MULTIUSER SUCCESSIVE REFINEMENT

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Multiuser Successive Refinement

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Abstract

In this work we propose a concept of successive refinement of information for multiple users. We give the achievable rate-distortion regions for the Gaussian case. The performance of the proposed scheme is shown to be superior to the naive time-sharing scheme using a point-to-point transmission approach. Our approach appears to be the source-coding dual to Cover's celebrated work on Broadcast channels [1].

1 Introduction

With the exponential growth in communication bandwidth and computing speed, more users are getting networked than ever before for seamless transmission of information between each other. In such scenarios, the management of information storage and transmission has to incorporate interaction of information signals of multiple users. We envisage one such situation in this report. Consider an encoder wishing to transmit an information source X to two users as shown in Fig. 1. The two receivers are connected to the encoder through errorless channels of unequal capacities. We wish to have a successively refinable transmission of information to these two receivers. We assume the broadcasting of information such that only a part of the information transmitted to the user with higher capacity is received by the user with lower capacity. This is a generalization of the successive refinement of information for a single user. Since the capacities of the channels are unequal, the refinement of the users must also be unequal and proportional to the capacities of the individual links.

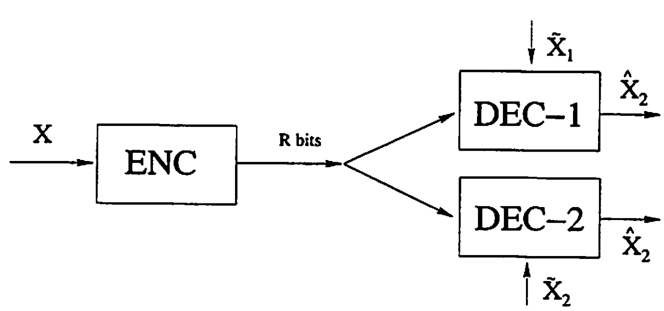

Figure 1: Multiuser successive refinement: encoder sends an information sequence at rate R bits/sample. This information is intended for both the decoders. The DEC_i has access to $\tilde{X}_i$, an optimal rate-distortion encoded version of X.

For simplicity assume the source X to be *i.i.d.* Gaussian with zero mean and unit variance. Let the capacities of the links to the decoders are αR and R bits/sample, where $0 \leq \alpha \leq 1$. In the beginning of the sequence of transmissions, the encoder sends conventional successively refinable source bits. Thus at the end of first transmission, the DEC_i (decoder-i, $i = 1, 2$) has a rate-distortion version $\tilde{X}_i$ of X such that $E(X - \tilde{X}_i)^2 \leq D_i$. As an illustration let us suppose that $D_1 = 2^{-2}$ and $D_2 = 2^{-4}$. This means that DEC_1 has 1 bit and DEC_2 has 2 bits worth of information about X (for the present case, we assume $\alpha = 0.5$). In the second transmission, the encoder wishes to upgrade the representation $\tilde{X}_i$ of the DEC_i in such a way that the user with a link of larger capacity is refined better, i.e. with better reconstruction fidelity, than the one with a link of lower capacity. Let us consider the problem from geometry as well as information theory.

A naive solution to this problem is obtained using time-sharing. In the first mode of transmission, the encoder sends information at the rate of R bits/sample intended only for user-2. Since the capacity of user-1 is less than R bits/sample, this information cannot be reliably transmitted to user-1. In the second mode of operation, the encoder sends information at a rate of αR bits/sample, which can be reliably transmitted to both the decoders. Using this approach, user-2 receives information at the lower capacity which is a waste of system resources. The encoder can use time-sharing between these two modes of transmission to choose any operating point in between.

In this report, we provide another solution to this problem which performs better. In a general case, ENC (the encoder) sends an information sequence at rate R bits/source sample such that this information is useful to both the decoders (a part of the information is meant to both the receivers and the rest is meant for the receiver with the larger capacity). *This formulation is a source coding counterpart of the concept of superposition of information considered for channel coding for multiple users by [1].* The formulation that we consider in this report is a unique interplay between the concepts of multiresolution source coding and source coding with side information. We pose the problem of multiuser successive refinement as a problem of source coding with multiple side information.

In this paper we study the asymptotic behaviour of the rate-distortion functions of two users. Let us consider the correlation between the source X and its representation $\tilde{X}_i$. Since the representation $\tilde{X}_i$ is obtained using an optimal rate-distortion encoding, we use the following test channel for the correlation. The test channel for the rate-distortion [2] function for the Gaussian case is shown in Fig. 2.

