POLYNOMIAL BOUNDED AND (APPARENTLY) NON POLYNOMIAL BOUNDED MATROID COMPUTATIONS

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ABSTRACT

There are known polynomial bounded algorithms for computing optimal intersections of two matroids, but no such algorithms have been discovered for problems involving three or more matroids. In this paper we prove that the maximum weighted intersection of k ($k \geq 4$) matroids over n elements can be found by computing the maximum weighted intersection of three matroids over 2kn elements. In other words, there exists a polynomial bounded algorithm for the intersection of three matroids if and only if there exists a polynomial bounded algorithm for the intersection of an arbitrary number of matroids. The problem reduction yielding this result is presented in terms of the more general matroid "parity" problem, and the relationship between the intersection problem and the parity problem is explained.

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1. Introduction

Edmonds [2,3] has shown that matroid theory provides a natural and useful generalization and explication of the optimization methods subsumed under the heading of network flow theory. Thus, for example, the minimum-cost network flow problem can be reduced to the weighted bipartite matching problem, and the matching problem can be viewed as a problem calling for the computation of a maximum-weight intersection of two partition matroids.

There are polynomial bounded algorithms for computing optimal intersections of two matroids, provided there exists a polynomial bounded algorithm for testing independence of arbitrary sets of elements in each of the matroids. (We assume that the reader is familiar with the notion of a polynomial bounded computation. Cf. [4].) Thus, for example, it is possible to efficiently compute the intersection of two graphic matroids or a graphic matroid and a partition matroid. (This latter is the "optimal branchings" problem [1].)

However, there is no known polynomial bounded algorithm for computing optimal intersections of three or more matroids. This fact is sometimes cited in connection with the traveling salesman problem, which calls for the intersection of one graphic and two partition matroids.

In this paper we present a problem reduction which proves the following. The maximum weighted intersection of k ($k \geq 4$) matroids over n elements can be found by computing the maximum weighted intersection of three matroids over 2kn elements. In other words, there exists a polynomial bounded algorithm for the intersection of three matroids if and only if there exists a polynomial bounded algorithm for the intersection of an arbitrary number of matroids.

We shall present the problem reduction in terms of the more general matroid "parity" problem. The technique is essentially the same as that employed to reduce the "exact covering problem" to the three dimensional assignment problem, so as to place the latter on Karp's [4] list of "complete" problems.

We note that the three-dimensional assignment problem asks for a maximum-cardinally intersection of three partition matroids, and is therefore considerably more specialized than the problem of finding a maximum-weight intersection of an arbitrary number of arbitrarily-structured matroids, yet the three dimensional assignment problem is "complete" in Karp's sense, and thus apparently one for which no polynomial bounded algorithm exists. We can therefore hold out small hope that a polynomial bounded algorithm will ever be discovered for the optimal intersection of three or more matroids.

2. Definitions

A matroid $M = (E, \mathcal{I})$ is a combinatorial structure in which E is a finite set of elements and $\mathcal{I}$ is a nonempty family of subsets of E (called independent sets) satisfying the axioms:

(2.1) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(2.2) If I_p and I_{p+1} are sets in $\mathcal{I}$ containing respectively p and p+1 elements, then there exists an element $e \in I_{p+1} - I_p$ such that $I_p \cup \{e\} \in \mathcal{I}$.

As examples of matroids, consider the following.

(2.3) Matric Matroids

Let A be a finite matrix, whose elements belong to an arbitrary field. Let E be the set of columns of the matrix and let $\mathcal{I}$ contain all subsets of linearly independent columns. $M = (E, \mathcal{I})$ is the matroid of the matrix A.

(2.4) Graphic Matroids

Let G be a finite graph. Let E be the set of arcs of G and let $\mathcal{I}$ contain all subsets of arcs which do not contain a cycle (each such subset is a tree or a forest of trees). $M = (E, \mathcal{I})$ is the (graphic) matroid of the graph G.

(2.5) Partition Matroids

Let E be an arbitrary finite set, and let π be an arbitrary partition of E. Let $\mathcal{I}$ contain all subsets of E which contain no more than one element from any block of π. $M = (E, \mathcal{I})$ is a matroid.

Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be matroids over the same set of n elements. A set I that is independent in both matroids, i.e. $I \in \mathcal{I}_1 \cap \mathcal{I}_2$, is said to be an intersection of M_1 and M_2. In some cases, we seek to find an intersection which contains as many elements as possible; this is the cardinality intersection problem. In other cases the elements in E are weighted by real numbers and we seek to find an intersection for which the sum of the weights of the elements is as large as possible; this is the weighted intersection problem.

Now let $M = (E, \mathcal{I})$ be a given matroid and let π be a partition which pairs the elements. I.e. each block of π contains exactly two elements e and $\bar{e}$, we call e the mate of $\bar{e}$, and vice versa. A set $A \subseteq E$ is said to be a parity set if, for each element e, $e \in A$ if and only if $\bar{e} \in A$. The cardinality parity problem is to find a parity set which contains as many elements as possible. The weighted parity problem is to find a parity set for which the sum of the weights of the elements is maximum, for some given weighting.

We generalize these problems in the natural way to intersection problems with k matroids and to parity problems where each element has k-1 mates. We speak of these as k-matroid intersection problems and k-parity problems, respectively.

3. Reduction of the k-Matroid Intersection Problem to the k-Parity Problem

We now illustrate the reduction of the k-matroid intersection problem to the k-parity problem.

