</u>. If each element of $\underset{\sim}{M}(\cdot)$ is a rational function, then $\underset{\sim}{M}(\cdot)$ is called a <u>rational matrix</u>. Note that a matrix which is positive real or bounded real can have no poles in the open RHP. Moreover, a standard result says that a bounded real rational matrix has no poles on the imaginary axis, and a positive real rational matrix can have at most simple poles on the imaginary axis [2]. These results will not be needed in this report; however, the following lemmas will be useful.

<u>(2.8) Lemma</u>. Let $\underset{\sim}{M}(\cdot)$ be an nxn rational matrix. Let Λ be a set of isolated points which contains all poles of $\underset{\sim}{M}(\cdot)$. Suppose that $\underset{\sim}{M}(s) + \underset{\sim}{M}^H(s) \geq 0$ for all $s \in$ (open RHP)$\setminus\Lambda$. Then $\underset{\sim}{M}(\cdot)$ is holomorphic in the open RHP, and $\underset{\sim}{M}(s) + \underset{\sim}{M}^H(s) \geq 0$ for <u>all</u> s in the open RHP.

<u>Proof</u>. For each $\underset{\sim}{w} \in \mathbb{C}^n$, define a rational function $f_{\underset{\sim}{w}}(\cdot)$ by

$$f_{\underset{\sim}{w}}(s) = \underset{\sim}{w}^H \underset{\sim}{M}(s) \underset{\sim}{w} = \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{w}_i w_j m_{ij}(s)$$

where w_i denotes the i-th component of $\underset{\sim}{w}$ and $m_{ij}(\cdot)$ denotes the ij-th element of $\underset{\sim}{M}(\cdot)$. Suppose s_0 is a pole of $\underset{\sim}{M}(\cdot)$, and choose $\underset{\sim}{w} \in \mathbb{C}^n$ such that $f_{\underset{\sim}{w}}(\cdot)$ has a pole at s_0. This can always be done, for if $m_{pp}(\cdot)$ has a pole at s_0, choose $w_p = 1$ and $w_i = 0$ for $i \neq p$. On the other hand, if no diagonal element has a pole at s_0, then $m_{pq}(\cdot)$ has a pole at s_0 for some p,q ($p \neq q$). Choose $w_p = 1$, $w_q = c$, and $w_i = 0$ for $i \neq p$ or q. By appropriate choice of the constant $c \neq 0$, the function $f_{\underset{\sim}{w}}(\cdot)$ will have a pole at s_0. Thus $f_{\underset{\sim}{w}}(\cdot)$ has a Laurent expansion in some neighborhood of s_0 as follows:

$$f_{\underset{\sim}{w}}(s) = \sum_{m=-k}^{\infty} a_m(s-s_0)^m$$

where $k \geq 1$ is the order of the pole of $f_{\underset{\sim}{w}}(\cdot)$ at s_0. For $|s-s_0|$ sufficiently small, the first term in the Laurent expansion dominates, and $\mathrm{Re}[a_{-k}(s-s_0)^{-k}]$ changes sign in every neighborhood of s_0. Thus $\mathrm{Re}\, f_{\underset{\sim}{w}}(s)$ changes sign in every neighborhood of s_0. Now $\mathrm{Re}\, f_{\underset{\sim}{w}}(s) = \frac{1}{2}\, \underset{\sim}{w}^H(\underset{\sim}{M}(s)+\underset{\sim}{M}^H(s))\underset{\sim}{w}$ (Lemma (2.2)). These facts, combined with the hypotheses of the lemma show that $\underset{\sim}{M}(\cdot)$ has no poles in the open RHP.

Finally, we deliberately stated the lemma so that Λ may contain points which are not poles of $\underset{\sim}{M}(\cdot)$. If $s_1 \in$ (open RHP) $\cap\, \Lambda$, then s_1 is not a pole of $\underset{\sim}{M}(\cdot)$. It follows from continuity that $\underset{\sim}{M}(s_1) + \underset{\sim}{M}^H(s_1) \geq 0$.

Q.E.D.

<u>(2.9) Lemma</u>. Let $\underset{\sim}{M}(\cdot)$ be an nxn rational matrix. Let Λ be a set of isolated points which contains all poles of $\underset{\sim}{M}(\cdot)$. Suppose that $\underset{\sim}{I} - \underset{\sim}{M}^H(s)\underset{\sim}{M}(s) \geq 0$ for all $s \in$ (open RHP)$\setminus\Lambda$. Then $\underset{\sim}{M}(\cdot)$ has no poles in the RHP, and $\underset{\sim}{I} - \underset{\sim}{M}^H(s)\underset{\sim}{M}(s) \geq 0$ for all s in the RHP.

<u>Proof</u>. If $\underset{\sim}{I} - \underset{\sim}{M}^H(s)\underset{\sim}{M}(s) \geq 0$, then each diagonal element of the matrix $\underset{\sim}{I} - \underset{\sim}{M}^H(s)\underset{\sim}{M}(s)$ must be nonnegative. Thus

$$1 - \sum_{i=1}^{n} |m_{ij}(s)|^2 \geq 0,\ j=1,2,\ldots,n,\quad s \in \text{(open RHP)}\setminus\Lambda,$$

where $m_{ij}(\cdot)$ is the ij-th element of $\underset{\sim}{M}(\cdot)$. This shows that $|m_{ij}(s)| \leq 1$ for all $s \in$ (open RHP)$\setminus\Lambda$ and for all i,j. But if s_0 is a pole of $m_{ij}(\cdot)$, then $|m_{ij}(s)| \to +\infty$ as $s \to s_0$. Therefore $\underset{\sim}{M}(\cdot)$ has no poles in the RHP. It follows from continuity that $\underset{\sim}{I} - \underset{\sim}{M}^H(s)\underset{\sim}{M}(s) \geq 0$ for all s in the RHP.

Q.E.D.

<u>(2.10) L^p spaces</u>. Let $\Phi \subset \mathbb{R}$ be a Lebesgue measurable set,

let $f: \Phi: \to \mathbb{C}$ be a measurable function, and for $p \geq 1$ define $\|f\|_p \triangleq (\int_\Phi |f|^p)^{1/p}$. Then, by definition, $f \in L^p(\Phi)$ if $\|f\|_p < +\infty$. If $f,g \in L^p(\Phi)$ and if $\alpha,\beta \in \mathbb{C}$, it turns out that $\alpha f + \beta g \in L^p(\Phi)$. Thus $L^p(\Phi)$ is a vector space. If $\underset{\sim}{f}: \Phi \to \mathbb{C}^n$, we say that $\underset{\sim}{f} \in L^{pn}(\Phi)$ if each component function of $\underset{\sim}{f}$ is an element of $L^p(\Phi)$. Similarly, if $\underset{\sim}{M}: \Phi \to \mathbb{C}^{nxn}$, we say that $\underset{\sim}{M} \in L^{p(nxn)}(\Phi)$ if each element of $\underset{\sim}{M}(\cdot)$ belongs to $L^p(\Phi)$. The space $L^p(\mathbb{R})$ will be denoted simply by L^p. Specifically, $f \in L^p$ if $\int_{-\infty}^{+\infty} |f(t)|^p dt < +\infty$.

(2.11) Fourier Transforms. A few standard results from the theory of Fourier transforms are listed here [3]. If $f \in L^1$, then the Fourier transform of f, denoted $\mathcal{F}[f]$, is defined for all $\omega \in \mathbb{R}$ by

$$\mathcal{F}[f](\omega) \triangleq \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt. \tag{1}$$

Since $f \in L^1$, the existence of the integral on the right-hand side of (1) is assured.

If $f \in L^2$, then there exists a function in L^2, which will also be called the Fourier transform of f and denoted by $\mathcal{F}[f]$, which satisfies the following condition:

$$\lim_{A\to+\infty} \int_{-\infty}^{+\infty} \left|\mathcal{F}[f](\omega) - \int_{-A}^{+A} f(t)e^{-j\omega t}dt\right|^2 d\omega = 0. \tag{2}$$

If there are two functions $\mathcal{F}[f]$ and $\hat{\mathcal{F}}[f]$ which satisfy (2), then $\mathcal{F}[f](\omega) = \hat{\mathcal{F}}[f](\omega)$ almost everywhere.[3] Following the usual convention, two functions which are equal almost everywhere are considered equivalent

[3]A property is said to hold almost everywhere if it holds everywhere except on a set of Lebesgue measure zero.

as members of any L^p space. Thus if $f \in L^2$, we can speak of _the_ Fourier transform $\mathcal{F}[f] \in L^2$. If $f \in L^1 \cap L^2$, then the Fourier transforms defined in (1) and (2) are equal almost everywhere.