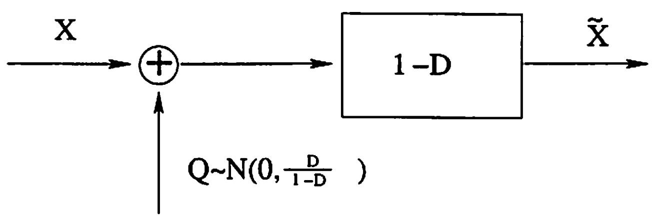

Figure 2: Backward test channel for the Gaussian source: the source is *i.i.d.* with variance 1 and is encoded such that the decoded fidelity is at D. The random variables X and Q are independent.

Let us define Y_i for $i = 1, 2$ as follows:

$$Y_i \triangleq \frac{1}{1 - D_i}\tilde{X}_i = X + Q_i \tag{1}$$

where Q_i is *i.i.d.* Gaussian with zero mean and variance σ_{qi}^2. Thus for the present example, $\sigma_{q1}^2 = 1/3$ and $\sigma_{q2}^2 = 1/15$. Y_i can be considered as side information correlated to the source X, available at the decoder. Since X is *i.i.d.* Gaussian, it is successively refinable [3].

Basic Idea: A multiresolution source codebook is partitioned into cosets of multiresolution channel codebook in two different ways suited for the two users.

Let $\theta, \beta \in \mathbb{R}$ be 2 parameters such that $1 > \theta, \beta > 0$ and $\theta \geq \beta$ (the choice of these parameters is governed by R and α). The code construction is described as follows: build successively refinable random-codes on the surface of L-dimensional hyper-spheres of radius $\sqrt{1-\theta^2}$ and $\sqrt{1-\theta^2+\beta^2}$ (the encoding and decoding are done in blocks of length L). Let these codes be denoted by $\mathbb{C}_1$ and $\mathbb{C}_2$ respectively.

$$2^{LR_{s1}} \triangleq \text{ the number of codewords in } \mathbb{C}_1 \triangleq \text{source-code-layer-1}$$

$$2^{LR_{s2}} \triangleq \text{ the number of codewords in } \mathbb{C}_2 \triangleq \text{source-code-layer-2}$$

where $R_{s1} \approx \frac{1}{2} log \frac{1}{\theta^2}$ and $R_{s2} \approx \frac{1}{2} log \frac{1}{\theta^2 - \beta^2}$. These values are obtained using the conventional single user rate-distortion functions. These codebooks are obtained by randomly distributing the given number of codewords uniformly on the surface of the given hyper-sphere. This is shown in Fig. 3.

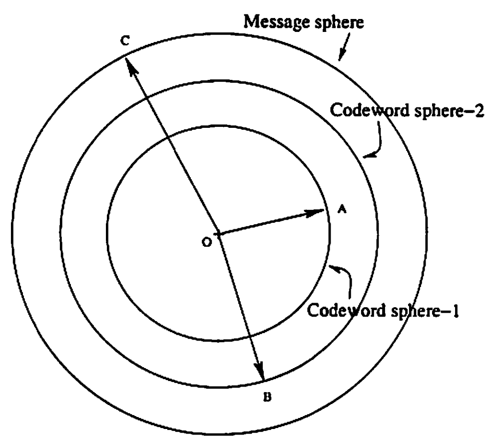

Figure 3: Codeword spheres: $||OA|| = \sqrt{1-\theta^2}$, $||OB|| = \sqrt{1-\theta^2+\beta^2}$ and $||OC|| = 1$. Codewords belonging to $\mathbb{C}_1$ are on the sphere with radius $\sqrt{1-\theta^2}$ and those belonging to $\mathbb{C}_2$ are on the sphere with radius $\sqrt{1-\theta^2+\beta^2}$.

Since X is successively refinable, we have the following arguments: given an outcome $\mathbf{x}$ of $\mathbf{X}$, encoder finds a codeword[1] $\mathbf{w_1} \in \mathbb{C}_1$ such that $d(\mathbf{x}, \mathbf{w_1}) \approx \theta$ and a codeword $\mathbf{w_2} \in \mathbb{C}_2$ such that $d(\mathbf{x}, \mathbf{w_2}) \approx \sqrt{\theta^2 - \beta^2}$. Before going to the details of encoding, we consider the structure of the optimal successively refinable codebooks. Let W_1 and W_2 denote the random variables associated with the codewords $\mathbf{w_1}, \mathbf{w_2}$ respectively, obtained by the encoder. We use the reverse test channel for the rate-distortion function of Gaussian sources as shown in Fig. 4. $\Phi_1 \sim N(0, \beta^2)$ and $\Phi_2 \sim N(0, \theta^2 - \beta^2)$ and Φ_1 and Φ_2 are independent. DEC_i has access to Y_i. Similar to the Wyner-Ziv theorem [4] for the Gaussian case, we partition source-code-layer-1 and source-code-layer-2 into cosets.

