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AN APPROACH TO THE SYSTEM IDENTIFICATION PROBLEM

by

D. J. Sakrison

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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An Approach to the System Identification Problem[†]

D. J. Sakrison^{*}

SUMMARY

The identification or modeling of a given plant or system seems to be of current interest in regard to control problems. In particular, attention is often focused upon the case in which the given system is assumed to be linear and time invariant with a rational transfer function whose order is known not to exceed some number n . In this case it is desirable to estimate the poles and zeros of the transfer function, or, alternatively, the coefficients of the numerator and denominator polynomials. Here we describe a method based on certain results in stochastic approximation and optimum filter theory. This method is computationally simple, ^{and} has a rate of convergence inversely proportional to the observation time. The method requires a knowledge of the correlation properties of the observation noise.

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I. INTRODUCTION

The identification or modeling of a given plant or system seems to be of current interest in regard to control problems. In particular, attention is often focused upon the case in which the given system is assumed to be linear and time-invariant with a rational transfer function whose order is known not to exceed some number n . In this case it is desired to estimate the poles and zeros of the transfer function, or alternatively, the coefficients of the numerator and denominator polynomials. One way of achieving this is through the use of an estimation method referred to as the instrumental variable method [1, 2]. In this paper we described an alternate method based on certain results in stochastic approximation and optimum filter theory. Our method is computationally simpler than the instrumental variable method, while also having a rate of convergence inversely proportional to the observation time, in common with the instrumental variable method [2]. Our method requires a knowledge of the correlation properties of the observation noise, while the instrumental variable method does not.

II. DESCRIPTION OF THE METHOD

We consider the situation shown in Fig. 1. The signal $U(t)$ is the input to the system, and the signals $N_1(t)$ and $N_2(t)$ represent measurement noise in the observations of the input and output signals,