(2.12) Theorem (Parseval). If f and g are elements of L^2, then

$$\int_{-\infty}^{+\infty} \overline{f(t)}\, g(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{\mathcal{F}[f](\omega)}\, \mathcal{F}[g](\omega)d\omega.$$

(2.13) Convolutions. If $f \in L^1$ and $g \in L^p$, $p \geq 1$, then the _convolution_ of f and g, denoted $f*g$, is defined by

$$f*g(t) \triangleq \int_{-\infty}^{+\infty} f(t-\tau)g(\tau)d\tau. \tag{1}$$

It turns out that $f*g(t)$ exists almost everywhere and that

$$\|f*g\|_p \leq \|f\|_1\|g\|_p. \tag{2}$$

If $f \in L^1 \cap L^2$ and $g \in L^1 \cap L^2$, then it follows from (2) that $f*g \in L^1 \cap L^2$. This fact will be useful in the sequel. Also, if $f \in L^1$ and $g \in L^1$, then (from (2)) $f*g \in L^1$, and it turns out that

$$\mathcal{F}[f*g] = \mathcal{F}[f]\,\mathcal{F}[g]. \tag{3}$$

The following observation will also be useful.

(2.14) Observation.[4] If $f(t)e^{-\sigma t} \in L^1 \cap L^2$ and $g(t)e^{-\sigma t} \in L^1 \cap L^2$, then

$$f*g(t)e^{-\sigma t} = \int_{-\infty}^{+\infty} [f(t-\tau)e^{-\sigma(t-\tau)}]g(\tau)e^{-\sigma\tau}d\tau,$$

and thus $f*g(t)e^{-\sigma t} \in L^1 \cap L^2$.

---

[4]Notations such as $f(t)e^{-\sigma t} \in L^1 \cap L^2$ are technically incorrect since $f(t)e^{-\sigma t}$ is a _number_, not a _function_. However, such notations are convenient and their meaning is clear.

(2.15) Locally L^p Spaces. Let $\Phi \subset \mathbb{R}$ be a Lebesgue measurable set. A function $f\colon \Phi \to \mathbb{C}$ is said to be locally L^p ($p \geq 1$) if $f \in L^p(E)$ for every bounded measurable set $E \subset \Phi$. A function $\underset{\sim}{f}\colon \Phi \to \mathbb{C}^n$ is said to be locally L^{pn} if each of the components of $\underset{\sim}{f}$ is locally L^p.

The set of nonnegative real numbers will be denoted by $\mathbb{R}_+ \triangleq \{t \in \mathbb{R}:\ t \geq 0\}$.

(2.16) Laplace Transforms. If $f\colon \mathbb{R}_+ \to \mathbb{C}$ is locally L^1, then the Laplace transform of f, denoted $\mathcal{L}[f]$, is defined by

$$\mathcal{L}[f](s) \triangleq \int_0^{+\infty} f(t)e^{-st}dt \tag{1}$$

where s is a complex variable. The domain of definition of $\mathcal{L}[f]$ is all $s \in \mathbb{C}$ such that the integrand in (1) is an element of $L^1(\mathbb{R}_+)$.

Let $f\colon \mathbb{R} \to \mathbb{C}$ be such that $f(t) = 0$ for $t < 0$. Suppose that $f(t)e^{-\sigma t} \in L^1$ for some $\sigma \in \mathbb{R}$. Then $\mathcal{L}[f](\sigma+j\omega) = \mathcal{F}[f(t)e^{-\sigma t}](\omega)$ for all $\omega \in \mathbb{R}$. This observation shows how one can apply results from the theory of Fourier transforms when dealing with Laplace transforms. For example, if $f,g\colon \mathbb{R} \to \mathbb{C}$ vanish for negative t and if $f(t)e^{-\sigma t}$, $g(t)e^{-\sigma t} \in L^1$ for some $\sigma \in \mathbb{R}$, then Observation (2.14) gives $\mathcal{L}[f*g](\sigma+j\omega) = \mathcal{F}[f*g(t)e^{-\sigma t}](\omega) = \mathcal{F}[f(t)e^{-\sigma t}](\omega)\,\mathcal{F}[g(t)e^{-\sigma t}](\omega) = \mathcal{L}[f](\sigma+j\omega)\,\mathcal{L}[g](\sigma+j\omega)$, all $\omega \in \mathbb{R}$.

The Fourier or Laplace transform of a vector-valued or matrix-valued function is obtained by taking the transform of each component. For example, if $\underset{\sim}{f} = [f_1,\ldots,f_n]^T\colon \mathbb{R}_+ \to \mathbb{C}^n$ is locally L^{1n}, then $\mathcal{L}[\underset{\sim}{f}] \triangleq [\mathcal{L}[f_1],\ldots,\mathcal{L}[f_n]]^T$. The preceeding results for Fourier and Laplace transforms have obvious extensions when dealing with vector-valued and matrix-valued functions, and these will be used freely in the sequel.

For example, Parseval's Theorem (2.12) gives the following observation:

(2.17) Observation. If $\underset{\sim}{f}(t)e^{-\sigma t} \in L^{1n} \cap L^{2n}$ and $\underset{\sim}{g}(t)e^{-\sigma t} \in L^{1n} \cap L^{2n}$ for some $\sigma \in \mathbb{R}$, and if $\underset{\sim}{f}(t) = \underset{\sim}{g}(t) = 0$ for $t < 0$, then

$$\int_{-\infty}^{+\infty} \underset{\sim}{f}^H(t)\underset{\sim}{g}(t)e^{-2\sigma t}dt = \int_{-\infty}^{+\infty} [\underset{\sim}{f}(t)e^{-\sigma t}]^H[\underset{\sim}{g}(t)e^{-\sigma t}]dt =$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\{\mathcal{F}[\underset{\sim}{f}(t)e^{-\sigma t}](\omega)\}^H\{\mathcal{F}[\underset{\sim}{g}(t)e^{-\sigma t}](\omega)\}d\omega =$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\{\mathcal{L}[\underset{\sim}{f}](\sigma+j\omega)\}^H\{\mathcal{L}[\underset{\sim}{g}](\sigma+j\omega)\}d\omega.$$

(2.18) Passivity. Consider a dynamical system (possibly nonlinear) with input $\underset{\sim}{u}(\cdot)$ and output $\underset{\sim}{y}(\cdot)$, which is described by ordinary differential equations of the form

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{f}(\underset{\sim}{x}(t),\underset{\sim}{u}(t)) \tag{1a}$$

$$\underset{\sim}{y}(t) = \underset{\sim}{g}(\underset{\sim}{x}(t),\underset{\sim}{u}(t)) \tag{1b}$$

where $\underset{\sim}{x}(t) \in \Sigma \subset \mathbb{R}^k$, $\underset{\sim}{u}(t) \in U \subset \mathbb{R}^n$, and $\underset{\sim}{y}(t) \in \mathbb{R}^m$. Σ is called the state space. It is assumed that the power flowing into the system at time t is a function of $\underset{\sim}{u}(t)$ and $\underset{\sim}{y}(t)$, denoted by $p(\underset{\sim}{u}(t),\underset{\sim}{y}(t))$. The following definition of passivity is essentially that given by Rohrer [4]. It is easily shown to be equivalent to Willems' [5] definition (see Appendix A).

(2.19) Definition (Passivity). The system described in Section (2.18) is passive if there exists a finite-valued function $E\colon \Sigma \to \mathbb{R}_+$ such that

$$\int_0^t p(\underset{\sim}{u}(\tau),\underset{\sim}{g}(\underset{\sim}{x}(\tau),\underset{\sim}{u}(\tau)))d\tau + E(\underset{\sim}{x}_0) \geq 0 \tag{1}$$

$\underset{\sim}{x}_0 \to$

for all $t \geq 0$, all $\underset{\sim}{x}_0 \in \Sigma$, and all admissible inputs $\underset{\sim}{u}(\cdot)$. The function $\underset{\sim}{x}(\cdot)$ appearing in (1) is the state trajectory corresponding to the input

$\underset{\sim}{u}(\cdot)$, and the notation $\int_{\substack{0 \\ \underset{\sim}{x}_0 \to}}^{t}$ indicates that $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$. A system which is not passive is <u>active</u>.

The integral appearing in (1) is simply the net energy flowing into the system over the time interval [0,t]. A negative value for this integral indicates a net extraction of energy from the system. These definitions allow us to classify the one-port described in the Introduction as <u>active</u>, since for any nonzero initial state, arbitrarily large amounts of energy can be extracted, and therefore the inequality (1) cannot be satisfied.