[1] This is to characterize the typicality in high-dimensional random coding.

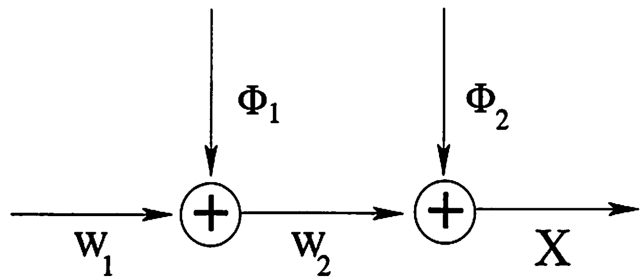

Figure 4: Forward test channel for the successive refinement of Gaussian sources. $W_1 \sim N(0, 1-\theta^2)$ $\Phi_1 \sim N(0, \beta^2)$ and $\Phi_2 \sim N(0, \theta^2 - \beta^2)$.

Note: For every codeword in $\mathbb{C}_1$, we can associate $2^{L(R_{s2}-R_{s1})}$ codewords in $\mathbb{C}_2$ (successive refinement). Thus the encoder first chooses a codeword $\mathbf{w_1} \in \mathbb{C}_1$ such that $d(\mathbf{x}, \mathbf{w_1}) \approx \theta$ and then it chooses a codeword $\mathbf{w_2} \in \mathbb{C}_2$ such that $\mathbf{w_2}$ is associated with $\mathbf{w_1}$ and $d(\mathbf{x}, \mathbf{w_2}) \approx \sqrt{\theta^2 - \beta^2}$. It can also be shown that with high probability $d(\mathbf{w_1}, \mathbf{w_2}) \approx \beta$.

We choose a subset of $\mathbb{C}_1$ where the codewords are drawn uniformly from $\mathbb{C}_1$. $\mathbb{C}_1$ is partitioned into cosets of this subset.

$$2^{LR_{c1}} \triangleq \begin{array}{c}\text{the number of codewords}\\ \text{in a coset in } \mathbb{C}_1\end{array} \triangleq \text{layer-1-coset}$$

and $R_{c1} \approx \frac{1}{2} log \frac{1+\sigma_{q1}^2}{\theta^2+\sigma_{q1}^2}$ = capacity of the channel between Y_1 and W_1. $R_1 \triangleq R_{s1} - R_{c1} = \frac{1}{2} log \frac{\theta^2+\sigma_{q1}^2}{\theta^2(1+\sigma_{q1}^2)}$. The common information meant for both the decoders is sent with rate R_1 bits/sample. The encoder sends the index of the coset containing the quantized codeword $\mathbf{w_1}$.

DEC_1: Finds $\mathbf{w_1}$ in the given coset such that $d(\mathbf{w_1}, \mathbf{y_1}) \approx \sqrt{\theta^2 + \sigma_{q1}^2}$ where $\mathbf{y_1}$ is the observed outcome of L-block of Y_1. Since $\sigma_{q1}^2 > \sigma_{q2}^2$, DEC_2 too can decode $\mathbf{y_2}$ in the same coset and obtain $\mathbf{w_1}$. DEC_1 obtains $\hat{\mathbf{x}}_1 \triangleq$ linear combination of $\mathbf{w_1}$ and $\mathbf{y_1}$ as the best representation of $\mathbf{x}$. The final distortion of DEC_1 is given by:

$$d_1 = \frac{\theta^2 \sigma_{q1}^2}{\theta^2 + \sigma_{q1}^2} \tag{2}$$

Now we focus on the channel between W_2 and Y_2 given W_1 is available (or the codeword $\mathbf{w_1}$ is decoded correctly). Since X is successively refinable, $\mathbf{w_2}$ can be thought of as a vector such that $d(\mathbf{w_1}, \mathbf{w_2}) \approx \beta$ and $d(\mathbf{w_2}, \mathbf{x}) \approx \sqrt{\theta^2 - \beta^2}$. Now the uncertainty at DEC_2 is one in $2^{L(R_{s2}-R_{s1})}$. At this stage, the DEC_2 has access to $\mathbf{w_1}$ and the set of codewords (with $2^{L(R_{s2}-R_{s1})}$ in number) that are associated with $\mathbf{w_1}$. The codeword $\mathbf{w_2}$ used by ENC cannot be reliably recovered by using $\mathbf{y_2}$ alone. ENC sends further information to DEC_2 at rate R_2 bits/sample so that DEC_2 is able to recover $\mathbf{w_2}$. In other words, the ENC partitions this set of $2^{L(R_{s2}-R_{s1})}$ codewords into cosets and sends the index of the coset containing the quantized codeword $\mathbf{w_2}$.