Let $M_1 = (E, \mathcal{J}_1)$, $M_2 = (E, \mathcal{J}_2)$, $\cdots$, $M_k = (E, \mathcal{J}_k)$ be the k matroids to be intersected. Let $E_1, E_2, \cdots, E_k$ be disjoint sets, where $E_i = \{e_1^{(i)}, e_2^{(i)}, \cdots, e_n^{(i)}\}$. Replace M_i by the isomorphic matroid $M_i' = (E_i, \mathcal{J}_i')$ over E_i, using the correspondence of e_j and $e_j^{(i)}$. Form the sum, $M' = (E', \mathcal{J}')$ of $M_1', M_2', \cdots, M_k'$, as follows:

$$E' = \bigcup_{i=1}^{k} E_i$$

$$\mathcal{I} = \{I_1' \cup I_2' \cup \cdots \cup I_k' \mid I_i' \in \mathcal{I}_i'\}.$$

It is a well-known fact of matroid theory that M' is a matroid. Moreover, it is possible to test for independence of a set in M' by means of a polynomial bounded algorithm if there is a polynomial-bounded algorithm for independence testing in each of the matroids M_i. (One can apply a two-matroid intersection algorithm [6] or a matroid partition algorithm [2] for this purpose.)

Define parity sets for M' in the obvious way. I.e. A is a parity set if for j = 1, 2, $\cdots$, n, either all k of the elements $e_j^{(1)}$, $e_j^{(2)}$, $\cdots$, $e_j^{(k)}$ belong to A, or none of them does. It is quite clear that there is a one-one correspondence between independent parity sets of M' and intersections of the k matroids $M_1, M_2, \cdots, M_k$.

In [7] the following problem reduction is illustrated. Let M = $(E, \mathcal{I})$ be a given matroid and let π be an arbitrary partition of its elements. An apparent generalization of the parity problem is obtained by asking for an independent set I which contains a maximum number of elements (or is of maximum weight), subject to the condition that I contains an <u>even</u> number of elements from each block of π. In fact, however, any problem of this type can be reduced to the 2-parity problem. That is, existing polynomial bounded computations for the 2-parity problem are sufficient to solve the apparently more general even-parity problem.

4. Reduction of k-Parity Problems to 3-Parity Problems

Let M = $(E, \mathcal{I})$ be the matroid of a k-parity problem, and let

$e_j^{(1)}, e_j^{(2)}, \cdots, e_j^{(k)}$ be a typical set of k elements, all of which or none of which must be contained in a parity set.

Let $E' \supseteq E$ be a set of 2n elements in which for each element $e_j^{(i)}$ of E there is a second element $e_j'^{(i)}$. Let $M_1 = (E', \mathcal{I}_1)$ be the matroid obtained from M by letting I_1 belong to $\mathcal{I}_1$ if and only if $I_1 \cap E \in \mathcal{I}$. Let $M_2 = (E', \mathcal{I}_2)$ be a partition matroid, where $I_2 \in \mathcal{I}_2$ if and only if not both $e_j^{(i)}$ and $e_j'^{(i)}$ belong to I_2, for all j. Let $M_3 = (E', \mathcal{I}_3)$ also be a partition matroid, where $I_3 \in \mathcal{I}_3$ if and only if, for all j, not both $e_j'^{(i)}$ and $e_j^{(i+1)}$ belong to I_3, $i = 1, 2, \cdots, k-1$, and not both $e_j'^{(k)}$ and $e_j^{(1)}$ belong to I_3.

As a consequence of this construction, a maximum-cardinality intersection of M_1, M_2 and M_3 contains exactly n elements. Moreover, there is a one-one correspondence between these n-element intersections and k-parity sets of M.

Suppose $w(e_j^{(i)})$ is the weight of element $e_j^{(i)}$ in the original weighted k-parity problem. In the new 3-matroid intersection problem let the weight of element $e_j^{(i)}$ be $w(e_j^{(i)}) + K$, and the weight of $e_j'^{(i)}$ be K, where K is a sufficiently large constant. Then it is the case that there is a one-one correspondence between maximum-weight k-parity sets of M and maximum-weight intersections of M_1, M_2 and M_3.

This construction establishes the following theorem.

<u>Theorem 4.1</u>

The weighted k-parity problem for a matroid with n elements reduces to a weighted intersection problem for three matroids over 2n elements, two of which are partition matroids. That is, there is a polynomial bounded algorithm for the weighted k-parity problem if and only if there is a polynomial bounded algorithm for the weighted 3-matroid intersection problem.

Taking the reduction of Section 3 into account, we have the following result.

<u>Corollary 4.2</u>

The weighted k-matroid intersection problem for n elements reduces to a weighted 3-matroid intersection problem for 2kn elements.

REFERENCES

[1] J. Edmonds, "Optimum branchings," Journal of Research of the National Bureau of Standards, 71B (1967) 233-240.

[2] J. Edmonds, "Submodular functions, matroids and certain polyhedra," Combinatorial Structures and their Applications, Proceedings of the Calgary International Conference, Gordon and Breach, N. Y. (1970) 69-87.

[3] J. Edmonds, "Matroids and the greedy algorithm," Mathematical Programming, 1 (1971) 127-136.

[4] R. M. Karp, "Reducibility among combinatorial problems," Proceedings of Symposium on Complexity of Computer Computations, to be published by Plenum Press, N. Y. (1972).

[5] E. L. Lawler, "Optimal matroid intersections," Combinatorial Structures and Their Applications, Proceedings of the Calgary International Conference, Gordon and Breach, N. Y. (1970) (Abstract only), 233.

[6] E. L. Lawler, "Matroid Intersection Algorithms," submitted for publication.

[7] E. L. Lawler, "Matroids with parity conditions: a new class of combinatorial optimization problems," submitted for publication.