AN ALGORITHM FOR REDUCING A LINEAR, TIME-INVARIANT DIFFERENTIAL SYSTEM TO STATE FORM

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ABSTRACT

This paper presents an algorithm for obtaining a state representation for linear, time-invariant, multiple-input, multiple-output, differential systems, of the form $L(p)y = M(p)u$.

INTRODUCTION

The problem of finding a state-space representation for a differential system has received wide attention. Thus, algorithms for obtaining a state-space representation for single input, single output systems can be found in numerous textbooks, such as those by Zadeh and Desoer,[1] Laning and Battin,[2] and Athans and Falb.[3] There are also a few algorithms available for linear RLC networks, such as those given by Bryant[4] and Kuh and Rohrer,[5] which can be considered to be particular forms of multiple-input, multiple-output differential systems. However, there does not seem to be a specific algorithm described in the literature

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which shows how to obtain a state space representation for a general, multiple input, multiple output system. In this paper we propose to help bridge this gap with an algorithm for reducing differential systems to a minimal state representation.

This algorithm can also be used in a two-stage procedure to obtain state representations for systems described by transfer functions with the first stage computing an associated differential system of minimal degree.

STATEMENT OF THE PROBLEM

Consider the differential system S with input $u(t) = (u^1(t), u^2(t) \dots \dots u^r(t))^T$ and output $y(t) = (y^1(t), y^2(t), \dots, y^n(t))^T$, described by the system of differential equations

$$S : L(p)y(t) = M(p)u(t), \tag{1}$$

where $L(p)$ is a $n \times n$ matrix, $M(p)$ is a $n \times r$ matrix, whose respective elements $\ell_{ij}(p)$, $m_{ij}(p)$ are finite polynomials in $p = d/dt$, the differentiation operator. It is assumed that the transfer function matrix $W(s) = L^{-1}(s)M(s)$ exists and that its elements $w_{ij}(s)$ are polynomial ratios with the degree of the denominator not smaller than the degree of the numerator.

It is proposed to construct from S a system $\tilde{S}$, with input $u(t) = (u^1(t), u^2(t), \dots, u^r(t))^T$ and output $\tilde{y}(t) = (\tilde{y}^1(t), \tilde{y}^2(t), \dots, \tilde{y}^n(t))^T$, of the form

$$\tilde{S} : \begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t) \\ \tilde{y}(t) = Cx(t) + Du(t) \end{array}, \tag{2}$$

where $x(t) = (x^1(t), x^2(t), \ldots, x^N(t))^T$ is a state vector, and A, B, C, and D are constant matrices of dimension $N \times N$, $N \times r$, $n \times N$, and $n \times r$, respectively. The dimension of the state N will be seen to be equal to the degree of $\det(L(s))$, and hence $\tilde{S}$ will be a minimal state representation of S. The system $\tilde{S}$ will have the following property. Let $u(t)$ be any input defined on $[t_0, \infty)$ and measurable in t, and let $\tilde{y}(t)$ be any output of $\tilde{S}$ satisfying the equations of $\tilde{S}$ for this $u(t)$. Then $\tilde{y}(t)$ also satisfies the equations of S for this $u(t)$. Conversely, let $y(t)$ be any output of S satisfying the equations of S for this $u(t)$, then there exists a unique initial state $x(t_0)$ such that $y(t)$ also satisfies the equations of $\tilde{S}$ for this $u(t)$.

Definition

We shall say that two systems S', S'', of the form of (1) or (2), are equivalent if and only if any output $y'(t)$ of S' corresponding to an arbitrary input $u(t)$, with $t \in [t_0, \infty)$, is an admissible output of S'' corresponding to the same input, and vice versa.

Thus, we propose to construct a completely observable system $\tilde{S}$ which is equivalent to S. The algorithm about to be given consists of two parts: in the first S is reduced to a specific triangular form using the Gauss elimination method, and in the second the state equations are constructed. The Gauss elimination method is well described by Zadeh and Desoer[1] and it is given here only to make the description of the algorithm complete.

TRIANGULARIZATION OF THE SYSTEM S

Let $Z(p)$ be any $n \times n$ matrix whose elements are finite polynomials in $p = d/dt$. We shall say that the system S is invariant under pre-multiplication by $Z(p)$ if and only if the system $S' : Z(p)L(p)y(t) = Z(p)M(p)u(t)$ is equivalent to S, i.e., if and only if S and S' are alternate mathematical descriptions for the same physical system.