We define a layer-1-coset of $\mathbb{C}_2$ as the set of all codewords $\mathbf{w_1} \in \mathbb{C}_2$ such that they are associated with the codewords of layer-1-coset of $\mathbb{C}_1$. Thus a layer-1-coset of $\mathbb{C}_2$ contains $2^{L(R_{c1}+R_{s2}-R_{s1})}$ codewords. We partition each layer-1-coset of $\mathbb{C}_2$ into layer-2-cosets. We need the following definition.

$$R_{ca} \approx \frac{1}{2} log \frac{\theta^2 + \sigma_{q2}^2}{\theta^2 - \beta^2 + \sigma_{q2}^2} = \frac{1}{2} log \left[1 + \frac{\beta^2}{\theta^2 - \beta^2 + \sigma_{q2}^2} \right] \tag{3}$$

R_{ca} is also nearly equal to the capacity of the test channel between W_2 and Y_2 given W_1. Each layer-1-coset of $\mathbb{C}_2$ is partitioned into layer-2-cosets each containing $2^{L(R_{ca}+R_{c1})} \triangleq 2^{LR_{c2}}$ codewords[2]. The encoder sends the coset index containing the quantized codeword $\mathbf{w}_2$ using R_2 bits/sample as defined below:

$$R_2 \triangleq R_{s2} - R_{s1} - R_{ca} = (R_{s2} - R_{c2}) - (R_{s1} - R_{c1}) \tag{4}$$

Thus the encoder sends the information at a total rate of $R = R_1 + R_2$ bits/sample. R_1 is useful to both DEC_1 and DEC_2 and R_2 is useful only for DEC_2.

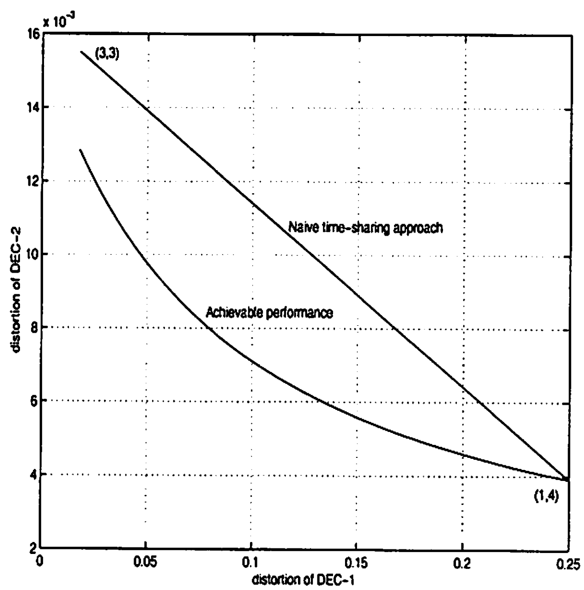

Figure 5: The achievable distortion pairs (d_1, d_2) with the rate of transmission $R = 2$ bits/sample. The performance of the naive time-sharing approach is also shown. The intial distortion points of the decoders are at $D_1 = 2^{-2}$ and $D_2 = 2^{-4}$. The final distortion point $(2^{-6}, 1/76)$ can be achieved with ENC sending the same 2 bits/sample of information to both DEC_1 and DEC_2. The conventional approach gives the final distortion point $(2^{-6}, 2^{-6})$.

DEC_2: It has access to $\mathbf{w}_1$. In a given layer-2-coset, for a given $\mathbf{w}_1$, there exists $2^{LR_{ca}}$ codewords on $\mathbb{C}_2$ which are associated with $\mathbf{w}_1$. Thus the uncertainty is one in $2^{LR_{ca}}$. DEC_2 finds $\mathbf{w}_2$ such that $d(\mathbf{w}_2, \mathbf{y}_2) \approx \sqrt{\theta^2 - \beta^2 + \sigma_{q2}^2}$ and finds $\hat{\mathbf{x}}_2 \triangleq$ linear combination of $\mathbf{y}_2$ and $\mathbf{w}_2$ as the best representation of $\mathbf{x}$. $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ are given by:

$$\hat{\mathbf{x}}_1 = \frac{\sigma_{q1}^2}{\theta^2 + \sigma_{q1}^2} \mathbf{w}_1 + \frac{\theta^2}{\theta^2 + \sigma_{q1}^2} \mathbf{y}_1 \tag{5}$$