III. PASSIVITY CONDITIONS FOR THE GENERAL NETWORK

<u>(3.1) Mathematical Representation of the Network</u>. Consider an n-port electrical network $\mathcal{N}$, with $\underset{\sim}{v}$ denoting the vector of port voltages and $\underset{\sim}{i}$ denoting the vector of port currents. It is assumed that there exists a mathematical representation of the n-port $\mathcal{N}$ involving variables $\underset{\sim}{u}$ and $\underset{\sim}{y}$ which are related to $\underset{\sim}{v}$ and $\underset{\sim}{i}$ by a nonsingular coordinate transformation matrix $\underset{\sim}{\Omega} \in \mathbb{R}^{2n \times 2n}$ as follows:

$$\begin{bmatrix} \underset{\sim}{v} \\ \hdashline \underset{\sim}{i} \end{bmatrix} = \underset{\sim}{\Omega} \begin{bmatrix} \underset{\sim}{y} \\ \hdashline \underset{\sim}{u} \end{bmatrix} = \left[\begin{array}{c:c} \underset{\sim}{a} & \underset{\sim}{b} \\ \hdashline \underset{\sim}{c} & \underset{\sim}{d} \end{array}\right] \begin{bmatrix} \underset{\sim}{y} \\ \hdashline \underset{\sim}{u} \end{bmatrix} \tag{1}$$

where $\underset{\sim}{\Omega}$ has been partitioned into four nxn submatrices $\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{c}$, and $\underset{\sim}{d}$ as shown on the right-hand side of (1). Likewise, $\underset{\sim}{\Omega}^{-1}$ is partitioned into four submatrices $\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma},\underset{\sim}{\delta} \in \mathbb{R}^{n \times n}$ as shown below:

$$\begin{bmatrix} \underset{\sim}{y} \\ \hdashline \underset{\sim}{u} \end{bmatrix} = \underset{\sim}{\Omega}^{-1} \begin{bmatrix} \underset{\sim}{v} \\ \hdashline \underset{\sim}{i} \end{bmatrix} = \left[\begin{array}{c:c} \underset{\sim}{\alpha} & \underset{\sim}{\beta} \\ \hdashline \underset{\sim}{\gamma} & \underset{\sim}{\delta} \end{array}\right] \begin{bmatrix} \underset{\sim}{v} \\ \hdashline \underset{\sim}{i} \end{bmatrix} . \tag{2}$$

The variables $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are called the <u>input</u> and <u>output</u>, respectively.

The input $\underset{\sim}{u}$ is considered to be the independent variable, and the output $\underset{\sim}{y}$ is considered to be the dependent variable. It is assumed that $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are related by the following linear, time-invariant, finite-dimensional state equations:

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}\underset{\sim}{x}(t) + \underset{\sim}{B}\underset{\sim}{u}(t) \tag{3a}$$

$$\underset{\sim}{x}(t_0) = \underset{\sim}{x}_0 \tag{3b}$$

$$\underset{\sim}{y}(t) = \underset{\sim}{C}\underset{\sim}{x}(t) + \underset{\sim}{D}\underset{\sim}{u}(t) \tag{3c}$$

where $\underset{\sim}{A} \in \mathbb{R}^{kxk}$, $\underset{\sim}{B} \in \mathbb{R}^{kxn}$, $\underset{\sim}{C} \in \mathbb{R}^{nxk}$, and $\underset{\sim}{D} \in \mathbb{R}^{nxn}$. Here the state space Σ is $\mathbb{C}^k$. The quantity $\underset{\sim}{x}_0 \in \mathbb{C}^k$ appearing in (3b) is called the <u>initial state at t_0</u>. In the special case when $t_0 = 0$, $\underset{\sim}{x}_0$ is simply called the <u>initial state</u>.

In general, the state equations for a linear time-invariant lumped network include terms involving derivatives of the input $\underset{\sim}{u}(\cdot)$ on the right-hand side of (3c) [6]. Such networks will not be considered in this report. Also, the assumption that $\underset{\sim}{\Omega}$ is a real matrix excludes the scattering representation with complex normalization [7]. However, the results in this report whose proofs do not use the assumption that $\underset{\sim}{A},\underset{\sim}{B},\underset{\sim}{C},\underset{\sim}{D}$, and $\underset{\sim}{\Omega}$ are real matrices will be noted. Such results remain valid for complex matrices. Finally, the above assumptions exclude the scattering representation with frequency-dependent normalization [7], for then the relation between $(\underset{\sim}{y},\underset{\sim}{u})$ and $(\underset{\sim}{v},\underset{\sim}{i})$ would be a convolution in the time domain.

Each input $\underset{\sim}{u}(\cdot)$ must belong to a set $\mathcal{U}$, called the set of admissible inputs. We shall take $\mathcal{U}$ to be the set of all locally L^{2n}

functions mapping $\mathbb{R}$ to $\mathbb{C}^n$.[5] Under these conditions, (3) has a unique solution for every initial state $\underset{\sim}{x}_0 \in \mathbb{C}^k$ at t_0 which is given by

$$\underset{\sim}{y}(t) = \underset{\sim}{C}e^{\underset{\sim}{A}(t-t_0)}\underset{\sim}{x}_0 + \int_{t_0}^{t} \underset{\sim}{C}e^{\underset{\sim}{A}(t-\tau)}\underset{\sim}{B}\underset{\sim}{u}(\tau)d\tau + \underset{\sim}{D}\underset{\sim}{u}(t). \tag{4}$$

The integral on the right-hand side of (4) exists, since $\underset{\sim}{u}(\cdot)$, being locally L^{2n}, is therefore locally L^{1n}. Moreover, it is clear from (4) that $\underset{\sim}{y}(\cdot)$ is locally L^{2n}.

Henceforth, the network described in this section will be referred to simply as <u>network $\mathcal{N}$</u>.

Complex-valued port variables $(\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot))$ are allowed as a mathematical convenience. However, Re $\underset{\sim}{v}(\cdot)$ and Re $\underset{\sim}{i}(\cdot)$ are considered to be the physical port variables. With this in mind, the definition of passivity (2.19) must be modified as follows:

<u>(3.2) Definition (Passivity)</u>. Network $\mathcal{N}$ is <u>passive</u> if there exists a finite-valued function $E\colon \mathbb{C}^k \to \mathbb{R}_+$ such that

$$\int_{\substack{0\\ \underset{\sim}{x}_0\to}}^{t} \mathrm{Re}[\underset{\sim}{v}^T(\tau)]\mathrm{Re}[\underset{\sim}{i}(\tau)]d\tau + E(\underset{\sim}{x}_0) \geq 0$$

for all $t \geq 0$, all $\underset{\sim}{x}_0 \in \mathbb{C}^k$, and all port voltage, port current pairs $(\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot))$ consistent with the initial state $\underset{\sim}{x}_0$. [The notation $\int_{\substack{0\\ \underset{\sim}{x}_0\to}}^{t}$ indicates that the network is in state $\underset{\sim}{x}_0$ at $t = 0$.] A network which is not passive is <u>active</u>.

---

[5] The locally L^{2n} functions were defined in Section II. The reader who is unfamiliar with these concepts may simply assume that $\mathcal{U}$ is all piecewise continuous functions, as done by Desoer [8]. Note that a piecewise continuous function is always locally L^{2n}.

The following equivalent condition for passivity is more useful.

(3.3) Lemma. Network $\mathcal{N}$ is passive if and only if there exists a finite-valued function $\hat{E}: \mathbb{C}^k \to \mathbb{R}_+$ such that

$$\operatorname{Re} \int_{\substack{0 \\ \underset{\sim}{x}_0 \to}}^{t} \underset{\sim}{v}^H(\tau)\underset{\sim}{i}(\tau)d\tau + \hat{E}(\underset{\sim}{x}_0) \geq 0 \tag{1}$$

for all $t \geq 0$, all $\underset{\sim}{x}_0 \in \mathbb{C}^k$, and all port voltage, port current pairs $(\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot))$ consistent with the initial state $\underset{\sim}{x}_0$.

Proof (Necessity). Definition (3.2) gives

$$\int_{\substack{0 \\ \underset{\sim}{x}_0 \to}}^{t} \operatorname{Re}[\underset{\sim}{v}^T(\tau)]\operatorname{Re}[\underset{\sim}{i}(\tau)]d\tau + E(\underset{\sim}{x}_0) \geq 0. \tag{2}$$

By linearity,

$$\int_{\substack{0 \\ \underset{\sim}{x}_0 \to}}^{t} \operatorname{Re}[j\underset{\sim}{v}^T(\tau)]\operatorname{Re}[j\underset{\sim}{i}(\tau)]d\tau + E(j\underset{\sim}{x}_0) \geq 0. \tag{3}$$

Now $\operatorname{Re}[j\underset{\sim}{v}(\tau)] = -\operatorname{Im}[\underset{\sim}{v}(\tau)]$, etc., thus (3) becomes

$$\int_{\substack{0 \\ \underset{\sim}{x}_0 \to}}^{t} \operatorname{Im}[\underset{\sim}{v}^T(\tau)]\operatorname{Im}[\underset{\sim}{i}(\tau)]d\tau + E(j\underset{\sim}{x}_0) \geq 0. \tag{4}$$

Note that $\operatorname{Re} \underset{\sim}{v}^H\underset{\sim}{i} = \operatorname{Re} \underset{\sim}{v}^T \operatorname{Re} \underset{\sim}{i} + \operatorname{Im} \underset{\sim}{v}^T \operatorname{Im} \underset{\sim}{i}$. Hence if $\hat{E}(\cdot)$ is defined by $\hat{E}(\underset{\sim}{x}_0) \triangleq E(\underset{\sim}{x}_0) + E(j\underset{\sim}{x}_0)$, then (2) and (4) give (1).

