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MODIFIED TWO STREAM INSTABILITY

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ABSTRACT

In the usual modified two stream instability (MTS) the drift between the electrons and ions is well above the ion thermal velocity. The real part of the frequency is well above ω_{ci}, justifying leaving the ions unmagnetized. This report considers lower drift velocities where the resulting lower frequencies require that the ion $\underline{v} \times \underline{B}$ force be kept. The more complicated dispersion relation is solved for a limited set of parameters. For drifts on the order of twice ion thermal speed, it is found that the MTS instability consists of distinct bands of interaction (in both ω and k), near each $n\omega_{ci}$, where the MTS waves interact with the Bernstein modes. For lower relative drifts, the instability vanishes. For larger drifts, the growth in bands remains and the usual MTS continuous complex ω roots appear, as found by others neglecting the ion $\underline{v} \times \underline{B}$ force.

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I. INTRODUCTION

In studies commonly made of the Modified Two Stream (MTS) instability, the possibility of ion cyclotron effects has been ignored through the use of a nonmagnetic approximation to the ions. Although a nonmagnetic approximation to the ions should be pretty good when the MTS growth rates are much larger than the ion cyclotron frequencies, discrepancies might be expected for low drift velocities where the frequencies become comparable. Also, even though MTS growth rates may be large, the saturation times of the MTS instability are usually on the order of ion cyclotron times, opening up the possibility of ion cyclotron effects again. For these reasons, a dispersion equation has been developed for the MTS instability which includes magnetic ion effects and a preliminary study has been made to determine how the inclusion of these effects will modify the results normally obtained for the MTS instability. The results at present indicate that, for small relative drift velocities, ion cyclotron effects do indeed become more important to the basic dispersion relation, thus opening the possibility of more accurate information on the threshold for initiation of the MTS instability. At larger values of the relative drift velocity, the MTS interaction also appears to excite growing Bernstein modes but with growth rates less than the MTS growth rate.

II. DISPERSION RELATION

Figure 1 indicates the basic model underlying the MTS interaction. Electrons are postulated to be drifting through the ions with a net velocity perpendicular to the magnetic field for some reason: E x B effects, collisionless shock, etc. Waves which have a small component k_z parallel to the magnetic field allow electron motion parallel to the magnetic field to interact with the perpendicular electron drift and the ions to produce the growing waves of the MTS instability. The interaction is essentially electrostatic and allows the dispersion relation to be written in the general form

$$\varepsilon\ (\underline{k},\omega) = 1 + \mu_e(\underline{k},\ \omega) + \mu_i(\underline{k},\omega) = 0 \quad , \tag{1}$$

where μ_i and μ_e are the respective susceptibilities of the ions and the electrons. The difference between the study presented here and what others have done is due to the inclusion of possible magnetic effects in the ion susceptibility μ_i .

For the electron susceptibility, we use the same one as McBride et al.,[1] in their study of the MTS interaction in the warm electron limit. This susceptibility is given by

$$\mu_e = \frac{1}{k^2\lambda_{de}^2}\left[1 - e^{-\lambda}I_o(\lambda)\left(1 + \frac{1}{2}Z'\left(\frac{\omega - k_\perp v_d}{\sqrt{2}\,k_z v_e}\right)\right)\right] \quad , \tag{2}$$

where $\lambda = k_\perp^2\rho_e^2$, I_o is the zeroth order modified Bessel function, and Z' is the derivative of the plasma dispersion function. ρ_e, λ_{de}, and v_e are respectively the electron Larmor radius, Debye length, and thermal velocity while v_d is the relative drift velocity between the electrons and ions. The only difference between the equation as written here and the one used by McBride et al., is that we wish to consider the electrons drifting relative to the ion frame of reference while McBride et al., operated in the electron frame of reference. A transformation to the doppler shifted frequency $\omega' = \omega - k_\perp v_d$ would bring the two formulations into correspondence. McBride's susceptibility is good in the range $\omega \ll \Omega_e$, and $k_z \rho_e \ll 1$. We shall want to restrict this somewhat further due to limitations on the ion susceptibility, and so require $k_\perp \rho_e \ll 1$. (In general, $k_z \ll k_\perp$ for the MTS interaction). This, however, allows us to make a small λ approximation in the $e^{-\lambda} I_o(\lambda)$ term of the susceptibility, which becomes $e^{-\lambda} I_o(\lambda) \simeq 1 - k_\perp^2 \rho_e^2$ and results in a simplification. For further simplification, we normalize the frequency to the ion plasma frequency, the wave number to the ion Debye length, lump the different parameters together and use $k_\perp/k \simeq 1$. The susceptibility then has the following form:

$$\bar{\mu}_e = I_r - \left[\frac{1}{K^2 T_r} - I_r\right] \frac{1}{2} Z'\left(\frac{1}{\sqrt{2K_r T_r}} \left(\frac{W}{K} - V_r\right)\right), \tag{3}$$

where we have used the new dimensionless variables defined by

$$W = \frac{\omega}{\omega_{pi}} ,$$

$$K = k \lambda_{di} ,$$

$$I_r = \frac{\omega_{pe}^2}{\Omega_e^2} ,$$

$$K_r = \frac{k_z^2}{k^2} \frac{\omega_{pe}^2}{\omega_{pi}^2} ,$$

$$T_r = \frac{T_e}{T_i} ,$$

$$V_r = \frac{v_d}{v_i} ,$$

Note that the normalization is such that a wave with phase velocity W/K = 1 is traveling at the ion thermal velocity. For zero drift velocity, the I_r term would be responsible for the cold plasma resonance at the lower hybrid frequency. The K_r term, on the other hand, is responsible for the MTS behavior. Examination shows it to be a measure of the component of wave parallel to the magnetic field (small K_r, small k_z, etc.) The MTS behavior arises because the term can be used to represent an 'effective' electron mass. For example, $K_r = 1$ occurs when

$$\frac{k_z^2}{k^2} \omega_{pe}^2 = \omega_{pi}^2$$

and the electrons have a characteristic frequency equal to the ion plasma frequency due to the small $\hat{z}$ component to their motion and thus an equal 'effective' mass (the result can best be seen by skipping ahead to Eq. 5 where the presence of the K_r factor in the cold electron approximation adds a term of the form $\frac{\omega^2_{char.}}{\omega^2}$ to the susceptibility). In terms of the effective mass concept, it is worth nothing that $K_r < 1$ implies that the electrons act more massive than the ions, while for $K_r > 1$ the electrons act less massive than the ions.

For the magnetic ion susceptibility, it was found convenient to use the formulation which Lampe et al.[2] used as an electron susceptibility in their study of the beam cyclotron instability. Applying Lampe's result to ions, for the weak condition $k_\perp \rho_i > 1$, one can write the sum over the cyclotron harmonics of the full susceptibility to give:

$$\mu_i = \frac{1}{k^2\lambda_{di}^2}\left[1 + \frac{1}{2}\zeta\left[Z(\zeta) - Z(-\zeta) + i\cot\left(\pi\frac{\omega_{pi}}{\Omega_i}\omega\right)\cdot\left(Z(\zeta) + Z(-\zeta)\right)\right]\right] \tag{4}$$

where

$$\zeta = \frac{1}{\sqrt{2}}\ \frac{\omega}{k} = \frac{\omega}{\sqrt{2}\,k\,v_i}$$

Although this susceptibility was developed for waves perpendicular to the magnetic field, we are only interested in small k_z where $k_\perp/k \simeq 1$. The approximation should be good under the condition that ion cyclotron damping is unimportant, which is given by[3]

$k_z \rho_i \ll 1$, which is equivalent to

$$k_\perp \rho_e \ll \sqrt{\frac{T_r}{K_r}} \; , \text{ or with the normalized wave number}$$

$$K \ll \frac{1}{\sqrt{K_r I_r}}$$

Since we are interested in the realm $K_r \lesssim 1$ and $I_r \simeq 1$, this is not a severe limitation, since it just restricts us to wavelengths longer than the ion Debye length, a condition we might have expected in any event. The same limit has already been used to simplify the electron mobility term. It is worth nothing that, in the susceptibility, all cyclotron harmonic effects have been absorbed into the cot term.

Taken together, the dispersion relation obtained from these susceptibilities is subject to the following conditions:

$$\omega < \Omega_e \; ,$$

$$k \rho_i > 1 \; , \text{ and}$$

$$k \lambda_{di} < 1 \; .$$

The susceptibilities have been tested by taking them to various limits. It can be shown for example, that the ion susceptibility leads to normal Landau damped plasma waves in the zero magnetic field limit, and the full dispersion reduces to the normal MTS approximation in the cold limit and to the lower hybrid resonance for zero drift velocity.