To triangularize S we shall use the following three $n \times n$ matrices: The matrix $T_{ij}[f(p)]$ whose principal diagonal elements are unity and whose off diagonal elements are zero, with the exception of the ijth which is equal to $f(p)$, a finite polynomial in p; the matrix U_{ij} whose principal diagonal elements are unity, except for the ith and the jth which are zero, and whose off diagonal elements are zero, with the exception of the ijth and the jith which are unity; the matrix $V_{ii}(c)$ whose off diagonal elements are zero and whose diagonal elements are unity except for the ith which is equal to c, a scalar. Thus,

$$T_{ij}[f(p)] = \begin{array}{c} \\ \\ \\ \\ \\ \\ i \\ \\ \\ \\ \\ \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ & & \cdot & \cdot & & & & & & \\ & & \cdot & & \cdot & & & & & \\ & & \cdot & & & \cdot & & & & \\ \cdot & \cdot & f(p) & \cdot & \cdot & \cdot & 1 & 0 & 0 & 0 \\ & & \cdot & & & & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdot & & & & 0 & 0 & 1 & 0 \\ 0 & 0 & \cdot & & & & 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\hspace{3em} j$$

$$
U_{ij} = \begin{array}{c} \\ \\ i \\ \\ \\ \\ \\ \\ j \\ \\ \\ \\ \end{array}
\begin{array}{c}
\left(\begin{array}{ccccccccccc}
1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 & 0 \\
 & & & 1 & & & & 0 & 0 & 0 & 0 \\
 & & & & 1 & & & & & & \\
\cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & \cdot & & & & \\
0 & 0 & 0 & & & & & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & & & & & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 1
\end{array}\right) \\
\begin{array}{ccccccccccc}
 & & i & & & & & & j & &
\end{array}
\end{array}
\tag{3}
$$

$$
V_{ii}(c) = \begin{array}{c}
\begin{array}{ccccccccccc}
 & & & & & i & & & & &
\end{array} \\
\left(\begin{array}{ccccccccccc}
1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \\
0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \\
0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 0 & c & 0 & \cdot & \cdot & \cdot & \cdot \\
 & & & & 0 & 0 & 1 & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & & & & & & 1 & 0 & 0 \\
0 & 0 & 0 & & & & & & 0 & 1 & 0 \\
0 & 0 & 0 & & & & & & 0 & 0 & 1
\end{array}\right)
\end{array}
$$

When $L(p)$ is premultiplied by $T_{ij}[f(p)]$, the $\underline{j}$th row of $L(p)$ is multiplied by $f(p)$ and added to the $\underline{i}$th row of $L(p)$, otherwise the product has the same rows as $L(p)$. The effect of premultiplication of $L(p)$ by U_{ij} is to exchange the $\underline{i}$th and $\underline{j}$th rows, and the effect of premultiplication by $V_{ii}(c)$ is to multiply the ith row by c, leaving the rest of $L(p)$ intact.

Theorem

The system S is invariant under premultiplication by $T_{ij}[f(p)]$, U_{ij}, or $V_{ii}(c)$, with $i \neq j$ any integers in $\{1, 2, \ldots, n\}$, $f(p)$ any finite polynomial in p, and c any finite scalar.

Proof

Since the determinants of $T_{ij}[f(p)]$, U_{ij} and $V_{ii}(c)$ are nonzero constants, their inverses exist and have elements which are constants (for U_{ij}, $V_{ii}(c)$) or polynomials in p. It now follows trivially that the system S is invariant under premultiplication by these matrices.

Algorithm

Perform the following operations on the matrices of the system S.

Step 1

Find among the nonzero element in the first column of the matrix $L(p)$, one which is of least degree. Suppose this element is in the ith row. Premultiply $L(p)$ and $M(p)$ by U_{1i} to obtain new matrices $L'(p)$, $M'(p)$, with the element in question in the first row and column of $L'(p)$. Rename the matrices $L'(p)$ and $M'(p)$ as $L(p)$, $M(p)$.