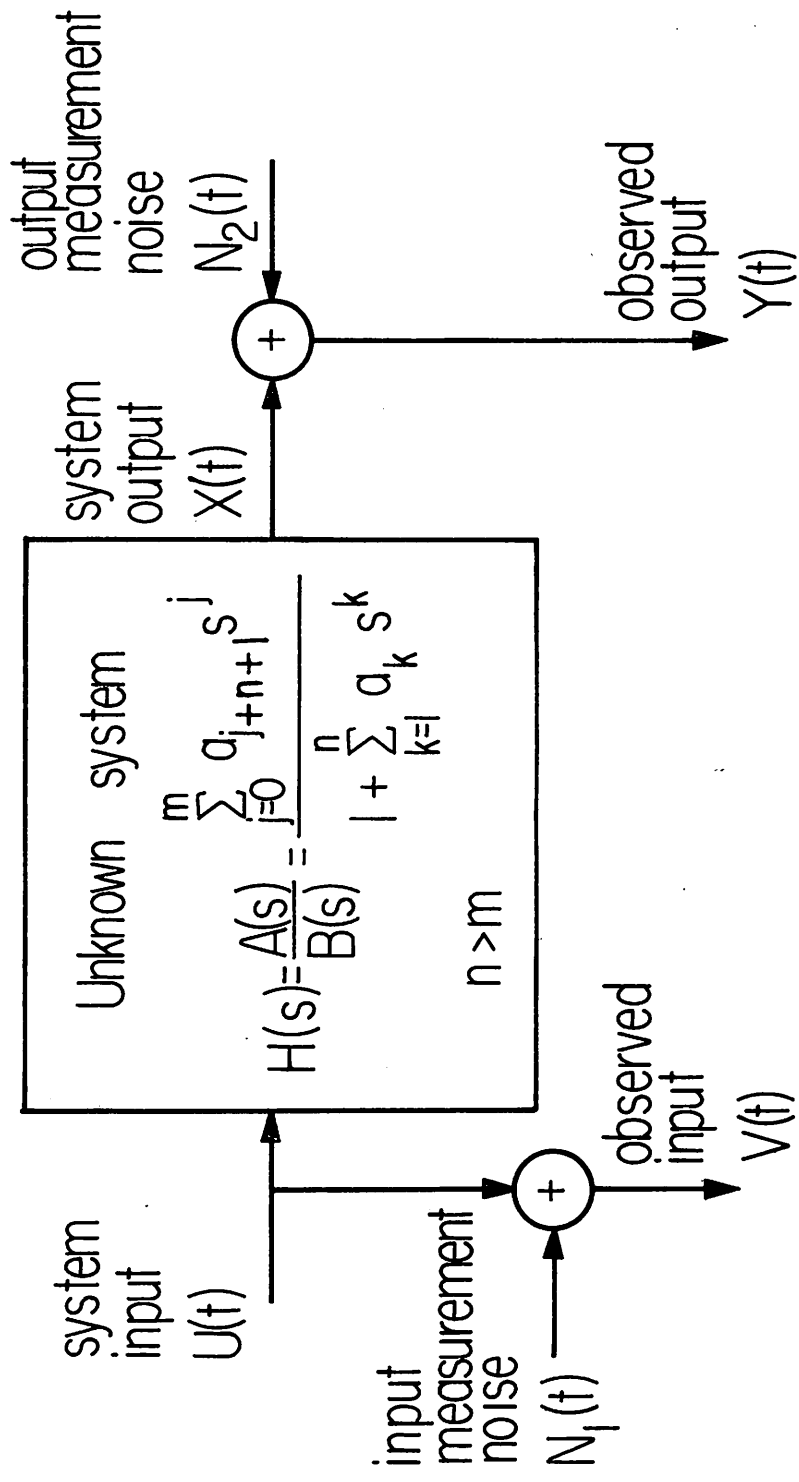


Fig. 1. Identification problem considered.

respectively. We assume that these three signals are mutually uncorrelated, stationary, random processes and that $N_1(t)$ and $N_2(t)$ are zero mean. Our method requires that the correlation functions of the processes $N_1(t)$ and $N_2(t)$ are known: we denote these two correlation functions by $\phi_{n_1}(\tau)$ and $\phi_{n_2}(\tau)$, respectively. In many cases, these correlation functions can be determined from a model of the noise based on a knowledge of its physical origin. In fact, the measurement noise may often be considered "white" compared to the spectrum of the process $U(t)$. If these correlation functions cannot be determined from a knowledge of the origin of the noise, they can be estimated from observations made in the absence of the system input. The transfer function of the unknown situation is assumed to be of the form

$$H(s) = \frac{A(s)}{B(s)} = \frac{\sum_{j=0}^m a_{n+1+j} s^j}{1 + \sum_{k=1}^n a_k s^k} \quad (1)$$

Before the identification procedure itself is discussed, it is useful to consider the configuration to be used in the identification procedure and to derive some relations that exist among the processes involved. This will provide motivation for the method to be used. The configuration used in the identification procedure was suggested to

the author by Professor O. J. M. Smith; results on the convergence of the procedure follow from some earlier work of the present author on filter design [3-5]. The identification procedure considered here is similar to the procedure considered by Ho and Lee [6] for the time-discrete case, but their work contains an oversight. The proof of convergence for their method requires that the $w_1(t)$ sequence in their appendix be independent for any time shift equal to an integral number of basic sample times. Thus, the proof of convergence applies to their identification procedure only if the same restriction applies to their $\omega(t)$ sequence in section II (contrary to their Eq. (2)). As can be seen from their Eq. (23), this greatly limits the system configurations to which their method applies. Professor Ho has informed the present author that the method will still work when the system has no numerator dynamics.