(Sufficiency). Suppose that (1) is satisfied. Let $(\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot))$ be a port voltage, port current pair corresponding to an initial state $\underset{\sim}{x}_0$ and an input $\underset{\sim}{u}(\cdot)$. Let $(\underset{\sim}{v}_R(\cdot),\underset{\sim}{i}_R(\cdot))$ be the port voltage, port current pair corresponding to the initial state $\operatorname{Re} \underset{\sim}{x}_0$ and the input $\operatorname{Re} \underset{\sim}{u}(\cdot)$. Since $\underset{\sim}{A},\underset{\sim}{B},\underset{\sim}{C},\underset{\sim}{D}$, and $\underset{\sim}{\Omega}$ a real matrices, it follows that $(\underset{\sim}{v}_R(\cdot),\underset{\sim}{i}_R(\cdot)) =$ $(\operatorname{Re} \underset{\sim}{v}(\cdot),\operatorname{Re} \underset{\sim}{i}(\cdot))$. Hence $\operatorname{Re} \underset{\sim}{v}_R^H\underset{\sim}{i}_R = \operatorname{Re} \underset{\sim}{v}^T \operatorname{Re} \underset{\sim}{i}$, and (1) gives

$$\int_0^t \mathrm{Re}\; \underset{\sim}{v}^T(\tau)\mathrm{Re}\; \underset{\sim}{i}(\tau)d\tau + \hat{E}(\mathrm{Re}\; \underset{\sim}{x}_0) \geq 0. \tag{5}$$

$\underset{\sim}{x}_0 \rightarrow$

If $E(\cdot)$ is defined by $E(\underset{\sim}{x}_0) = \hat{E}(\mathrm{Re}\; \underset{\sim}{x}_0)$, then Definition (3.2) is satisfied.

Q.E.D.

(3.4) Remark. The assumption that $\underset{\sim}{A},\underset{\sim}{B},\underset{\sim}{C},\underset{\sim}{D}$, and $\underset{\sim}{\Omega}$ are real matrices was used in the sufficiency part of the above proof and was not used in the necessity part.

(3.5) Notation. Let $\lambda(\underset{\sim}{A})$ denote the set of eigenvalues of $\underset{\sim}{A}$, i.e., $\lambda(\underset{\sim}{A}) \triangleq \{s \in \mathbb{C}: \det(s\underset{\sim}{I}-\underset{\sim}{A}) = 0\}$.

(3.6) Definition. The network function $\underset{\sim}{G}: \mathbb{C}\backslash\lambda(\underset{\sim}{A}) \rightarrow \mathbb{C}^{nxn}$ for network $\mathcal{N}$ is defined by

$$\underset{\sim}{G}(s) \triangleq \underset{\sim}{C}(s\underset{\sim}{I}-\underset{\sim}{A})^{-1}\underset{\sim}{B} + \underset{\sim}{D}.$$

Let $\{\lambda_i\}_{i=1}^q$ denote the set of poles of $\underset{\sim}{G}(\cdot)$.[6] Let $\sigma_c = \max\{\mathrm{Re}\; \lambda_i: 1 \leq i \leq q\}$. The following basic facts from linear system theory will be needed. Each element of the matrix $\underset{\sim}{C}e^{t\underset{\sim}{A}}\underset{\sim}{B}$ has the form $\sum_{i=1}^q p_i(t)e^{\lambda_i t}$, where $p_i(\cdot)$ is a polynomial (possibly constant). Hence $(\underset{\sim}{C}e^{t\underset{\sim}{A}}\underset{\sim}{B})e^{-\sigma t} \in L^{1(nxn)}(\mathbb{R}_+) \cap L^{2(nxn)}(\mathbb{R}_+)$ for all $\sigma > \sigma_c$. Therefore $\mathcal{L}[\underset{\sim}{C}e^{t\underset{\sim}{A}}\underset{\sim}{B}]$ exists in the half plane $\{s \in \mathbb{C}: \mathrm{Re}(s) > \sigma_c\}$, and in fact $\mathcal{L}[\underset{\sim}{C}e^{t\underset{\sim}{A}}\underset{\sim}{B}](s) = \underset{\sim}{C}(s\underset{\sim}{I}-\underset{\sim}{A})^{-1}\underset{\sim}{B}$ in this half plane.

---

[6] Note that $\{\lambda_i\}_{i=1}^q \subset \lambda(\underset{\sim}{A})$, but it is not necessarily true that $\{\lambda_i\}_{i=1}^q = \lambda(\underset{\sim}{A})$. However, $\{\lambda_i\}_{i=1}^q = \lambda(\underset{\sim}{A})$ in the special case when the state equations (3) in Section (3.1) are minimal, i.e., both controllable and observable [2]. This fact will not be needed in this report.

(3.7) Lemma. Let $s_0 \in \mathbb{C} \setminus \lambda(\underset{\sim}{A})$ and let $\underset{\sim}{w} \in \mathbb{C}^n$. If the input to network $\mathcal{N}$ is $\underset{\sim}{u}(t) = \underset{\sim}{w}e^{s_0 t}$ for $t \geq 0$, then there exists an initial condition $\underset{\sim}{x}_0 \in \mathbb{C}^k$ such that the output is $\underset{\sim}{y}(t) = \underset{\sim}{G}(s_0)\underset{\sim}{w}e^{s_0 t}$ for $t \geq 0$, where $\underset{\sim}{G}(\cdot)$ is the network function (Def. (3.6)).

Proof. The desired initial condition is $\underset{\sim}{x}_0 = (s_0\underset{\sim}{I}-\underset{\sim}{A})^{-1}\underset{\sim}{B}\underset{\sim}{w}$. In fact, the corresponding state trajectory $\underset{\sim}{x}(\cdot)$ is

$$\underset{\sim}{x}(t) = (s_0\underset{\sim}{I}-\underset{\sim}{A})^{-1}\underset{\sim}{B}\underset{\sim}{w}e^{s_0 t}, \quad t \geq 0. \tag{1}$$

To see this, note that

$$\begin{aligned}\dot{\underset{\sim}{x}}(t) &= s_0(s_0\underset{\sim}{I}-\underset{\sim}{A})^{-1}\underset{\sim}{B}\underset{\sim}{w}e^{s_0 t} \\ &= \underset{\sim}{A}(s_0\underset{\sim}{I}-\underset{\sim}{A})^{-1}\underset{\sim}{B}\underset{\sim}{w}e^{s_0 t} + (s_0\underset{\sim}{I}-\underset{\sim}{A})(s_0\underset{\sim}{I}-\underset{\sim}{A})^{-1}\underset{\sim}{B}\underset{\sim}{w}e^{s_0 t} \\ &= \underset{\sim}{A}\underset{\sim}{x}(t) + \underset{\sim}{B}\underset{\sim}{u}(t), \quad t \geq 0.\end{aligned}$$

Using the solution for $\underset{\sim}{x}(\cdot)$ from (1) gives

$$\begin{aligned}\underset{\sim}{y}(t) &= \underset{\sim}{C}\underset{\sim}{x}(t) + \underset{\sim}{D}\underset{\sim}{u}(t) \\ &= \underset{\sim}{C}(s_0\underset{\sim}{I}-\underset{\sim}{A})^{-1}\underset{\sim}{B}\underset{\sim}{w}e^{s_0 t} + \underset{\sim}{D}\underset{\sim}{w}e^{s_0 t} \\ &= \underset{\sim}{G}(s_0)\underset{\sim}{w}e^{s_0 t}, \quad t \geq 0.\end{aligned}$$

Q.E.D.

Obviously, Lemma (3.7) remains valid for complex matrices.

(3.8) Definition. The initial condition described in Lemma (3.7) will be called an appropriate initial condition.

