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ON THE STABILITY OF NONLINEAR SAMPLED-DATA FEEDBACK SYSTEMS

by

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Summary

This work is a generalization of Tsypkin's stability criterion for a class of time-varying nonlinear sampled-data feedback systems. Also some sufficient conditions for the response to any bounded input sequence to be bounded are presented. In this paper, no assumptions are made concerning the internal dynamics of the linear subsystem, except that its input-output relation is of the form of convolution. The essence of the proof is to consider the nonlinear system as a perturbation of a stable linear system.

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I. Description of the System

The system considered in this paper is a sampled-data feedback system of the type shown in Fig. 1. It is assumed that its linear and nonlinear parts, G and N, are separated by an ideal sampler [6]. The sampling period, T, is assumed to be constant. Since the theory of distributions is well established, there is no difficulty in interpreting expressions involving Dirac-δ functions. Thus, if the input to the sampler is $x(t)$, its output will be $x^*(t) = \sum_{n=0}^{\infty} x(n)\delta(t-nT)$. For brevity, $\{x(n)\}$ will be used to denote the sequence $\{x(0), x(1), \ldots, x(n), \ldots\}$. Using z-transform, the behavior of the sampled-data system is characterized only at the sampling instants. This is usually enough in practical applications, because the hidden oscillation can be easily predicted and avoided by changing the sampling period. Therefore, in this paper, only the performance at the sampling instants will be considered.

N is a memoryless, time-varying nonlinear element, continuous with respect to its two arguments; furthermore there are two constants β_1 and β_2 such that $\beta_2 > 0$, $\beta_2 > \beta_1$ and

(N.1) $\quad \beta_1 \le \frac{\varphi(\sigma, n)}{\sigma} \le \beta_2, \qquad \text{for } \sigma \neq 0$

$\varphi(0, n) = 0, \qquad$ for all n (the set of natural numbers).

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N is a memoryless, time-varying nonlinear element, continuous with respect to its two arguments; furthermore there are two constants β_1 and β_2 such that $\beta_2 > 0$, $\beta_2 > \beta_1$ and

$$\beta_1 \leq \frac{\phi(\sigma, n)}{\sigma} \leq \beta_2, \quad \text{for } \sigma \neq 0$$

$$\phi(0, n) = 0, \quad \text{for all } n \text{ (the set of natural numbers).}$$

G is a nonanticipative, linear, time-invariant subsystem; it usually consists of a zero-order hold [7] and a linear plant. Let $g(t)$ be its unit impulse response, $g^*(t)$ its sampled impulse response; then $g^*(t) = \sum_{n=0}^{\infty} g(n)\delta(t - nT)$. Thus the input-output relation of G is given by

$$\text{(G.1)} \quad y(n) = z(n) + \sum_{m=0}^{n} g(n-m)x(m)$$

where $\{x(n)\}$ is the input sequence, $\{y(n)\}$ its output sequence, $\{z(n)\}$ its sampled zero-input response.

It is assumed that $\{z(n)\}$ satisfies the following condition:

(G.2) For all initial states, $\{z(n)\} \in \ell^2$, i.e., $\sum_{n=0}^{\infty} |z(n)|^2 < \infty$.

It is noted that $z(t) \in L^2_{(0,\infty)}$ does not imply $\{z(n)\} \in \ell^2$. However, we have $\{z(n)\} \in \ell^2$ if $z(t) \in L^2_{(0,\infty)} \cap L^\infty_{(0,\infty)}$ and is piecewise continuous.