III. RESULTS

Preliminary investigation of the dispersion relation has been carried out for a variety of situations. Although it has been found useful to make approximations to Z' in the electron susceptibility, the complete Z functions were used in the ion susceptibility in all cases (in fact difficulties arose in earlier attempts to use an expansion of Z for the ions due to convergence problems for large values of the argument). Because of computer overflow problems, it was also found useful to use a trigonometric identity for calculating the cot function,

$$\cot(x + iy) = \frac{\sin(2x) - i \sinh(2y)}{\cosh(2y) - \cos(2x)} .$$

All studies were made with $T_r = \frac{T_e}{T_i} = 1$ and $I_r = \frac{\omega_{pe}^2}{\Omega_e^2} = 1$ as being representative of the parameter range for the MTS behavior. With $I_r = 1$, one has $\sqrt{\frac{m_i}{m_e}} = \frac{\omega_{pi}}{\Omega_i}$ which was chosen to be equal to 4.5 (giving a mass ratio of 20.25) in order to simplify finding roots at multiple cyclotron harmonics.

Two general regimes were studied. The first case made use of a large argument expansion of the Z' function in the electron susceptibility. This stopped all electron damping effects and made it possible to isolate the pure ion cyclotron modifications to the MTS instability. The full Z' was also used in the electron susceptibility but with more incomplete results.

<u>CASE I</u>. Large Argument Expansion in the Electron Susceptibility

For large argument, one has

$$\frac{1}{2} Z'\left(\frac{W/K - V_r}{\sqrt{2\, K_r T_r}}\right) \approx \frac{K_r T_r}{\left(W/K - V_r\right)^2} \quad . \tag{5}$$

We should note that this expansion is always valid in the cold electron limit $T_r \to 0$ (except for those waves which have phase velocity exactly equal to the relative drift velocity $W/K = V_r$). For equal electron and ion temperatures, $T_r = 1$, and the validity of the expansion depends on the magnitudes of K_r and W/K relative to V_r. The range of parameters where the large argument expansion can cause difficulties is shown by the shaded diagonal area of Fig. 2, along with the limitations imposed by the basic dispersion relation shown by the surrounding shaded areas. The significance of its diagonal nature is that waves whose phase velocity are sufficiently close to the relative drift velocity of the electrons can be damped by electron motion along field lines. The width of the damping area, where the large argument expansion breaks down, is related to the magnitude of K_r, as we would expect, since K_r is a function of the wavelength along the magnetic field. For small K_r the wavelength along the magnetic field is long and little damping occurs, while for larger K_r the wave length is shorter and a greater range of waves can be damped. Although we donot want to forget about these possibilities of electron damping, it is worthwhile considering the electrons as cold for a while since much insight can be gained in how the MTS interaction evolves in regimes where ion cyclotron effect dominate.

Figures 3 and 4 show the evolution of the MTS interaction at two different values of K_r, $K_r = 1.0$ and $K_r = 0.1$ respectively. Each figure is broken up into a series of plots of the dispersion relation for increasing values of relative drift velocity. Figures 3a and 4a, which show the dispersion relation for $V_r = 0$, reveal the normal ion Bernstein modes and compare quite well with the results of previous studies.[4] A comparison of Fig. 3a with 4a indicates that the effect of larger values of K_r and thus k_z is to raise the frequencies of the Bernstein modes at the lower cyclotron harmonics. What happens when there is an increasing relative drift velocity is best evidenced by comparing the series in Fig. 3 which has a little more complete data. The series shows the first negative frequency cyclotron harmonic gradually shifting in the positive frequency direction under the influence of the relative drift velocity of the electrons. As it shifts far enough so that it can interact with the positive frequency cyclotron harmonics, coupling occurs and growing modes arise, at first as bands in K which then merge into a continuous region of growth for a relative drift velocity around $\tilde{V}_r > 2$.

Going through the series in Fig. 4 with smaller K_r and k_z and thus greater 'effective' electron mass, one can see the initiation of the unstable waves at lower relative drift velocities. The band structure shows up more clearly than in Fig. 3 and one can see that stable MTS waves exist between bands of unstable waves.

The lines representing those waves whose phase velocities are equal to the relative drift velocities are shown on the figures, and indicate the general area where electron damping effects could be important. Since the locus of growing roots generally follows the drift velocity lines, it is evident that damping effects can cause important modifications to the results shown in the two figures. In this regard, it is worth noting that the roots shown in Fig. 4 approach the drift velocity lines more closely than those of Fig. 3. This would appear to be related to the fact that the electrons of Fig. 4 with smaller K_r have a greater 'effective' mass. It is expected that the full dispersion would show damping to predominate at the lower drift velocities but that the information on the two figures should be an aid toward gaining accurate information on the threshold of the MTS interaction.