The configuration used in the identification procedure is shown in Fig. 2. The transfer functions $G_1(s)$ and $G_2(s)$ are given respectively by

$$G_1(s) = \sum_{j=0}^m \alpha_{j+n+1} \frac{s^j}{Q(s)} ; \quad G_2(s) = \frac{1}{Q(s)} + \sum_{k=1}^n \alpha_k \frac{s^k}{Q(s)} , \quad (2)$$

in which $Q(s)$ is chosen only to guarantee that the response of $s^n/Q(s)$ to either $N_1(t)$ or $N_2(t)$ has a finite mean-square value. The quantities α_j , $j = 1, \dots, n+m+1$, simply denote generic values that the

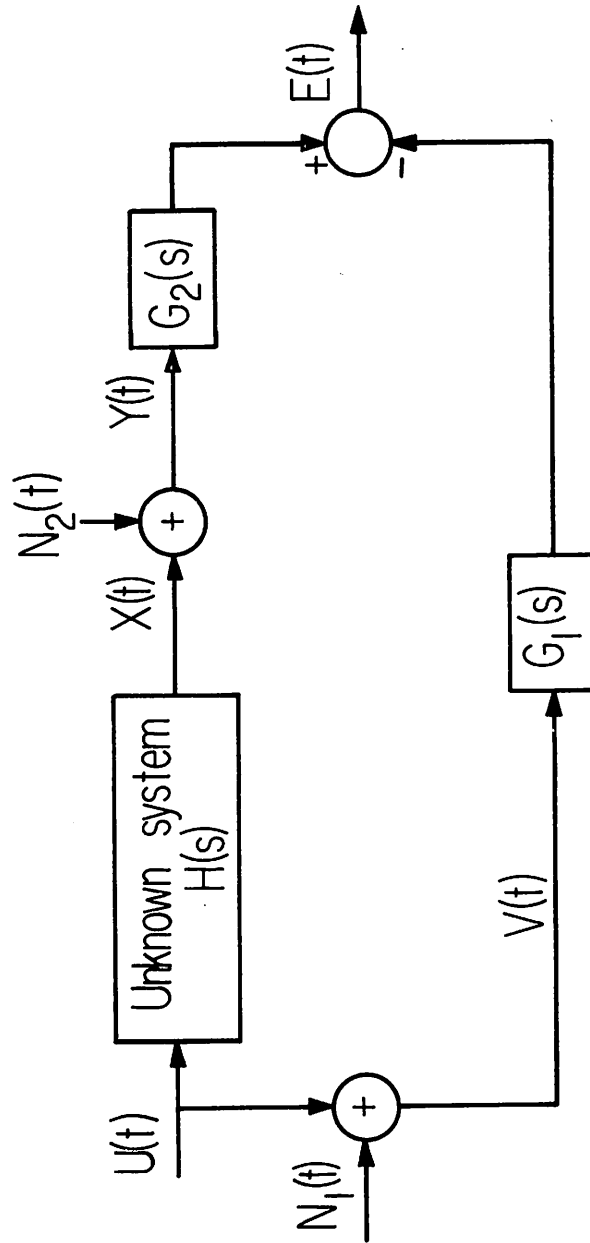


Fig. 2. Configuration used in identification procedure.

parameters in $G_1(s)$ and $G_2(s)$ may take. We denote by \hat{a}_j the estimate of a_j that we will generate. Thus, a_j denotes simply a variable nonrandom parameter while \hat{a}_j is a random variable. We denote the $m+n+1$ tuple of parameters $(a_1, a_2, \dots, a_{n+m+1})$ by the $n+m+1$ dimensional vector \underline{a} , and the $n+m+1$ tuple of estimates $(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_{n+m+1})$ by the vector random variable $\hat{\underline{a}}$. It is convenient to define the following processes:

$$\begin{aligned}
 Q_u(t) &= \text{response of } 1/Q(s) \text{ to the process } U(t) , \\
 Q_x(t) &= \text{response of } 1/Q(s) \text{ to the process } X(t) , \\
 Z(t) &= \text{response of } 1/Q(s) B(s) \text{ to the process } U(t) , \\
 P_1(t) &= \text{response of } 1/Q(s) \text{ to the process } N_1(t) , \text{ and} \\
 P_2(t) &= \text{response of } 1/Q(s) \text{ to the process } N_2(t) .
 \end{aligned}$$