<u>Definition</u>

$$G(z) \triangleq \mathcal{Z}[g(n)] = \sum_{n=0}^{\infty} g(n) z^{-n}$$

$$\bar{\beta} \triangleq \frac{\beta_1 + \beta_2}{2}$$

$$H(z) \triangleq \frac{\bar{\beta} G(z)}{1 + \bar{\beta} G(z)}$$

$$h^*(t) \triangleq \mathcal{Z}^{-1}[H(z)] = \sum_{n=0}^{\infty} h(n)\delta(t - nT)$$

$$\tilde{\varphi}(\sigma, n) \triangleq \varphi(\sigma, n) - \overline{\beta}\sigma$$

$$\|h\|_1 \triangleq \sum_{n=0}^{\infty} |h(n)|$$

$$\|z\|_p \triangleq \left(\sum_{n=0}^{\infty} |z(n)|^p\right)^{1/p}, \quad 1 \le p < \infty, \quad \|z\|_\infty \triangleq \sup_n |z(n)|$$

II. Sufficient Conditions for Stability

Recently, Tsypkin [1, 2] applied the famous Popov method to establish the sufficient conditions for the stability of the system shown in Fig. 1. Jury and Lee [3] extended his result to obtain less conservative sufficient conditions by adding restrictions on the slope of the nonlinearity. Though not stated explicitly, Tsypkin considered only the linear subsystem with rational transfer function. In this paper we remove this restriction and present a different proof which is analogous to that of Sandberg [4].

Theorem 1

In the sampled-data system shown in Fig. 1, N is a memoryless, time-varying nonlinear element satisfying (N. 1), G is a nonanticipative linear time-invariant subsystem satisfying (G. 1) and (G. 2). If

(i) $\{g(n)\} \in \ell^1$, i.e., $\sum_{n=0}^{\infty} |g(n)| < \infty$

(ii) $\inf_{|z| \ge 1} |1 + \beta G(z)| > 0$, for $\beta_1 \le \beta \le \beta_2$

(iii) $\gamma \triangleq \dfrac{\beta_2 - \beta_1}{2} \max_{-\pi < \omega T \le \pi} \left| \dfrac{G(e^{i\omega T})}{1 + \overline{\beta} G(e^{i\omega T})} \right| < 1$

then for any input $\{u(n)\} \in \ell^2$, and for any initial state, the output sequence $\{y(n)\}$ is an element of ℓ^2, hence is a bounded sequence which tends to zero as $n \to \infty$.

Remarks

(a) $G(z) = \sum_{n=0}^{\infty} g(n) z^{-n}$ with $\{g(n)\} \in \ell^1$, in general, cannot be put in a closed form nor is it a rational function of z. Although $G(z)$ is analytic for $|z| > 1$, nothing can be said about the singularities inside the unit circle; for example, the singularities need not be isolated, therefore the residue theorem is generally not applicable.

(b) It is interesting to note, if (ii) is not satisfied, a non-anticipative linear part G may result in an anticipative closed loop system. For example, let the nonlinear part N be linear, let the slope of its characteristic $\overline{\beta}$ be 1, and let $G(z) = -1 + z^{-1}$; then the closed loop transfer function is $H(z) = -z + 1$, i.e., $H(z)$ is anticipative.

(c) If the system is linear, then $\gamma = 0$ and the condition (iii) drops out; conditions (i) and (ii) are then the necessary and sufficient conditions for the linear system to be stable.

Proof of Theorem 1

Before we start to prove the theorem, we cite some well-known facts. Consider the linearized sampled-data system as shown in Fig. 2; then the closed loop impulse transfer function is $H(z) = \dfrac{\overline{\beta} G(z)}{1 + \overline{\beta} G(z)}$.

Since $1 + \overline{\beta} G(z) = (1 + \overline{\beta} g(0)) + \sum_{n=1}^{\infty} \overline{\beta} g(n) z^{-n} \triangleq \sum_{n=0}^{\infty} \overline{g}(n) z^{-n}$, and since

$\{g(n)\} \in \ell^1$ implies $\{\overline{g}(n)\} \in \ell^1$, it follows that $1 + \overline{\beta}G(z)$ is analytic for $|z| > 1$ and continuous for $|z| = 1$. Thus, condition (ii) implies that $(1 + \overline{\beta}G(z))^{-1}$ is analytic for $|z| > 1$ and continuous for $|z| = 1$; therefore it has a Laurent expansion of the form $\sum_{n=0}^{\infty} b(n)z^{-n}$ valid for $|z| > 1$. Wiener's result [5] implies that $\{b(n)\} \in \ell^1$. Because the convolution of two ℓ^1 sequences is an ℓ^1 sequence, it follows that $\{h(n)\} \in \ell^1$.