(3.9) Definition. Network $\mathcal{N}$ is said to be controllable if, given any two states $\underset{\sim}{x}_0, \underset{\sim}{x}_1 \in \mathbb{C}^k$, there exists an input $\underset{\sim}{u}(\cdot)$ and finite times $t_0, t_1 \in \mathbb{R}$ with $t_1 \geq t_0$ such that the corresponding state trajectory $\underset{\sim}{x}(\cdot)$ satisfies $\underset{\sim}{x}(t_0) = \underset{\sim}{x}_0$ and $\underset{\sim}{x}(t_1) = \underset{\sim}{x}_1$. [Since the network is time invariant, we may always choose $t_0 = 0$ or any other convenient value.]

(3.10) Lemma.

(i) If network $\mathcal{N}$ is passive, then

$$\mathrm{Re} \int_{\underset{\sim}{0}+}^{t} \underset{\sim}{v}^H(\tau)\underset{\sim}{i}(\tau) \geq 0 \tag{1}$$

for all $t \geq 0$ and all port voltage, port current pairs $(\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot))$ consistent with the zero initial condition.

(ii) If network $\mathcal{N}$ is controllable and condition (1) is satisfied, then network $\mathcal{N}$ is passive.

Proof of (i). Suppose that

$$\mathrm{Re} \int_{\underset{\sim}{0}+}^{t_1} \underset{\sim}{v}^H(\tau)\underset{\sim}{i}(\tau)d\tau = -K < 0 \tag{2}$$

for some $t_1 > 0$ and for some pair $(\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot))$ consistent with the zero initial condition. Let $r \in \mathbb{R}$. By the linearity of the zero state response, $(\hat{\underset{\sim}{v}}(\cdot),\hat{\underset{\sim}{i}}(\cdot) \triangleq (r\underset{\sim}{v}(\cdot),r\underset{\sim}{i}(\cdot))$ is a port voltage, port current pair consistent with the zero initial condition. Hence, from (2)

$$\mathrm{Re} \int_{\underset{\sim}{0}+}^{t_1} \hat{\underset{\sim}{v}}^H(\tau)\hat{\underset{\sim}{i}}(\tau)d\tau = r^2\mathrm{Re} \int_{\underset{\sim}{0}+}^{t_1} \underset{\sim}{v}^H(\tau)\underset{\sim}{i}(\tau)d\tau = -r^2K \tag{3}$$

Since $r \in \mathbb{R}$ can be chosen arbitrarily large, it is impossible to define a finite-valued function $\hat{E}(\cdot)$ as in Lemma (3.3). Thus network $\mathcal{N}$ is active.

Proof of (ii). Suppose that network $\mathcal{N}$ is controllable and condition (1) is satisfied. Let $\underset{\sim}{x}_0 \in \mathbb{C}^k$. By controllability, there exists an input $\underset{\sim}{u}_0(\cdot) \in \mathcal{U}$ and a finite time $-t_0 \leq 0$ such that the corresponding state trajectory satisfies $\underset{\sim}{x}(-t_0) = \underset{\sim}{0}$ and $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$. Suppose that an arbitrary input $\underset{\sim}{u}(\cdot) \in \mathcal{U}$ is applied over $[0,+\infty)$. Let $(\underset{\sim}{v}_0(\cdot),\underset{\sim}{i}_0(\cdot))$ and $(\underset{\sim}{v}(\cdot),\underset{\sim}{i}(\cdot))$ denote the corresponding port voltage, port current pairs over $[-t_0,0)$ and $[0,+\infty)$, respectively. Since the network is time invariant, condition (1) can be rewritten

$$\operatorname{Re}\underset{\underset{\sim}{0}\to\underset{\sim}{x}_0}{\int_{-t_0}^{0}} \underset{\sim}{v}_0^H(\tau)\underset{\sim}{i}_0(\tau)d\tau + \operatorname{Re}\underset{\underset{\sim}{x}_0\to}{\int_{0}^{t}} \underset{\sim}{v}^H(\tau)\underset{\sim}{i}(\tau)d\tau \geq 0 \tag{4}$$

for $t \geq 0$. Note that the first integral in (4) is finite, since $\underset{\sim}{u}_0(\cdot)$ and $\underset{\sim}{y}_0(\cdot)$, and hence $\underset{\sim}{v}_0(\cdot)$ and $\underset{\sim}{i}_0(\cdot)$, are locally L^{2n}. Thus we can define a finite-valued function $\hat{E}(\cdot)$ as in Lemma (3.3). This shows that network $\mathcal{N}$ is passive.

Q.E.D.

(3.11) Remark. The example in the Introduction is an active, uncontrollable network which looks like an ideal one-port short circuit when the initial state is zero. Thus it satisfies condition (1) of Lemma (3.10). This shows that the controllability hypothesis in part (ii) of Lemma (3.10) is not superfluous.

(3.12) Remark. The proof of part (i) of Lemma (3.10) did not use, either directly or indirectly, the assumption that $\underset{\sim}{A},\underset{\sim}{B},\underset{\sim}{C},\underset{\sim}{D}$, and $\underset{\sim}{\Omega}$ are real matrices. Thus part (i) of Lemma (3.10) remains valid when $\underset{\sim}{A},\underset{\sim}{B},\underset{\sim}{C},\underset{\sim}{D}$, and $\underset{\sim}{\Omega}$ are complex matrices.

(3.13) Definition. The dissipation matrix $\underset{\sim}{\mathcal{D}}: \mathbb{C}\setminus\lambda(\underset{\sim}{A}) \to \mathbb{C}^{n\times n}$ for network $\mathcal{N}$ is defined by

$$\underset{\sim}{\mathcal{D}}(s) \triangleq [\underset{\sim}{a}\underset{\sim}{G}(s)+\underset{\sim}{b}]^H[\underset{\sim}{c}\underset{\sim}{G}(s)+\underset{\sim}{d}]$$

where $\underset{\sim}{G}(\cdot)$ is the network function (Def. (3.6)) and $\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{c}$, and $\underset{\sim}{d}$ are the matrices appearing in (1) of Section (3.1).

(3.14) Theorem (Necessary Condition for Passivity). If network $\mathcal{N}$ is passive, then the dissipation matrix satisfies

$$\mathcal{D}(s) + \mathcal{D}^H(s) \geq 0 \text{ for all } s \in (\text{open RHP}) \setminus \lambda(A).$$

Proof. Let $s_0 \in (\text{open RHP}) \setminus \lambda(A)$, let $w \in \mathbb{C}^n$ be arbitrary, and suppose that the input to network $\mathcal{N}$ is $u(t) = we^{s_0 t}$ for $t \geq 0$. Moreover, suppose that the network starts at the appropriate initial condition x_0 corresponding to the input $u(\cdot)$ (Def. (3.8)). Thus the output is $y(t) = G(s_0)we^{s_0 t}$ for $t \geq 0$ (Lemma (3.7)). From (1) of Section (3.1) we obtain

$$v(t) = [aG(s_0)+b]we^{s_0 t}, \quad t \geq 0 \tag{1}$$

$$i(t) = [cG(s_0)+d]we^{s_0 t}, \quad t \geq 0. \tag{2}$$

Since the network is passive, it follows from (1), (2), and Lemma (3.3) that

$$\int_0^t \mathrm{Re}\{w^H[aG(s_0)+b]^H[cG(s_0)+d]w\}e^{2\sigma_0 \tau} d\tau + \hat{E}(x_0) \geq 0 \tag{3}$$

where $\sigma_0 = \mathrm{Re}\, s_0 > 0$. Recalling the definition of $\mathcal{D}(\cdot)$ and performing the integration in (3) gives

$$\mathrm{Re}[w^H \mathcal{D}(s_0)w]\left(\frac{e^{2\sigma_0 t}-1}{2\sigma_0}\right) + \hat{E}(x_0) \geq 0. \tag{4}$$

Condition (4) can be satisfied for all $t \geq 0$ only if

$$\mathrm{Re}[w^H \mathcal{D}(s_0)w] \geq 0. \tag{5}$$

It follows from Lemma (2.2) that $\mathcal{D}(s_0) + \mathcal{D}^H(s_0) \geq 0$.

Q.E.D.

(3.15) Remark. The proof of Theorem (3.14) does not use, directly or indirectly, the assumption that $\underset{\sim}{A},\underset{\sim}{B},\underset{\sim}{C},\underset{\sim}{D}$, and $\underset{\sim}{\Omega}$ are real matrices. Thus Theorem (3.14) remains valid when $\underset{\sim}{A},\underset{\sim}{B},\underset{\sim}{C},\underset{\sim}{D}$, and $\underset{\sim}{\Omega}$ are complex matrices.