Let us now consider the mean-square value of the "error" signal $E(t)$ defined in Fig. 2 when the parameters in G_1 and G_2 have the non-random value \underline{a} . By simple manipulations it can be shown that this quantity may be expressed in terms of the above processes as

$$E\{E_t^2\} = m_1(\underline{a}) + m_2(\underline{a}) + m_3(\underline{a}) \tag{3}$$

in which the functions m_1 , m_2 , and m_3 are given by

$$\begin{aligned}
m_{1\sim}(\alpha) &= E \left\{ \left[\sum_{j=0}^m \alpha_{j+n+1} \frac{d^j}{dt^j} P_1(t) \right]^2 \right\} \\
&= \sum_{j=0}^m \sum_{k=0}^m \alpha_{j+n+1} \alpha_{k+n+1} (-1)^k \frac{d^{j+k}}{d\tau^{j+k}} \phi_{P_1}(\tau) \Big|_{\tau=0}, \quad (4)
\end{aligned}$$

$$\begin{aligned}
m_{2\sim}(\alpha) &= E \left\{ \left[P_2(t) + \sum_{k=1}^n \alpha_k \frac{d^k}{dt^k} P_2(t) \right]^2 \right\} \\
&= \phi_{P_2}(0) + \sum_{k=1}^n \alpha_k (-1)^k \frac{d^k}{d\tau^k} \phi_{P_2}(\tau) \Big|_{\tau=0} \\
&\quad + \sum_{j=1}^n \sum_{k=1}^n \alpha_j \alpha_k (-1)^k \frac{d^{j+k}}{d\tau^{j+k}} \phi_{P_2}(\tau) \Big|_{\tau=0}, \quad (5)
\end{aligned}$$

and

$$\begin{aligned}
m_{3\sim}(\alpha) &= E \left\{ \left[Q_x(t) + \sum_{k=1}^n \alpha_k \frac{d^k}{dt^k} Q_x(t) - \sum_{j=0}^m \alpha_{j+n+1} \frac{d^j}{dt^j} Q_u(t) \right]^2 \right\} \\
&= E \left\{ \left[\sum_{j=0}^m (a_{j+n+1} - \alpha_{j+n+1}) \frac{d^j Z(t)}{dt^j} \right. \right. \\
&\quad \left. \left. + \sum_{j=0}^m \sum_{k=1}^n (\alpha_k a_{j+n+1} - a_k \alpha_{j+n+1}) \frac{d^{j+k} Z(t)}{dt^{j+k}} \right]^2 \right\}. \quad (6)
\end{aligned}$$

Let us now make some observations concerning these functions. First, note that $m_1(\underline{\alpha})$, $m_2(\underline{\alpha})$, and $m_3(\underline{\alpha})$ are all nonnegative quadratic functions of $\underline{\alpha}$ and from Eq. (6) we see that

$$m_3(\underline{\alpha}) = 0 \quad \text{for} \quad \underline{\alpha} = \underline{a} \quad (7)$$

in which \underline{a} is the $m+n+1$ tuple of true parameter values, $(a_1, a_2, \dots, a_{m+n+1})$. Shortly we state a condition guaranteeing that $m_3(\underline{\alpha})$ is (strictly) positive definite. In this case, it is possible to show from previous work on filter optimization [3] that $m_3(\underline{\alpha})$ is unimodal and that its gradient satisfies

$$k_0 \|\underline{\alpha} - \underline{a}\|^2 \leq (\text{grad } m_3(\underline{\alpha}), \underline{\alpha} - \underline{a}) \leq k'_0 \|\underline{\alpha} - \underline{a}\|^2 \quad \text{for some } k_0 \text{ and } k'_0$$

$$0 < k_0 \leq k'_0 < \infty \quad (8)$$

By $\text{grad } m_3(\underline{\alpha})$ we mean the $m+n+1$ dimensional vector whose components are the partial derivatives of m_3 with respect to the components of $\underline{\alpha}$. The notation $\|\cdot\|$ and (\cdot, \cdot) stands, respectively, for the Euclidean norm and inner product. Equation (7) and inequality (8) suggest that a gradient-seeking procedure could be used to find the value $\underline{\alpha} = \underline{a}$ if the function $m_3(\underline{\alpha})$ could be observed. Although we cannot observe the function