Let $\hat{y}(n)$ be the output of the linearized system; then

$$\hat{y}(n) = z(n) + \sum_{m=0}^{n} h(n - m)[u(m) - z(m)] . \tag{1}$$

Since $\{h(n)\} \in \ell^1$, if $\{z(n)\} \in \ell^2$ and $\{u(n)\} \in \ell^2$ then $\{\hat{y}(n)\} \in \ell^2$.

Introducing $\hat{y}(n)$ and $\tilde{\varphi}$, it is easy to see that (G.1) can be written as

$$y(n) = \hat{y}(n) + \frac{1}{\overline{\beta}} \sum_{m=0}^{n} h(n - m)\tilde{\varphi}[u(m) - y(m), m] . \tag{2}$$

Let $\sigma(n) \triangleq u(n) - y(n)$, $\hat{\sigma}(n) \triangleq u(n) - \hat{y}(n)$, then Eq. (2) becomes

$$\sigma(n) = \hat{\sigma}(n) - \frac{1}{\overline{\beta}} \sum_{m=0}^{n} h(n - m)\tilde{\varphi}[\sigma(m), m] . \tag{3}$$

Given the sequence $\{\hat{\sigma}(n)\}$, $\{\sigma(n)\}$ may be calculated through the recursive equation (3). The more interesting interpretation of Eq. (3) is to consider it as the equation of a feedback system shown in Fig. 3: $\{\frac{1}{\overline{\beta}}h(n)\}$ is the impulse response of the linear subsystem, $\{\hat{\sigma}(n)\}$ is its zero-input response and $\{\sigma(n)\}$ is the output of the entire system.

Let us show that if condition (ii) is satisfied, the sequence $\{\sigma(k)\}$ is such that $\sigma(n)$ is finite for all n. The only possibility for $\sigma(n)$ to be infinite for some n is that $-\frac{1}{\overline{\beta}}h(0)\tilde{\varphi}[\sigma(n), n] = \sigma(n)$. Since

$h(0) = \lim_{z\to\infty} H(z) = \lim_{z\to\infty} \frac{\overline{\beta}G(z)}{1+\overline{\beta}G(z)} = \frac{\overline{\beta}g(0)}{1+\overline{\beta}g(0)}$, it is equivalent to

$\frac{g(0)}{1+\overline{\beta}g(0)}\tilde{\varphi}[\sigma(n), n] = -\sigma(n)$, or $g(0) = \frac{-1}{\overline{\beta} + \frac{\tilde{\varphi}[\sigma(n), n]}{\sigma(n)}}$. Since

$\left|\frac{\tilde{\varphi}[\sigma(n), n]}{\sigma(n)}\right| \le \frac{\beta_2 - \beta_1}{2}$, it implies $\frac{-1}{\beta_1} \le g(0) \le \frac{-1}{\beta_2}$ for $\beta_1 > 0$; $g(0) \le \frac{-1}{\beta_2}$

for $\beta_1 = 0$; and $g(0) \ge \frac{1}{\beta_1}$, $g(0) \le \frac{-1}{\beta_2}$ for $\beta_2 < 0$. But these cases are ruled out by condition (ii) when $z \to \infty$ ($\lim_{z\to\infty} |1 + \beta G(z)| = |1 + \beta g(0)| > 0$ for $\beta_1 \le \beta \le \beta_2$), therefore $|\sigma(n)| < \infty$ for all finite n.