(3.16) Remark. If network $\mathcal{N}$ is passive, then Theorem (3.14) shows that $\underset{\sim}{\mathcal{D}}(s) + \underset{\sim}{\mathcal{D}}^H(s) \geq 0$ for all $s \in$ (open RHP)$\setminus\lambda(\underset{\sim}{A})$. Moreover, if $\underset{\sim}{A},\underset{\sim}{B},\underset{\sim}{C},\underset{\sim}{D}$, and $\underset{\sim}{\Omega}$ are real matrices, then $\underset{\sim}{\mathcal{D}}(\sigma)$ is a real matrix for all real $\sigma \notin \lambda(\underset{\sim}{A})$. However, $\underset{\sim}{\mathcal{D}}(\cdot)$ is not necessarily a positive real matrix, because it is not necessarily holomorphic in the open RHP. This is illustrated in Example (3.17) below.

(3.17) Example. Suppose that network $\mathcal{N}$ is a one-port described by state equations of the form

$$\dot{x} = \frac{1}{2}x + u \tag{1a}$$

$$y = x \tag{1b}$$

where (v,i) is related to (y,u) by

$$\begin{bmatrix} v \\ i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} y \\ u \end{bmatrix}. \tag{2}$$

The network function for this network is $G(s) = (s-\frac{1}{2})^{-1}$ (note that $G(\cdot)$ has a pole in the open RHP). The dissipation matrix for this network is seen from Definition (3.13) to be

$$\mathcal{D}(s) = \overline{[1 \cdot G(s) + 0]}\,[1 \cdot G(s) + 1] = \frac{1}{\bar{s}-\frac{1}{2}}\left(\frac{1}{s-\frac{1}{2}} + 1\right)$$

$$= \frac{1}{\bar{s}-\frac{1}{2}}\left(\frac{s+\frac{1}{2}}{s-\frac{1}{2}}\right) = \frac{s+\frac{1}{2}}{|s-\frac{1}{2}|^2}. \tag{3}$$

Note that $\mathcal{D}(\cdot)$ is not a rational function. $\mathcal{D}(\cdot)$ is not holomorphic in the open RHP, in fact, it is not even meromorphic there. Moreover, $|\mathcal{D}(s)| \to +\infty$ as $s \to \frac{1}{2}$. Thus it has a "pole-like" singularity at $s = \frac{1}{2}$. From (1) and (2) we obtain

$$\dot{x} = \frac{1}{2}x + u = \frac{1}{2}x + (i-y) = \frac{1}{2}x + (i-x) = -\frac{1}{2}x + i \tag{4}$$

$$v = y = x. \tag{5}$$

It will follow from the results in Section IV that this network is passive, because (4) and (5) show that it has a controllable hybrid representation with a positive real hybrid matrix $\hat{G}(s) = (s+\frac{1}{2})^{-1}$. Finally, note that Re $\mathcal{D}(s) \geq 0$ for all s in the open RHP where $\mathcal{D}(\cdot)$ is defined, in agreement with Theorem (3.14).

<u>(3.18) Theorem (Sufficient Condition for Passivity)</u>. Suppose that

(a) network $\mathcal{N}$ is controllable,

(b) the dissipation matrix (Def. (3.13)) satisfies $\underset{\sim}{\mathcal{D}}(s) + \underset{\sim}{\mathcal{D}}^H(s) \geq 0$ for all $s \in$ (open RHP)$\setminus \lambda(\underset{\sim}{A})$,

(c) the network function $\underset{\sim}{G}(\cdot)$ (Def. (3.6)) has no poles in the open RHP, and

(d) $\underset{\sim}{a}^T\underset{\sim}{c} = \underset{\sim}{0}_{nxn}$ (see (1), Section (3.1)),

then network $\mathcal{N}$ is passive.

<u>Proof</u>. From Lemma (3.10) and (1) of Section (3.1), it is sufficient to show that

$$\text{Re}\int_{\underset{\sim}{0}-}^{t_1} \underset{\sim}{v}^H(t)\underset{\sim}{i}(t)dt = \text{Re}\int_{\underset{\sim}{0}-}^{t_1} [\underset{\sim}{a}\underset{\sim}{y}(t)+\underset{\sim}{b}\underset{\sim}{u}(t)]^H[\underset{\sim}{c}\underset{\sim}{y}(t)+\underset{\sim}{d}\underset{\sim}{u}(t)]dt \geq 0 \tag{1}$$

for all $t_1 \geq 0$ and for all inputs $\underset{\sim}{u}(\cdot) \in \mathcal{U}$, where $\underset{\sim}{y}(\cdot)$ is the corresponding output for zero initial state. In the following argument it is assumed that $t_1 \geq 0$ is <u>arbitrary</u>, but <u>fixed</u>. Let $\underset{\sim}{u}(\cdot) \in \mathcal{U}$ be an arbitrary input over $[0,t_1]$ and let $\underset{\sim}{y}(\cdot)$ denote the corresponding output over $[0,t_1]$ for zero initial state. Define an input $\hat{\underset{\sim}{u}}(\cdot) \in \mathcal{U}$ as follows:

$$\hat{\underset{\sim}{u}}(t) \triangleq \underset{\sim}{u}(t), \ t \in [0,t_1] \tag{2a}$$

$$\triangleq \underset{\sim}{0}, \ t < 0, \ t > t_1. \tag{2b}$$

Let $\hat{\underset{\sim}{y}}(\cdot)$ denote the output corresponding to the input $\hat{\underset{\sim}{u}}(\cdot)$ for zero initial state. From (4) of Section (3.1) it follows that

$$\hat{\underset{\sim}{y}}(t) = \underset{\sim}{y}(t), \ t \in [0,t_1] \tag{3a}$$

$$= \underset{\sim}{0}, \ t < 0. \tag{3b}$$

Note that $\hat{\underset{\sim}{y}}(t)$ does not necessarily vanish for $t > t_1$. However, since $\underset{\sim}{a}^T\underset{\sim}{c} = \underset{\sim}{0}_{nxn}$ (assumption (d)), if follows from (2) and (3) that

$$[\underset{\sim}{a}\hat{\underset{\sim}{y}}(t)+\underset{\sim}{b}\hat{\underset{\sim}{u}}(t)]^H[\underset{\sim}{c}\hat{\underset{\sim}{y}}(t)+\underset{\sim}{d}\hat{\underset{\sim}{u}}(t)]$$

$$= [\underset{\sim}{a}\underset{\sim}{y}(t)+\underset{\sim}{b}\underset{\sim}{u}(t)]^H[\underset{\sim}{c}\underset{\sim}{y}(t)+\underset{\sim}{d}\underset{\sim}{u}(t)], \ t \in [0,t_1] \tag{4a}$$

$$= 0, \ t < 0, \ t > t_1. \tag{4b}$$

Note that $\hat{\underset{\sim}{u}}(t)e^{-\sigma t} \in L^{1n} \cap L^{2n}$ for all $\sigma \in \mathbb{R}$, and so $\hat{\underset{\sim}{U}}(s) \triangleq \mathcal{L}[\hat{\underset{\sim}{u}}](s)$ exists for all $s \in \mathbb{C}$. The assumption that $\underset{\sim}{G}(\cdot)$ has no poles in the open RHP guarantees that $(\underset{\sim}{C}e^{t\underset{\sim}{A}}\underset{\sim}{B})e^{-\sigma t} \in L^{1(nxn)}(\mathbb{R}_+) \cap L^{2(nxn)}(\mathbb{R}_+)$ for all $\sigma > 0$. Since the initial condition is zero, (4) of Section (3.1) and Observation (2.14) show that $\hat{\underset{\sim}{y}}(t)e^{-\sigma t} \in L^{1n} \cap L^{2n}$ for all $\sigma > 0$, moreover,

$$\mathcal{L}[\hat{\underset{\sim}{y}}](s) = \underset{\sim}{G}(s)\hat{\underset{\sim}{U}}(s) \text{ for } \mathrm{Re}(s) > 0. \tag{5}$$

It follows from (1),(4),(5), and Observation (2.17) that

$$\begin{aligned}
&\mathrm{Re}\int_0^{t_1} \underset{\sim}{v}^H(t)\underset{\sim}{i}(t)e^{-2\sigma t}dt \\
&\quad = \mathrm{Re}\int_{-\infty}^{+\infty} [\underset{\sim}{a}\hat{\underset{\sim}{y}}(t)+\underset{\sim}{b}\hat{\underset{\sim}{u}}(t)]^H[\underset{\sim}{c}\hat{\underset{\sim}{y}}(t)+\underset{\sim}{d}\hat{\underset{\sim}{u}}(t)]e^{-2\sigma t}dt \\
&\quad = \mathrm{Re}\,\frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{\underset{\sim}{U}}^H(\sigma+j\omega)\,\underset{\sim}{\mathcal{D}}(\sigma+j\omega)\,\hat{\underset{\sim}{U}}(\sigma+j\omega)d\omega \\
&\quad \geq 0, \text{ for } \sigma > 0. \qquad (6)
\end{aligned}$$

The last inequality follows from assumption (b). Now $|\underset{\sim}{v}^H(t)\underset{\sim}{i}(t)e^{-2\sigma t}| \leq |\underset{\sim}{v}^H(t)\underset{\sim}{i}(t)|$, $t \geq 0$, and $\int_0^{t_1}|\underset{\sim}{v}^H(t)\underset{\sim}{i}(t)|dt < +\infty$ since $\underset{\sim}{v}(\cdot)$ and $\underset{\sim}{i}(\cdot)$ are locally L^{2n}. Let $\langle\sigma_n\rangle$ be a sequence with $\sigma_n > 0$ and $\sigma_n \to 0$. It follows from (6) and the Lebesgue Dominated Convergence Theorem [3] that

$$\mathrm{Re}\int_0^{t_1} \underset{\sim}{v}^H(t)\underset{\sim}{i}(t)dt = \lim_{n\to\infty} \mathrm{Re}\int_0^{t_1} \underset{\sim}{v}^H(t)\underset{\sim}{i}(t)e^{-2\sigma_n t} \geq 0. \qquad (7)$$

Q.E.D.