Step 2

For $i = 2, 3, \ldots, n$, divide $\ell_{11}(p)$ into $\ell_{i1}(p)$ to obtain

$$\ell_{i1}(p) = \ell_{11}(p)q_{i1}(p) + r_{i1}(p), \qquad i = 2, 3, \dots, n, \tag{4}$$

where $q_{i1}(p)$ is the quotient polynomial and $r_{i1}(p)$ is the remainder polynomial, with degree strictly less than the degree of $\ell_{11}(p)$. Now, multiply both sides of (1), in succession, by the matrices $T_{i1}[-q_{i1}(p)]$, where $i = 2, 3, \dots, n$. This results in a matrix $M'(p)$ and in a matrix $L'(p)$ whose first column is

$$(\ell_{11}, r_{21}, r_{31}, \dots, r_{n1})^T, \tag{5}$$

in terms of the quantities appearing in (4). Rename the matrices $L'(p)$ $M'(p)$ as $L(p)$ and $M(p)$.

Step 3

If the elements $\ell_{21}, \ell_{31}, \dots, \ell_{n1}$ are not identically zero, repeat Step 1 and Step 2 again and again until the remainders $r_{i1}(p)$, as given by (4), are identically zero for $i = 2, 3, \dots, n$. Since all the polynomials are of finite degree and since each iteration of Step 1 and 2 lowers the degree of the element $\ell_{11}(p)$, it is clear that this is a finite procedure. Again rename the matrices as $L(p)$ and $M(p)$.

Step 4

Find a nonzero element among $\ell_{22}, \ell_{32}, \dots, \ell_{n2}$ which is of least degree (second column, last $n-1$ rows). Suppose it is in the ith row. Premultiply both sides of (1) by U_{2i} to bring it to the second row and rename the matrices of the products as $L(p)$ and $M(p)$, respectively. Carry out Steps 2 and 3 with the index 2 replacing the index 1 in all operations. We now have a matrix $L(p)$ whose elements in the first two columns, below the principal diagonal, are zero. Proceed in a similar fashion to obtain a system S whose matrix L(p) has only zero elements below the principal diagonal, i.e.,

$$\begin{pmatrix} \ell_{11}(p) & \ell_{12}(p) & \cdots & \cdots & \ell_{1n}(p) \\ 0 & \ell_{22}(p) & \cdots & \cdots & \ell_{2n}(p) \\ 0 & 0 & \ell_{33}(p) & \cdots & \ell_{3n}(p) \\ & & & & \\ 0 & 0 & 0 & \cdots & \ell_{nn}(p) \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ \vdots \\ y^n \end{pmatrix} = \begin{pmatrix} m_{11}(p) & m_{12}(p) & \cdots & m_{1r}(p) \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ m_{n1}(p) & \cdots & \cdots & m_{nr}(p) \end{pmatrix} \begin{pmatrix} u^1 \\ \vdots \\ \vdots \\ u^r \end{pmatrix} \tag{6}$$

Step 5

Now force, in each column, the off diagonal elements of the matrix L(p) to be lower degree polynomials than the diagonal element. To achieve this, divide $\ell_{22}(p)$ into $\ell_{12}(p)$, to obtain

$$\ell_{12}(p) = \ell_{22}(p)\, q_{12}(p) + r_{12}(p), \tag{7}$$

where $q_{12}(p)$ is the quotient polynomial and $r_{12}(p)$ is the remainder polynomial, with degree strictly less than the degree of $\ell_{22}(p)$. Now multiply both sides of (6) by the matrix $T_{12}[-q_{12}(p)]$ and rename the respective matrix products as L(p) and M(p). Next, divide $\ell_{33}(p)$ into $\ell_{13}(p)$, $\ell_{23}(p)$ to obtain

$$\ell_{i3}(p) = \ell_{33}(p)q_{i3}(p) + r_{i3}(p), \qquad i=1, 2 \tag{8}$$

where q_{i3} is the quotient polynomial and r_{i3} is the remainder polynomial with degree strictly less than $\ell_{33}(p)$. Now multiply both sides of (1) by $T_{i3}[-q_{i3}(p)]$, $i=1, 2$, and rename the matrix products as L(p) and M(p), respectively. Proceed in a similar fashion to reduce the degree of the off diagonal elements in the remaining $n-3$ columns of the matrix L(p).

With the system S reduced to the equivalent form in which the matrix L(p) is upper triangular, with the degree of the off diagonal elements in each column lower than the degree of the corresponding diagonal element, we are ready to derive the state equations of S.