Define $\{\sigma_N(n)\}$ as the sequence $\{\sigma(n)\}$ truncated after its N-th term, and $\Sigma_N(z)$ its corresponding z-transform, i.e.,

$$\begin{aligned} \sigma_N(n) &= \sigma(n) \qquad \text{if } n \le N \\ &= 0 \qquad \text{if } n > N \end{aligned}$$

and

$$\Sigma_N(z) \triangleq \sum_{n=0}^{\infty} \sigma_N(n) z^{-n} = \sum_{n=0}^{N} \sigma(n) z^{-n}.$$

Similarly for $\hat{\sigma}_N(n)$ and $\hat{\Sigma}_N(z)$. Multiplying z^{-n} and taking the summation from 0 to some fixed N, (3) becomes

$$\sum_{n=0}^{N} \sigma(n) z^{-n} = \sum_{n=0}^{N} \hat{\sigma}(n) z^{-n} - \frac{1}{\beta}\left[\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} h(n-m)\tilde{\varphi}(\sigma_N(m), m)\right) z^{-n}\right.$$

$$\left. - \sum_{n=N+1}^{\infty}\left(\sum_{m=0}^{n} h(n-m)\tilde{\varphi}(\sigma_N(m), m)\right) z^{-n}\right]$$

or

$$\Sigma_N(z) = \hat{\Sigma}_N(z) - \frac{1}{\beta} H(z)\tilde{\Phi}_N(z) + V(z)$$

where

$$\tilde{\Phi}_N(z) = \sum_{n=0}^{\infty} \tilde{\varphi}(\sigma_N(n), n) z^{-n}$$

$$V(z) = \frac{1}{\beta}\sum_{n=N+1}^{\infty}\left(\sum_{m=0}^{n} h(n-m)\tilde{\varphi}(\sigma_N(m), m)\right) z^{-n}.$$

$V(z)$ is the z-transform of the zero-state response [starting from (N+1)-th sampling instant] as a result of $\{\tilde{\varphi}(\sigma_N(m), m)\}$ being applied to a system whose impulse response is $\left\{\frac{1}{\beta} h(n)\right\}$. Since $\{\sigma_N(n)\}$ is of finite length, $\Sigma_N(z)$, $\hat{\Sigma}(z)$, and $\tilde{\Phi}_N(z)$ are polynomials, hence are all convergent for $|z| \geq 1$. $V(z)$ is also convergent for $|z| \geq 1$, since it is the z-transform of the convolution of two ℓ^1 sequences. It is convenient here to change the variable z to $e^{i\omega T}$ and apply Minkowski's inequality to Eq. (3); then

$$\left(\int_{-\pi/T}^{\pi/T} |\Sigma_N(e^{i\omega T}) - V(e^{i\omega T})|^2 d\omega\right)^{1/2} \leq \left(\int_{-\pi/T}^{\pi/T} |\hat{\Sigma}_N(e^{i\omega T})|^2 d\omega\right)^{1/2}$$

$$+ \frac{1}{\beta}\left(\int_{-\pi/T}^{\pi/T} |H(e^{i\omega T})\tilde{\Phi}_N(e^{i\omega T})|^2 d\omega\right)^{1/2}.$$

Using Parseval's equality, the fact that $\Sigma_N(z)$ and $V(z)$ have no power of z^{-1} in common, condition (iii), and the fact that $|\tilde{\varphi}(\sigma, n)| \le \frac{\beta_2 - \beta_1}{2}|\sigma|$, we obtain

$$\left(\sum_{n=0}^{N}|\sigma(n)|^2\right)^{1/2} = \left(\sum_{n=0}^{\infty}|\sigma_N(n)|^2\right)^{1/2} \le \left(\sum_{n=0}^{\infty}|\sigma_N(n)|^2 + \sum_{n=0}^{\infty}|v(n)|^2\right)^{1/2}$$

$$= \left(\sum_{n=0}^{\infty}|\sigma_N(n) - v(n)|^2\right)^{1/2} \le \frac{\beta_2-\beta_1}{2\bar{\beta}} \max_{\omega}|H(e^{i\omega T})| \left(\sum_{n=0}^{\infty}|\sigma_N(n)|^2\right)^{1/2}$$

$$+ \left(\sum_{n=0}^{\infty}|\hat{\sigma}_N(n)|^2\right)^{1/2} \le \gamma\left(\sum_{n=0}^{N}|\sigma(n)|^2\right)^{1/2} + \left(\sum_{n=0}^{N}|\hat{\sigma}(n)|^2\right)^{1/2}.$$