[2]Each set of codewords in $\mathbb{C}_2$ that are associated with the given $\mathbf{w}_1$ is partitioned into cosets each containing $2^{LR_{ca}}$ codewords. Since a layer-1-coset of $\mathbb{C}_2$ contains $2^{R_{c1}}$ codewords, there are a total of $2^{LR_{c2}}$ in a layer-2-coset in $\mathbb{C}_2$

$$\hat{\mathbf{x}}_2 = \frac{\sigma_{q2}^2}{\theta^2 - \beta^2 + \sigma_{q2}^2}\mathbf{w}_2 + \frac{\theta^2 - \beta^2}{\theta^2 - \beta^2 + \sigma_{q2}^2}\mathbf{y}_2 \tag{6}$$

The final distortion of DEC_2 is given by:

$$d_2 = \frac{(\theta^2 - \beta^2)\sigma_{q2}^2}{\theta^2 - \beta^2 + \sigma_{q2}^2} \tag{7}$$

and R_2 is given by

$$R_2 \approx \frac{1}{2} log \frac{(\theta^2 + \sigma_{q1}^2)(\theta^2 - \beta^2 + \sigma_{q2}^2)}{(1 + \sigma_{q1}^2)(\theta^2 - \beta^2)(\theta^2 + \sigma_{q2}^2)} \tag{8}$$

We have:

$$R_1 = \alpha R \text{ and } R_2 = (1 - \alpha)R, \quad 0 \le \alpha \le 1 \tag{9}$$

Let us consider the following example:

Example: Let $R = 2$ bits/sample.

Case 1: $\alpha = 0$. This implies that $\theta = 1$ and $\beta = \sqrt{\frac{240}{241}}$. We have $d_1 = 2^{-2}$ and $d_2 = 2^{-8}$.

Case 2: $\alpha = 1$. This implies that $\theta = \sqrt{\frac{1}{61}}$ and $\beta = 0$. We have $d_1 = 2^{-6}$ and $d_2 = \frac{1}{76}$.

Note: It can be noted that for this case, eventhough ENC is sending information at $R = 2$ bits/sample to upgrade the distortion level of DEC_1 from 2^{-2} to 2^{-6} (which is the minimum rate required for this refinement), DEC_2 upgrades using the same information, its version $\tilde{X}_2$ of X to $\hat{X}_2$ with distortion level from 2^{-4} to $\frac{1}{76}$.

Case 3: For any given α, $d_1 = \frac{1}{4.2^{4\alpha}}$ and $d_2 = \frac{2^{4\alpha}}{192+64.2^{4\alpha}}$ implying that $d_2 = \frac{1}{64+768d_1}$. Using the naive time-sharing argument we can achieve $d_2 = \frac{21-64d_1}{1280}$. The comparison of the performances under two schemes are shown in Fig. 5.

Conjecture: Information theoretically we conjecture the following achievable rate-region: $Y \longrightarrow X \longrightarrow W_1, W_2$.

$$R_1 \ge I(X; W_1) - I(Y_1, W_1) \tag{10}$$

$$R_2 \ge I(X; W_1, W_2) - I(X; W_1) - I(Y_2, W_2|W_1) \tag{11}$$

such that $E[d(g_1(W_1, Y_1), X)] \le d_1$ and $E[d(g_2(W_1, W_2, Y_2), X)] \le d_2$, where the functions g_1 and g_2 give the reconstruction values. We have

$$R \ge I(X; W_1, W_2) - I(Y_2; W_2|W_1) - I(Y_1; W_1) \tag{12}$$

(R, d_1, d_2) as defined above is achievable.

2 Conclusions

In this report, we introduced the concept of multiuser successive refinement. For the Gaussian case, it is shown that the achievable performance is better than that of the time-sharing approach. The future work includes information theoretic proof of the achievable rate-distortion region and practical constructions of quantizers for the problem.

References

[1] T. Cover, "Broadcast channels," *IEEE Trans. on Inform. Theory*, vol. IT-18, pp. 2–14, Jan. 1972.

[2] T. M. Cover and J. A. Thomas, *Elements of information theory.* New York: Wiley, 1991.

[3] W. H. Equitz and T. M. Cover, "Successive refinement of information," *IEEE Trans. on Inform. Theory*, vol. 37, pp. 269–275, Mar. 1991.

[4] A. D. Wyner, "The rate-distortion function for source coding with side information at the decoder-II: General sources," *Inform. Contr.*, vol. 38, pp. 60–80, 1978.