REDUCTION OF THE SYSTEM S TO ITS STATE EQUIVALENT $\tilde{S}$

Let ν_i be the degree of the polynomial $\ell_{ii}(p)$, where $i=1, 2, \ldots, n$. Then, since we have assumed that the transfer function $W(s) = L^{-1}(s)M(s)$ exists and has elements which are ratios of polynomials with the degree of the denominator no smaller than the degree of the numerator, it follows that the degree of the polynomials $m_{ij}(p)$, $j=1, 2, \ldots, r$, is no greater than ν_i, $i=1, 2, \ldots, n$. Hence we may assume that the elements of the matrices L(p), M(p), in (6) may be written in the form

$$\ell_{ji}(p) = a_{ji}^{0} + a_{ji}^{1}p + \ldots + a_{ji}^{\nu_i}p^{\nu_i}, \quad i=1, 2, \ldots, n, \quad j=1, 2, \ldots, i \tag{9}$$

$$m_{ij}(p) = b_{ij}^{0} + b_{ij}^{1}p + \ldots + b_{ij}^{\nu_i} p^{\nu_i}, \quad i=1, 2, \ldots, n, \quad j=1, 2, \ldots, r \tag{10}$$

Referring back to (6), it is seen that

$$\mathrm{Det}(L(p)) = \ell_{11}(p)\,\ell_{22}(p)\ldots\ell_{nn}(p), \tag{11}$$

and hence the order of the system S is $\nu_1 + \nu_2 + \ldots + \nu_n$. We now show how to obtain ν_i state variables, $x_i^1, x_i^2, \ldots, x_i^{\nu_i}$, from the scalar equation for the output y^i, where $i=1, 2, \ldots, n$.

Again referring to (6), we see that the scalar differential equation for y^n is

$$(a_{nn}^{0} + a_{nn}^{1}p + \ldots + a_{nn}^{\nu_n} p^{\nu_n})\, y^n(t) = \sum_{j=1}^{r} (b_{nj}^{0} + b_{nj}^{1}p + \ldots + b_{nj}^{\nu_n} p^{\nu_n})\, u^j(t) \tag{12}$$

We now use a standard algorithm (p. 231, [1]) to associate states with the output y^n. Let $x_n^1, x_n^2, \ldots, x_n^{\nu_n}$ be defined as follows :

$$x_n^1 = a_{nn}^{\nu_n} y^n - \sum_{j=1}^{r} b_{nj}^{\nu_n} u^j$$

$$x_n^2 = a_{nn}^{\nu_n} p y^n - \sum_{j=1}^{r} b_{nj}^{\nu_n} p u^j + a_{nn}^{\nu_n - 1} y^n - \sum_{j=1}^{r} b_{nj}^{\nu_n - 1} u^j$$

....

$$x_n^{\nu_n} = a_{nn}^{\nu_n} p^{\nu_n - 1} y^n - \sum_{j=1}^{r} b_{nj}^{\nu_n} p^{\nu_n - 1} u^j + a_{nn}^{\nu_n - 1} p^{\nu_n - 2} y^n - \sum_{j=1}^{r} b_{nj}^{\nu_n - 1} p^{\nu_n - 2} u^j + \ldots$$

(continued)

$$\ldots + a_{nn}^{1} y^{n} - \sum_{j=1}^{r} b_{nj}^{1} u^{j}. \qquad (13)$$

Solving the above system of equations we obtain:

$$\dot{x}_n(t) = A_{nn} x_n(t) + B_n u(t) \qquad (14)$$

$$y^n(t) = C_{nn} x_n(t) + D_n u(t) \qquad (15)$$

where $x_n = (x_n^1, x_n^2, \ldots, x_n^{\nu_n})$, A_{nn} is a $\nu_n \times \nu_n$ matrix, B_n is a $\nu_n \times r$ matrix, C_{nn} is a $1 \times \nu_n$ matrix, and D_n is a $1 \times r$ matrix, each with components as shown below:

$$A_{nn} = \frac{1}{a_{nn}^{\nu_n}} \begin{pmatrix} -a_{nn}^{\nu_n - 1} & a_{nn}^{\nu_n} & 0 & \cdots & 0 \\ -a_{nn}^{\nu_n - 2} & 0 & a_{nn}^{\nu_n} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_{nn}^{1} & 0 & 0 & \cdots & a_{nn}^{\nu_n} \\ -a_{nn}^{0} & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad (16)$$