Since $\hat{\sigma}(n) = u(n) - \hat{y}(n)$, therefore $\{u(n)\} \in \ell^2$ and $\{y(n)\} \in \ell^2$ imply $\{\hat{\sigma}(n)\} \in \ell^2$; thus

$$\left(\sum_{n=0}^{N}|\sigma(n)|^2\right)^{1/2} \le \frac{1}{1-\gamma}\left(\sum_{n=0}^{N}|\hat{\sigma}(n)|^2\right)^{1/2} \le \frac{1}{1-\gamma}\left(\sum_{n=0}^{\infty}|\hat{\sigma}(n)|^2\right)^{1/2} < \infty \qquad (4)$$

i.e., $\{\sigma(n)\} \in \ell^2$. But $y(n) = u(n) - \sigma(n)$, it follows that $\{y(n)\} \in \ell^2$; hence $\{y(n)\}$ is a bounded sequence which tends to zero as $n \to \infty$. Q.E.D.

In case $G(z)$ is a rational function of z, the condition (i) in the theorem can be relaxed.

<u>Corollary 1</u>

In the theorem, we replace (i) by (i');

(i') $G(z)$ is a rational function of z

then the same conclusion holds.

Proof

(i') and (ii) imply that the rational function $H(z) = \frac{\beta G(z)}{1 + \beta G(z)}$ is analytic for $|z| > a$ where a is a number < 1; then $\{h(n)\} \in \ell^1$, as can easily be seen from the residue theorem. The rest of the proof is the same as in the theorem. Q.E.D.

Remarks

(a) It is easy to see that if $G(z)$ is a rational function of z, condition (ii) is equivalent to the Nyquist criterion. More precisely, if $G(z)$ has q poles outside the unit circle, then $1 + \beta G(z) \neq 0$, $\beta_1 \leq \beta \leq \beta_2$ for $|z| \geq 1$, if and only if the Nyquist plot of $G(z)$ [8] does not go through the critical interval $\left[\left(-\frac{1}{\beta_1}, 0\right), \left(-\frac{1}{\beta_2}, 0\right)\right]$ and encircles it q times in the clockwise direction.

Let $G(z)$ in Fig. 1 be a rational function of z with q poles outside the unit circle, and let $\beta_2 > \beta_1 > 0$, then the Corollary says: if the Nyquist plot of $G(z)$ [8] lies outside the disk with center at $\left(-\frac{1}{2}(\beta_1^{-1} + \beta_2^{-1}), 0\right)$ and radius $\frac{1}{2}(\beta_1^{-1} - \beta_2^{-1})$, and encircles the disk q times in the clockwise direction, then the system is stable, i.e., the output sequence is bounded and tends to zero as $n \to \infty$.

(b) The boundedness of $\|y\|_2$ can be given in terms of the norms of the input sequence and the zero-input sampled response. From Eq. (4), we obtain

$$\|\sigma\|_2 \leq \frac{1}{1-\gamma}\|\hat{\sigma}\|_2. \qquad (5)$$

Since $\|\sigma\|_2 \geq \|y\|_2 - \|u\|_2$, $\|\hat{\sigma}\|_2 \leq \|u\|_2 + \|\hat{y}\|_2$, and from Eq. (1), $\|\hat{y}\|_2 \leq \|z\|_2 + \|h\|_1\|u\|_2 + \|h\|_1\|z\|_2$, Eq. (5) becomes

$$\|y\|_2 - \|u\|_2 \leq \frac{1}{1-\gamma}(1 + \|h\|_1)(\|u\|_2 + \|z\|_2)$$

or

$$\|y\|_2 \leq \frac{2 - \gamma + \|h\|_1}{1-\gamma}\|u\|_2 + \frac{1 + \|h\|_1}{1-\gamma}\|z\|_2. \tag{6}$$