<u>(3.19) Remarks</u>. The example given in the Introduction shows that assumption (a) is not superfluous. However, the passive network in Example (3.17) satisfies assumptions (a) and (b) but not (c) and (d). Therefore assumptions (c) and (d) are stronger than necessary, but they are adequate for our purposes. Finally, since $\underset{\sim}{G}(\cdot)$ is assumed to be holomorphic in the open RHP, assumption (b) becomes $\underset{\sim}{\mathcal{D}}(s) + \underset{\sim}{\mathcal{D}}^H(s) \geq 0$ for <u>all</u> s in the open RHP.

Appendix B gives a condition under which a network representation as described in Section (3.1) involving a pair of variables $(\underset{\sim}{y}_1,\underset{\sim}{u}_1)$ can be transformed into a representation of the same form involving another pair of variables $(\underset{\sim}{y}_2,\underset{\sim}{u}_2)$.

IV. THE HYBRID REPRESENTATION

(4.1) Definition. The representation described in Section (3.1) is said to be a hybrid representation for network $\mathcal{N}$ if, for each k, $1 \leq k \leq n$, one of the following two conditions is satisfied:

(i) $v_k = y_k$ and $i_k = u_k$, or

(ii) $v_k = u_k$ and $i_k = y_k$,

where v_k is the k-th component of $\underset{\sim}{v}$, etc.

(4.2) Observation. Thus, for a hybrid representation, the matrix $\underset{\sim}{\Omega}$ (see (1) of Section (3.1)) has the following form:

$$\underset{\sim}{\Omega} = \left[\begin{array}{c|c} \underset{\sim}{a} & \underset{\sim}{b} \\ \hline \underset{\sim}{b} & \underset{\sim}{a} \end{array}\right] \tag{1}$$

where $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are nxn diagonal matrices:

$$\underset{\sim}{a} = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}, \quad \underset{\sim}{b} = \begin{bmatrix} b_{11} & & 0 \\ & \ddots & \\ 0 & & b_{nn} \end{bmatrix}. \tag{2}$$

Moreover, for each k, $1 \leq k \leq n$, the constants a_{kk} and b_{kk} satisfy one of the following two conditions:

(i) $a_{kk} = 1$ and $b_{kk} = 0$, or

(ii) $a_{kk} = 0$ and $b_{kk} = 1$.

From these properties, it is easy to verify that $\underset{\sim}{\mathcal{D}}(s) = \underset{\sim}{G}^H(s)\underset{\sim}{a} + \underset{\sim}{b}\underset{\sim}{G}(s)$. Since $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{I}$, it follows that $\underset{\sim}{\mathcal{D}}(s) + \underset{\sim}{\mathcal{D}}^H(s) = \underset{\sim}{G}(s) + \underset{\sim}{G}^H(s)$.

(4.3) Terminology. The network function $\underset{\sim}{G}(\cdot)$ for a hybrid representation will be called a hybrid matrix.

(4.4) Theorem (Necessary and Sufficient Passivity Conditions for the Hybrid Representation). Suppose that network $\mathcal{N}$ has a hybrid representation.

(i) If network $\mathcal{N}$ is passive, then the hybrid matrix $G(\cdot)$ is a positive real matrix.

(ii) If network $\mathcal{N}$ is controllable and its hybrid matrix $G(\cdot)$ is positive real, then network $\mathcal{N}$ is passive.

Proof of (i). From Observation (4.2) and Theorem (3.14), $\mathcal{D}(s) + \mathcal{D}^H(s) = G(s) + G^H(s) \geq 0$ for all $s \in (\text{open RHP})\setminus\lambda(A)$. Thus, from Lemma (2.8), $G(\cdot)$ is holomorphic in the open RHP and $G(s) + G^H(s) \geq 0$ for all s in the open RHP. Finally, since A, B, C, and D are real matrices, it follows that $G(\sigma)$ is a real matrix for real, positive σ. Thus $G(\cdot)$ is a positive real matrix (Def. (2.5)).

Proof of (ii). Note that $a^T c = a^T b = ab = 0_{nxn}$. From Observation (4.2), $\mathcal{D}(s) + \mathcal{D}^H(s) = G(s) + G^H(s)$. Thus it follows that all the conditions of Theorem (3.18) are satisfied, and therefore network $\mathcal{N}$ is passive.

Q.E.D.

(4.5) Remark. The example given in the Introduction shows that the assumption of controllability in part (ii) of Theorem (4.4) is not superfluous.

V. THE SCATTERING REPRESENTATION

Although the sufficient condition for passivity given in Theorem (3.18) has a rather limited range of applicability, the necessary condition given in Theorem (3.14) can be applied to any network which has a representation of the form described in Section (3.1). As noted in Remark (3.15), Theorem (3.14) is valid even for complex matrices. In this section we will utilize Theorem (3.14) to obtain a necessary passivity condition for the scattering representation.

(5.1) Definition. The representation described in Section (3.1) is said to be a scattering representation for network $\mathcal{N}$ if $\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{c}$, and $\underset{\sim}{d}$ (see (1) of Section (3.1)) are nxn diagonal matrices with the following form:

$$\underset{\sim}{a} = \underset{\sim}{b} = \begin{bmatrix} \sqrt{r_1} & & & \\ & \sqrt{r_2} & & \\ & & \ddots & \\ & & & \sqrt{r_n} \end{bmatrix}$$

$$\underset{\sim}{d} = -\underset{\sim}{c} = \begin{bmatrix} \frac{1}{\sqrt{r_1}} & & & \\ & \frac{1}{\sqrt{r_2}} & & \\ & & \ddots & \\ & & & \frac{1}{\sqrt{r_n}} \end{bmatrix}$$

where the real positive constants r_k are called the port normalizing numbers.

(5.2) Observation. Thus, for a scattering representation, $\underset{\sim}{\mathcal{D}}(s) = \underset{\sim}{I} - \underset{\sim}{G}^H(s)\underset{\sim}{G}(s) + \underset{\sim}{G}^H(s) - \underset{\sim}{G}(s)$ and $\underset{\sim}{\mathcal{D}}(s) + \underset{\sim}{\mathcal{D}}^H(s) = 2[\underset{\sim}{I}-\underset{\sim}{G}^H(s)\underset{\sim}{G}(s)]$.

(5.3) Terminology. The network function $\underset{\sim}{G}(\cdot)$ for a scattering representation will be called a scattering matrix.

(5.4) Theorem (Necessary Passivity Condition for the Scattering Representation). Suppose that network $\mathcal{N}$ has a scattering representation. If network $\mathcal{N}$ is passive, then the scattering matrix $\underset{\sim}{G}(\cdot)$ is a bounded real matrix.

Proof. From Observation (5.2) and Theorem (3.14), $\frac{1}{2}[\mathcal{D}(s)+\mathcal{D}^H(s)] = I - G^H(s)G(s) \geq 0$ for all $s \in$ (open RHP)$\setminus\lambda(A)$. Thus, from Lemma (2.9), $G(\cdot)$ is holomorphic in the open RHP and $I - G^H(s)G(s) \geq 0$ for all s in the open RHP. Finally, since A,B,C, and D are real matrices, it follows that $G(\sigma)$ is real for real, positive σ. Thus $G(\cdot)$ is a bounded real matrix (Def. (2.6).

Q.E.D.

VI. CONCLUDING REMARKS

This report has presented rigorous derivations for frequency domain passivity conditions. A very general necessary condition for passivity was presented in Theorem (3.14), and a restricted sufficient condition was presented in Theorem (3.18). It is possible that assumptions (a) and (b) alone of Theorem (3.18) are sufficient to guarantee passivity, although the authors were unable to find a rigorous proof for this conjecture.