$$m_3(\underline{\alpha}) = E \{ E_t^2 \} - m_1(\underline{\alpha}) - m_2(\underline{\alpha}) , \quad (9)$$

we can observe E_t^2 , and $m_2(\underline{\alpha})$ and $m_3(\underline{\alpha})$ are known functions of $\underline{\alpha}$.
 Further, since E_t^2 is quadratic in $\underline{\alpha}$,

$$\text{grad } E \{ E_t^2 \} = E \{ \text{grad } E_t^2 \} . \quad (10)$$

The random process

$$\underline{Y}(t, \underline{\alpha}) = \text{grad} \left[E^2(t) \right] - \text{grad } m_1(\underline{\alpha}) - \text{grad } m_2(\underline{\alpha}) \quad (11)$$

can be easily generated since $\text{grad } E^2(t)$ can be formed in a simple manner from $E(t)$ and the signals $P_1(t)$, $Q_x(t)$, $P_2(t)$ and $Q_u(t)$, while $\text{grad } m_1(\underline{\alpha})$ and $\text{grad } m_2(\underline{\alpha})$ are known linear functions of $\underline{\alpha}$. Moreover, $\underline{Y}_t(\underline{\alpha})$ is a random variable whose mean value is $\text{grad } m_3(\underline{\alpha})$. Thus a stochastic approximation procedure (in particular the Robbins-Monro method) which properly averages out the random fluctuations can be used to estimate \underline{a} , the value of $\underline{\alpha}$ for which $\text{grad } m_3(\underline{\alpha}) = 0$.

In particular, one could pick $\hat{\underline{\alpha}}(0)$ arbitrarily and generate the estimate $\hat{\underline{\alpha}}(t)$ by the differential equation

$$\frac{d\hat{\underline{\alpha}}(t)}{dt} = -a(t) \underline{Y}[t, \hat{\underline{\alpha}}(t)] . \quad (12)$$

Equivalently, if $\hat{\alpha}(t)$ were to be generated by digital computation equipment, $\text{grad } E^2(t)$ could be sampled periodically (we take the sample time here to be unity for notational convenience) and $\hat{\alpha}(t)$ could be generated by the recursion equation

$$\hat{\alpha}(t+1) = \hat{\alpha}(t) - a_t Y[t+1, \hat{\alpha}(t+1)] \quad (13)$$

$$t = 0, 1, 2, \dots$$

In the continuous-time case we would choose the function $a(t)$ to satisfy

$$\int_0^{\infty} a^2(t) dt < \infty ; \quad \int_0^{\infty} a(t) dt = \infty , \quad (14)$$

and in the discrete-time case

$$\sum_{n=0}^{\infty} a_n^2 < \infty ; \quad \sum_{n=0}^{\infty} a_n = \infty . \quad (15)$$

To guarantee convergence of $\hat{\alpha}(t)$ to \underline{a} , we need to impose some regularity properties on the processes involved. Let the vector valued random process $\underline{S}(t)$ be defined as

$$\underline{S}(t) = \left(Q_u(t), \frac{d}{dt} Q_u(t), \dots, \frac{d^m}{dt^m} Q_u(t), Q_x(t), \dots, \frac{d^n}{dt^n} Q_x(t), \right. \\ \left. P_1(t), \dots, \frac{d^m}{dt^m} P_1(t), P_2(t), \dots, \frac{d^n}{dt^n} P_2(t) \right) \quad (16)$$

We then assume the following conditions on $\underline{Z}(t)$:

1. Each component of $\underline{Z}(t)$ has variance less than or equal to some value σ^2 , $\sigma < \infty$.
2. Consider predicting $\underline{S}(t + \tau)$ by any functional on the past of this process, $\underline{S}(s)$, $s \leq t$. Let $\hat{\underline{S}}(t + \tau)$ denote the minimum mean-square error prediction. Then the minimum mean-square prediction error, $E\left\{\|\underline{S}_{t+\tau} - \hat{\underline{S}}_{t+\tau}\|^2\right\}$, must approach its asymptotic value at least as fast as $1/\tau^2$ for large values of the prediction time τ . This rate of decrease is hardly a stringent requirement: if for example, $U(t)$, $N_1(t)$, and $N_2(t)$ were Gaussian processes with rational spectra, then this prediction error would decrease exponentially with τ .
3. The set of random variables

$$Q_x(t), \frac{d}{dt} Q_x(t), \dots, \frac{d^n}{dt^n} Q_x(t), Q_u(t), \frac{d}{dt} Q_u(t), \dots, \frac{d^m}{dt^m} Q_u(t)$$

are a linearly independent set, i. e., the correlation coefficient between any one of these random variables and any linear combination of the rest is unequal to ± 1 . Further, we assume that each of these random variables has nonzero variance. This will guarantee that $m_3(\underline{g})$ is (strictly) positive definite.

The above three assumptions, together with Eqs. (8) and (14) (or (15) for the discrete-time case) guarantee that $\hat{\alpha}(t)$ converges to \hat{a} in the mean-square sense,

$$\lim_{t \rightarrow \infty} E \left\{ \left\| \hat{\alpha}_t - \hat{a} \right\|^2 \right\} = 0 . \quad (17)$$

In fact, if we choose $a(t)$ to be

$$a(t) = A/(t + 1), \quad t > 0 \quad (18)$$

in which A is sufficiently large, then it can be shown that

$$E \left\{ \left\| \alpha_t - \underline{a} \right\|^2 \right\} \leq K/t, \quad K < \infty . \quad (19)$$

Proofs of these two statements for the continuous-time procedure for Eq. (13) have been presented [4] in a slightly different but equivalent formulation. Relevant material is also presented in references [3], [5], and [6].

The computational features of the method should be pointed out. The differential equation described by Eq. (13) is easily implemented with analog computation equipment, while the recursion relation of Eq. (14) can be easily implemented with a modest digital computation

facility. The only difficulty in either case is generating $Y(t, \underline{\alpha})$. However, this poses no real problem since $\text{grad } m_1(\underline{\alpha})$ and $\text{grad } m_2(\underline{\alpha})$ are known, constant linear functions of $\underline{\alpha}$, while

$$\frac{\partial}{\partial \alpha_k} E^2(t) = E(t) \cdot \left[\frac{d^k}{dt^k} \left[P_2(t) + Q_x(t) \right] \right] \quad (20)$$

$k = 1, 2, \dots, n,$

and

$$\frac{\partial}{\partial a_{k+n+1}} E^2(t) = E(t) \left[\frac{d^k}{dt^k} \left[P_1(t) + Q_u(t) \right] \right] \quad (21)$$

$k = 0, 1, 2, \dots, m.$

The signal $E(t)$ is available using the configuration shown in Fig. 2 while the signals

$$\frac{d^j}{dt^j} \left[P_1(t) + Q_u(t) \right] \quad \text{and} \quad \frac{d^k}{dt^k} \left[P_2(t) + Q_x(t) \right]$$

are, respectively, the responses of $s^j/Q(s)$ to $V(t)$ and $s^k/Q(s)$ to $Y(t)$. A block diagram for generating $Y[t, \hat{\underline{\alpha}}(t)]$ and $\hat{\underline{\alpha}}(t)$ is shown in Fig. 3 for the continuous-time estimation algorithm of Eq. (12).

The double arrows (\Rightarrow) indicate vector-valued signals ($m+n+1$ tuples).