If $h(n) \geq 0$ for all n, then $\|h\|_1 = \max_{\omega} \|H(e^{i\omega T})\| = \frac{2\bar{\beta}\gamma}{\beta_2 - \beta_1} \leq \frac{2\bar{\beta}}{\beta_2 - \beta_1}$. In this case, Eq. (6) becomes

$$\|y\|_2 \leq \frac{2 - \gamma + \frac{2\bar{\beta}\gamma}{\beta_2-\beta_1}}{1-\gamma}\|u\|_2 + \frac{1 + \frac{2\bar{\beta}}{\beta_2-\beta_1}}{1-\gamma}\|z\|_2$$

$$\leq \frac{2\beta_2}{(1-\gamma)(\beta_2 - \beta_1)}(\|u\|_2 + \|z\|_2).$$

<u>Theorem 2</u>

In the sampled-data system shown in Fig. 1, N is a memoryless, time-varying nonlinear element satisfying (N.1), G is a nonanticipative linear time-invariant subsystem satisfying (G.1). If for all initial states, $\{z(n)\} \in \ell^{\infty}$, i.e., $\|z\|_{\infty} < \infty$, and if

(i) $\{g(n)\} \in \ell^1$

(ii) $\inf_{|z| \geq 1} |1 + \beta G(z)| > 0$, for $\beta_1 \leq \beta \leq \beta_2$

(iii) $\alpha = \frac{\beta_2 - \beta_1}{2\bar{\beta}}\|h\|_1 < 1$

then for any $\{u(n)\} \in \ell^{\infty}$, and any initial state, the output sequence $\{y(n)\}$ is an element of ℓ^{∞} (the response to any bounded input sequence is bounded).

Remarks

(a) This theorem is the discrete analog to the one in [9].

(b) Except for some special cases, it is not easy to obtain the frequency domain interpretation for condition (iii).

(c) This theorem may be stated more generally, i.e., if $\{z(n)\} \in \ell^{p}$, $1 \leq p \leq \infty$, for all initial states, and if (i) (ii) (iii) hold, then for any $\{u(n)\} \in \ell^{p}$, and any initial state, the output $\{y(n)\} \in \ell^{p}$.

Proof of Theorem 2

Since condition (ii) implies that $\{h(n)\}$ is nonanticipative, and since φ is memoryless, from Eq. (3), we have

$$\sigma_N(n) = \hat{\sigma}_N(n) - \frac{1}{\beta} \sum_{m=0}^{n} h(n-m)\tilde{\varphi}[\sigma_N(m), m] . \tag{7}$$

Since $\{h(n)\} \in \ell^{1}$, $\{u(n)\} \in \ell^{\infty}$ and $\{z(n)\} \in \ell^{\infty}$, it follows that $\{\hat{\sigma}(n)\} \in \ell^{\infty}$. We have shown that $\sigma(n) < \infty$ for all finite n, therefore we may take the norm and apply Minkowski's inequality to Eq. (7),

$$\|\sigma_N\|_{\infty} \leq \|\hat{\sigma}_N\|_{\infty} + \frac{1}{\beta} \cdot \|h * \tilde{\varphi}_N\|_{\infty}.$$

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Since condition (ii) implies that $\{h(n)\}$ is nonanticipative, and since ϕ is memoryless, from Eq. (3), we have

$$y_N(n) = \hat{y}_N(n) - \frac{1}{\beta}\sum_{m=0}^{n} h(n-m)\tilde{\phi}[e_N(m), m] . \qquad (7)$$

Since $\{h(n)\} \in \ell^1$, $\{u(n)\} \in \ell^\infty$ and $\{z(n)\} \in \ell^\infty$, it follows that $\{\hat{y}(n)\} \in \ell^\infty$. We have shown that $e(n) < \infty$ for all finite n, therefore we may take the norm and apply Minkowski's inequality to Eq. (7),

$$\|y_N\|_\infty \le \|\hat{y}_N\|_\infty + \frac{1}{\beta}\|h * \tilde{\phi}_N\|_\infty .$$