APPENDIX

A. EQUIVALENT PASSIVITY DEFINITIONS

It was stated in Section II that the definition of passivity (2.19) is equivalent to Willems' [5] definition. In the context of Section II, Willems' definition reduces to the following: The system described in Section (2.18) is passive[7] if there exists a finite-valued function $S: \Sigma \to \mathbb{R}_+$ such that

$$\int_{0 \atop \underset{\sim}{x}_0 \to}^{t} p(\underset{\sim}{u}(\tau), \underset{\sim}{g}(\underset{\sim}{x}(\tau), \underset{\sim}{u}(\tau)))d\tau + S(\underset{\sim}{x}_0) \geq S(\underset{\sim}{x}(t)) \tag{1}$$

for all $t \geq 0$, all $\underset{\sim}{x}_0 \in \Sigma$, and all admissible inputs $\underset{\sim}{u}(\cdot)$. Willems then defines the available storage $S_a(\cdot)$ as follows

$$S_a(\underset{\sim}{x}_0) \triangleq \sup_{t \geq 0} \{-\int_{0 \atop \underset{\sim}{x}_0 \to}^{t} p(\underset{\sim}{u}(\tau), \underset{\sim}{g}(\underset{\sim}{x}(\tau), \underset{\sim}{u}(\tau)))d\tau\}$$

where the supremum is taken over all inputs $\underset{\sim}{u}(\cdot)$ and all $t \geq 0$. In Theorem 1 of Part I of his paper [5], Willems shows that a system is passive according to his definition if and only if $S_a(\underset{\sim}{x}_0) < +\infty$ for all $\underset{\sim}{x}_0 \in \Sigma$.

Since $S(\underset{\sim}{x}(t)) \geq 0$, it is clear that that condition (1) implies Definition (2.19) (choose $E(x_0)=S(x_0)$). Conversely, if Definition (2.19) is satisfied, then

$$-\int_{0 \atop \underset{\sim}{x}_0 \to}^{t} p(\underset{\sim}{u}(\tau), \underset{\sim}{g}(\underset{\sim}{x}(\tau), \underset{\sim}{u}(\tau)))d\tau \leq E(\underset{\sim}{x}_0) < +\infty.$$

[7] Willems uses the term dissipative rather than passive, because he allows functions $p(\cdot,\cdot)$ which may not have the physical significance of power.

It follows that $S_a(\mathbf{x}_0) < +\infty$ for all $\mathbf{x}_0 \in \Sigma$. Thus Willems' definition is equivalent to Definition (2.19).

B. THE TRANSFORMATION OF NETWORK REPRESENTATIONS

Suppose that there exists a network representaticn of the form described in Section (3.1) involving a pair of variables $(\mathbf{y}_1,\mathbf{u}_1)$ as follows:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}_1 \tag{1a}$$

$$\mathbf{y}_1 = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}_1 \tag{1b}$$

where

$$\left[\begin{array}{c} \mathbf{y}_1 \\ \hline \mathbf{u}_1 \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{\alpha}_1 & \boldsymbol{\beta}_1 \\ \hline \boldsymbol{\gamma}_1 & \boldsymbol{\delta}_1 \end{array}\right] \left[\begin{array}{c} \mathbf{v} \\ \hline \mathbf{i} \end{array}\right] . \tag{2}$$

Let $(\mathbf{y}_2,\mathbf{u}_2)$ be another pair of variables which is related to $(\mathbf{v},\mathbf{i})$ as follows:

$$\left[\begin{array}{c} \mathbf{v} \\ \hline \mathbf{i} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{a}_2 & \mathbf{b}_2 \\ \hline \mathbf{c}_2 & \mathbf{d}_2 \end{array}\right] \left[\begin{array}{c} \mathbf{y}_2 \\ \hline \mathbf{u}_2 \end{array}\right] . \tag{3}$$

<u>(B.1) Lemma</u>. Let $(\mathbf{y}_1,\mathbf{u}_1)$ and $(\mathbf{y}_2,\mathbf{u}_2)$ be as described above. Then a network representation of the form described in Section (3.1) involving the pair $(\mathbf{y}_2,\mathbf{u}_2)$ exists if

$$\det[\boldsymbol{\alpha}_1\mathbf{a}_2+\boldsymbol{\beta}_1\mathbf{c}_2-\mathbf{D}(\boldsymbol{\gamma}_1\mathbf{a}_2+\boldsymbol{\delta}_1\mathbf{c}_2)] \neq 0.$$

<u>Proof</u>. Note that

$$\left[\begin{array}{c} \mathbf{y}_1 \\ \hline \mathbf{u}_1 \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{\alpha}_1\mathbf{a}_2+\boldsymbol{\beta}_1\mathbf{c}_2 & \boldsymbol{\alpha}_1\mathbf{b}_2+\boldsymbol{\beta}_1\mathbf{d}_2 \\ \hline \boldsymbol{\gamma}_1\mathbf{a}_2+\boldsymbol{\delta}_1\mathbf{c}_2 & \boldsymbol{\gamma}_1\mathbf{b}_2+\boldsymbol{\delta}_1\mathbf{d}_2 \end{array}\right] \left[\begin{array}{c} \mathbf{y}_2 \\ \hline \mathbf{u}_2 \end{array}\right] \triangleq \left[\begin{array}{c|c} \mathbf{M} & \mathbf{N} \\ \hline \mathbf{P} & \mathbf{Q} \end{array}\right] \left[\begin{array}{c} \mathbf{y}_2 \\ \hline \mathbf{u}_2 \end{array}\right] . \tag{4}$$

Using (4) in (1b) gives

$$(\underset{\sim}{M}-\underset{\sim}{D}\underset{\sim}{P})\underset{\sim}{y}_2 = \underset{\sim}{C}\underset{\sim}{x} + (\underset{\sim}{D}\underset{\sim}{Q}-\underset{\sim}{N})\underset{\sim}{u}_2. \tag{5}$$

By assumption, $\det(\underset{\sim}{M}-\underset{\sim}{D}\underset{\sim}{P}) \neq 0$. Thus (5) becomes

$$\underset{\sim}{y}_2 = (\underset{\sim}{M}-\underset{\sim}{D}\underset{\sim}{P})^{-1}\underset{\sim}{C}\underset{\sim}{x} + (\underset{\sim}{M}-\underset{\sim}{D}\underset{\sim}{P})^{-1}(\underset{\sim}{D}\underset{\sim}{Q}-\underset{\sim}{N})\underset{\sim}{u}_2. \tag{6}$$

Using (4) and (6), (1a) becomes

$$\dot{\underset{\sim}{x}} = [\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{P}(\underset{\sim}{M}-\underset{\sim}{D}\underset{\sim}{P})^{-1}\underset{\sim}{C}]\underset{\sim}{x} + [\underset{\sim}{B}\underset{\sim}{P}(\underset{\sim}{M}-\underset{\sim}{D}\underset{\sim}{P})^{-1}(\underset{\sim}{D}\underset{\sim}{Q}-\underset{\sim}{N})+\underset{\sim}{B}\underset{\sim}{Q}]\underset{\sim}{u}_2. \tag{7}$$

Q.E.D.

Although we assumed in the main body of this report that the matrices involved were real, the results of this appendix are valid for complex matrices also.

REFERENCES

[1] R. V. Churchill, et al., Complex Variables and Applications, 3rd ed. New York: McGraw-Hill, 1974.

[2] B. D. O. Anderson and S. Vongpanitlerd, Network Analysis and Synthesis: A Modern Systems Theory Approach. Englewood Cliffs, NJ: Prentice-Hall, 1973.

[3] W. Rudin, Real and Complex Analysis, 2nd ed. New York: McGraw-Hill, 1974.

[4] R. A. Rohrer, "Lumped Network Passivity Criteria," IEEE Trans. Circuit Theory, vol. CT-15, pp. 24-30, March 1968.

[5] J. C. Willems, "Dissipative Dynamical Systems--Part I: General Theory," Arch. Ration. Mech. Anal., vol. 45, no. 5, pp. 321-351, 1972.

____________, "Dissipative Dynamical Systems--Part II: Linear Systems with Quadratic Supply Rates," ibid., pp. 352-393.

[6] L. O. Chua and P.-M. Lin, Computer-Aided Analysis of Electronic Circuits: Algorithms and Computational Techniques, Englewood Cliffs, NJ: Prentice-Hall, 1975.

[7] N. Balabanian and T. A. Bickart, Electrical Network Theory. New York: John Wiley, 1969.

[8] C. A. Desoer, Notes for a Second Course on Linear Systems. New York: D. Van Nostrand, 1970.