$$B_n = \frac{1}{a_{nn}^{\nu_n}} \begin{pmatrix} (a_{nn}^{\nu_n} b_{n1}^{\nu_n - 1} - a_{nn}^{\nu_n - 1} b_{n1}^{\nu_n}), & (a_{nn}^{\nu_n} b_{n2}^{\nu_n - 1} - a_{nn}^{\nu_n - 1} b_{n2}^{\nu_n}) & \ldots & (a_{nn}^{\nu_n} b_{nr}^{\nu_n - 1} - a_{nn}^{\nu_n - 1} b_{nr}^{\nu_n}) \\ \cdots & \cdots & \cdots & \cdots \\ (a_{nn}^{\nu_n} b_{n1}^{0} - a_{nn}^{0} b_{n1}^{\nu_n}), & (a_{nn}^{\nu_n} b_{n2}^{0} - a_{nn}^{0} b_{n2}^{\nu_n}) & \ldots & (a_{nn}^{\nu_n} b_{nr}^{0} - a_{nn}^{0} b_{nr}^{\nu_n}) \end{pmatrix}$$

(17)

$$C_n = (1/a_{nn}^{\nu_n}, 0, \ldots, 0), \qquad D_n = (b_{n1}^{\nu_n}/a_{nn}^{\nu_n}, b_{n2}^{\nu_n}/a_{nn}^{\nu_n}, \ldots, b_{nr}^{\nu_n}/a_{nn}^{\nu_n}) \tag{18}$$

Now let us proceed to the next scalar differential equation in (6), i.e.,

$$\ell_{n-1,n-1}(p)y^{n-1} = -\ell_{n-1,n}(p)y^n + \sum_{j=1}^{r} m_{n-1,j}(p)u^j \tag{19}$$

The set of equations (13) are now used to eliminate y^n and its derivatives from (19), resulting in an expression of the form

$$\ell_{n-1,n-1}(p)y^{n-1} = \sum_{i=1}^{\nu^n} e_{n-1,n}^i x_n^i + \sum_{j=1}^{r} [\overline{m}_{n-1,j}(p) + m_{n-1,j}(p)]\, u^j, \tag{20}$$

where the $e_{n-1,j}^i$ are scalars and $\overline{m}_{n-1,j}(p)$ are polynomials in p with degree no greater than that of $\ell_{n-1,n-1}(p)$. The fact that no derivatives of previous state variables occur in (20) was ensured by forcing the degree of the off diagonal elements in each column to be less than the degree of the corresponding diagonal element, while the property of the degrees of $m_{n-1,j}(p)$ is due to the assumption on $W(s) = L^{-1}(s)M(s)$. Now, treating the variables x_n^i in (20) as additional inputs, we use a substitution similar to (13) to introduce the following ν_{n-1} state variables, $x_{n-1}^1, x_{n-1}^2, \ldots, x_{n-1}^{\nu_{n-1}}$, which satisfy a set of equations of the form

$$\dot{x}_{n-1}(t) = A_{n-1,n-1}x_{n-1}(t) + A_{n-1,n}x_n(t) + B_{n-1}u(t) \tag{21}$$

$$y^{n-1} = C_{n-1,n-1}x_{n-1}(t) + C_{n-1,n}x_n(t) + D_{n-1}u(t) \tag{22}$$

Where $A_{n-1,n-1}$ is a $\nu_{n-1} x \nu_{n-1}$ matrix, $A_{n-1,n}$ is a $\nu_{n-1} x \nu_n$ matrix, B_{n-1} is a $\nu_{n-1} x r$ matrix, and $C_{n-1,n-1}$, $C_{n-1,n}$, and D_{n-1} are matrices of respective dimension $1 x \nu_{n-1}$, $1 x \nu_n$ and $1 x r$.