Then, from

$$\|h*\tilde{\varphi}_N\|_\infty = \sup_n \left| \sum_{m=0}^{n} h(n-m)\tilde{\varphi}(\sigma_N(m), m) \right| \le \sup_n \left[\sum_{m=0}^{n} |h(n-m)| \, |\tilde{\varphi}(\sigma_N(m), m)| \right]$$

$$\le \frac{\beta_2 - \beta_1}{2} \sup_n \left[\sum_{m=0}^{n} |h(n-m)| \, |\sigma_N(m)| \right] \le \frac{\beta_2 - \beta_1}{2} \|\sigma_N\|_\infty \|h\|_1$$

and condition (iii), we obtain

$$\|\sigma_N\|_\infty \le \frac{1}{1-\alpha} \|\hat{\sigma}_N\|_\infty \le \frac{1}{1-\alpha} \|\hat{\sigma}\|_\infty < \infty .$$

Q.E.D.

Corollary 2

In Theorem 2, we replace (i) by (i')

(i') $G(z)$ is a rational function of z

then the same conclusion holds.

III. Conclusions

Stability criteria for a class of time-varying nonlinear sampled-data systems are obtained. In the ℓ^2 case, because of Passeval's equality, we are able to obtain the stability criterion in the frequency domain. The impulse transfer function of the linear subsystem is not assumed to be a rational function; therefore these criteria are applicable to a very broad class of systems.

In the ℓ^2 case, if $G(z)$ is a rational function of z, the stability region in the $G(z)$-plane is the same as in [2], though the analytic form is different. In this paper we use the linearized closed-loop sampled impulse response and some well-known facts about linear systems. Therefore, we can apply Minkowski's inequality directly to

$$\|h\varphi(e_N)\|_\infty = \sup_n \Big|\sum_{m=0}^{n} h(n-m)\varphi(e_N(m), m)\Big| \le \sup_n \sum_{m=0}^{n} |h(n-m)|\,|\varphi(e_N(m), m)|$$

$$\le \frac{\beta_2-\beta_1}{2} \sup_n \sum_{m=0}^{n} |h(n-m)|\,|e_N(m)| \le \frac{\beta_2-\beta_1}{2} \|e_N\|_\infty \|h\|_1$$

and condition (iii), we obtain

$$\|e_N\|_\infty \le \frac{1}{1-\alpha}\|u_N\|_\infty \le \frac{1}{1-\alpha}\|u\|_\infty < \infty .$$ Q. E. D.

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III. Conclusions

Stability criteria for a class of time-varying nonlinear sampled-data systems are obtained. In the ℓ^2 case, because of Parseval's equality, we are able to obtain the stability criterion in the frequency domain. The impulse transfer function of the linear subsystem is not assumed to be a rational function; therefore those criteria are applicable to a very broad class of systems.

In the ℓ^∞ case, if $G(z)$ is a rational function of z, the stability region in the $G(z)$-plane is the same as in [3], though the analytic form is different. In this paper we use the linearized closed-loop sampled impulse response and some well-known facts about linear systems. Therefore, we can apply Minkowski's inequality directly to the z-transform of the convolution equation which characterizes the system.

Acknowledgments

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References

[1] Y. Z. Tsypkin, "On the stability in the large of nonlinear sampled-data systems," Dokl. Akad. Nauk., Vol. 145, pp. 52-55; July, 1962.

[2] Y. Z. Tsypkin, "Frequency criteria for the absolute stability of nonlinear sampled-data systems," Avtomatika i Telemekhanika, Vol. 25, No. 3, pp. 281-289; March, 1964.

[3] E. I. Jury and B. W. Lee, "On the stability of a certain class of nonlinear sampled-data systems," IEEE Trans. on Automatic Control, Vol. AC-9, pp. 51-61; Jan., 1964.