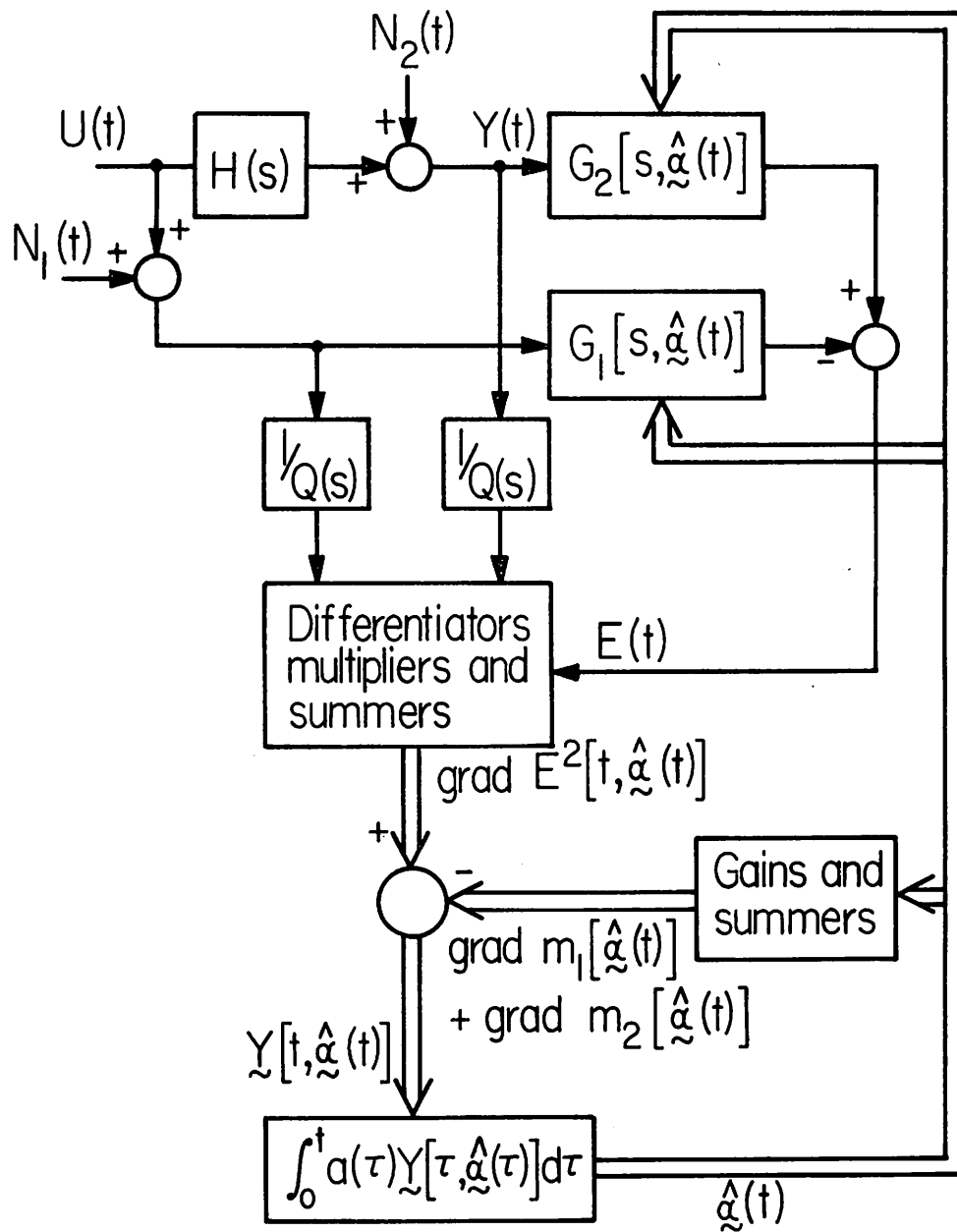


Fig. 3. Configuration used in continuous-time estimation procedure.

III. CONCLUSION

We have presented a computationally practical method for identifying a linear, time-invariant system with rational transfer function whose order is known to be less than some number n . Our procedure is predicated on the assumption that the system input and output are both observable in the presence of noise ^{processes} with known correlation functions. The estimate generated by our procedure converges to the true value in the mean-square sense at a rate inversely proportional to the observation time.

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