The same process is now applied to the third line from the bottom of (6) and continued until all the n equations of (6) are similarly treated. Putting it all together into a single matrix form, we get the desired minimal state equivalent of our original system S:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \cdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & \cdots & & A_{1n} \\ 0 & A_{22} & \cdots & & A_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_{n-1,n-1} & A_{n-1,n} \\ 0 & 0 & \cdots & & A_{nn} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_{n-1}(t) \\ x_n(t) \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \\ \cdots \\ B_{n-1} \\ B_n \end{pmatrix} u(t) \tag{23}$$

$$\begin{pmatrix} y^1(t) \\ y^2(t) \\ \cdots \\ y^n(t) \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & \cdots & C_{1n} \\ 0 & C_{22} & C_{23} & \cdots & C_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & C_{nn} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2 \\ \cdots \\ D_n \end{pmatrix} u(t) \tag{24}$$

Note that the dimension of the system (23) is $\sum_{i=1}^{n} \nu_i$, the same as the dimension of S. The matrices in (23), (24) are defined in an obvious manner. This terminates the exposition of the algorithm. We conclude this paper with an example to demonstrate its application.

Example

Consider the system

$$S: \begin{pmatrix} p^2+3p+1 & 2p+3 \\ p^3+3p^2+p & 3p^2+3p+6 \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ p+1 & p+3 \end{pmatrix} \begin{pmatrix} u^1 \\ u^1 \end{pmatrix} \tag{25}$$

Multiplying through by $T_{21}[-p]$, (i.e., multiplying the first equation by $-p$ and add in the result to the second equation) we get

$$S: \begin{pmatrix} p^2+3p+1 & 2p+3 \\ 0 & p^2+6 \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & p+3 \end{pmatrix} \begin{pmatrix} u^1 \\ u \end{pmatrix} \tag{26}$$

We note that the system now is of the form (6) and that Step 5 of the algorithm in the previous section has been carried out. Hence, we may proceed to construct the state equations. We have, for y^2,

$$p^2 y^2 + 0py^2 + 6y^2 = u^1 + (p+3)u^2 \tag{27}$$

Therefore, following the formulas just obtained, we let

$$\dot{x}_2^1 = x_2^2 + u^2 \tag{28}$$

$$\dot{x}_2^2 = -6x_2^1 + 3u^2 + u^1 \tag{29}$$

$$y^2 = x_2^1 \tag{30}$$

We note that this is consistent with (27). Now, proceding to the next equation we find that

$$(p^2 + 3p + 1)\, y^1 = -(2p + 3)\, y^2 + u^1 \tag{31}$$

From (28) to (30) we get that

$$-(2p + 3)\, y^2 = -(2x_2^2 + 3x_2^1) - 2u^2 \tag{32}$$

Hence

$$(p^2 + 3p + 1)\, y^1 = (2x_2^2 + 3x_2^1) - 2u^2 + u^1 \tag{33}$$

Now, let

$$\dot{x}_1^1 = -3x_1^1 + x_1^2 \tag{34}$$

$$\dot{x}_1^2 = -x_1^1 + u^1 - 2u^2 - (3x_2^1 + 2x_2^2) \tag{35}$$

$$y^1 = x_1^1 \tag{36}$$

We see again that the last three equations are consistent with (4). We may now write down the state equations of the system:

$$\begin{pmatrix} \dot{x}_1^1 \\ \dot{x}_1^2 \\ \dot{x}_2^1 \\ \dot{x}_2^2 \end{pmatrix} = \begin{pmatrix} -3 & -1 & 0 & 0 \\ -1 & 0 & -3 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & 0 \end{pmatrix} \begin{pmatrix} x_1^1 \\ x_1^2 \\ x_2^1 \\ x_2^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & -2 \\ 0 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u^1 \\ u_2 \end{pmatrix} \tag{37}$$

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1^1 \\ x_1^2 \\ x_2^1 \\ x_2^2 \end{pmatrix} \tag{38}$$

CONCLUSION

The assumption that the transfer function $W(s) = L^{-1}(s)M(s)$ of the differential system has elements with numerator polynomials of degree no higher than the corresponding denominator polynomials is only necessary to ensure that the resulting state representation be of the form (2). When this assumption is not satisfied, the algorithm can be extended, in an obvious manner, to obtain state representations of the form: $x = Ax + Bu$, $y = Cx + Du + Eu + Fu + \ldots$.

The algorithm was written in a form particularly suitable for digital computer implementation. The reader using it for hand calculation may find it helpful to refer to (12) and (13) at each step following Eq. (19).

The main advantage of the algorithm is that it gives a completely unambiguous and easily implementable way for transcribing systems in differential equation form into a more desirable state form.

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