[4] I. W. Sandberg, "Frequency-domain criteria for the stability of nonlinear feedback systems," Proc. 1964 National Electronics Conference, pp. 737-741.

[5] N. Wiener, The Fourier Integral and Certain of its Application, Cambridge University Press; 1933 (p. 91).

[6] E. I. Jury, Sampled-Data Control Systems, John Wiley and Sons, Inc. New York; 1958 (p. 6).

[7] E. I. Jury, op. cit. (p. 48).

[8] E. I. Jury, op. cit. (p. 44, Fig. 1.33).

[9] I. W. Sandberg, "A condition for the $\mathcal{L}_\infty$-stability of feedback systems containing a single time-varying nonlinear element," Bell Syst. Tech. J., Vol. 43, pp. 1815-1817; July, 1964.

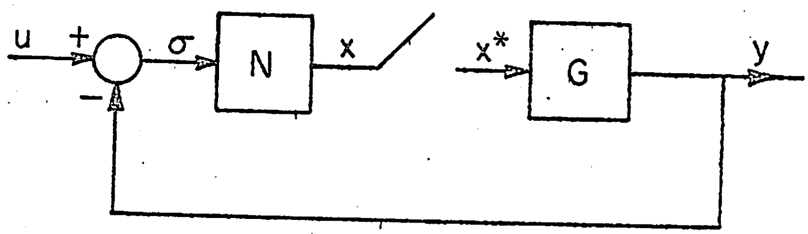

Fig. 1. Time-varying nonlinear sampled-data feedback system.

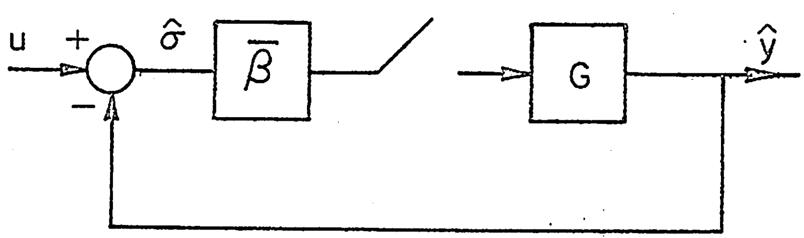

Fig. 2. Linearized sampled-data feedback system.

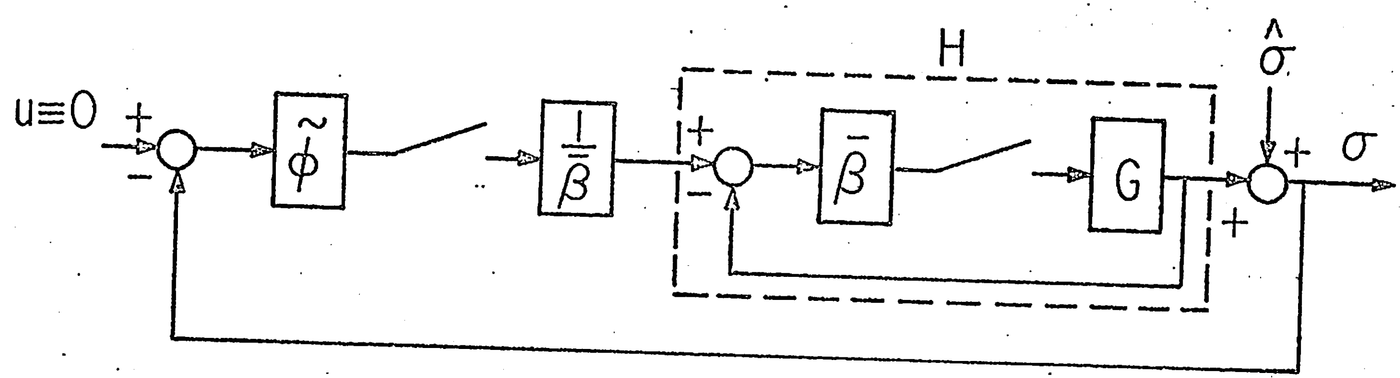

Fig. 3. System interpretation of Eq. (3).