At larger values of drift velocity where the unstable bands have merged, the results look quite similar to what one obtains from the normal MTS cold dispersion relation, shown in Fig. 5. A comparison of Fig. 5 with Fig. 3f, 3g, and 3h shows the widths of the unstable bands to be about the same, but that the maximum growth rates are less than those predicted by the cold dispersion with the differences being greater at the lower drift velocities. For these larger values of the relative drift velocity, one still sees the presence of the Bernstein modes but it was not possible to get roots in areas near the dominant MTS interaction and thus impossible to tell exactly how the two regimes couple. Studies done with the full Z' in the electron susceptibility shed some light on this however.

Case II: Complete Electron Susceptibility

Results of this part of the study are rather scanty, but some comment can be made in reference to Figs. 6, 7 and 8 which show the growing modes for $K_r = 0.01$, 0.1, and 1.0, respectively, and fairly large drift velocities. A distinguishing feature of all three figures is that the MTS interaction excites growing Bernstein modes. In each of the figures the growth rates for each harmonic are shown with their baseline attached to the harmonic for which they apply. Fig. 6 with $K_r = 0.01$ and $V_r = 2$ shows no MTS behavior at all, only bands of excited cyclotron harmonics whose locus generally follows the line representing waves whose phase velocity matches the drift velocity. Fig. 7 on the other hand, with $K_r = 0.1$ and $V_r = 4.0$, is in a regime where the MTS interaction dominates. The phase velocities of the MTS waves are higher than those of the Bernstein modes, resulting in low coupling and only small growth rates for cyclotron harmonics. Fig. 8 with $K_r = 1.0$ and $V_r = 4.0$, shows a regime where both cyclotron and MTS effects are important. Although the MTS interaction dominates, the maximum growth of the excited cyclotron harmonics is more than 50% of the maximum MTS growth. These results are only indicative of the cyclotron effects to be observed and further study is needed to delineate the regions of importance and the instability thresholds.

SUMMARY

A dispersion relation has been obtained that includes magnetic ion effects on the MTS interaction. A preliminary study of its effects indicate that magnetic ion effects should be important in determining the threshold of the MTS interaction and that excitation of growing cyclotron modes add a new facet to the MTS interaction at somewhat larger values of drift velocity.

ACKNOWLEDGMENTS

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4. R. W. Fredricks, J. Plasma Physics, 2, Pt. 2, 197 (1968).

FIGURE CAPTIONS

Fig. 1 The MTS interaction is produced when electrons drift across a magnetic field relative to the ions. Waves with a small component $\underset{\sim}{k}_z$ parallel to the magnetic field are unstable.

Fig. 2 Shaded regions show areas where the dispersion relation may be invalid. Diagonal shaded region in the center indicates where the large argument expansion of Z' used in Case I breaks down.

Fig. 3 Plots of the Case I dispersion relation for $K_r = 1.0$ and various relative drift velocities. Frequencies are normalized to the ion cyclotron frequency and wave numbers are normalized to the ion Debye length. The diagonal lines show where phase velocities are equal to the relative drift velocities. Solid lines indicate stable waves; for instability, the real part of the frequency is represented by dashed lines and the imaginary part by dotted lines. (a) through (h) for $V_r = 0, 0.25, 0.5, 0.75, 1.0, 2.0, 4.0, 8.0$, respectively.

Fig. 4 Same general diagram as Fig. 3 except that here $K_r = 0.1$, with (a) through (h) for $V_r = 0, 0.25, 0.5, 0.75, 1.0, 1.5, 2.0\ 4.0$.

Fig. 5 Real and imaginary parts of the roots of the cold MTS dispersion relation for $K_r = 1.0$ and $V_r = 2, 4$, and 8. Roots are drawn to same scale as in Fig. 3.

Fig. 6 Roots to full dispersion with $K_r = 0.01$, and $V_r = 2.0$. Growing roots of cyclotron harmonics are shown with baselines attached to the harmonics for which they apply.

Fig. 7 Roots to full dispersion with $K_r = 0.1$ and $V_r = 4.0$.

Fig. 8 Roots to full dispersion with $K_r = 1.0$ and $V_r = 4.0$.

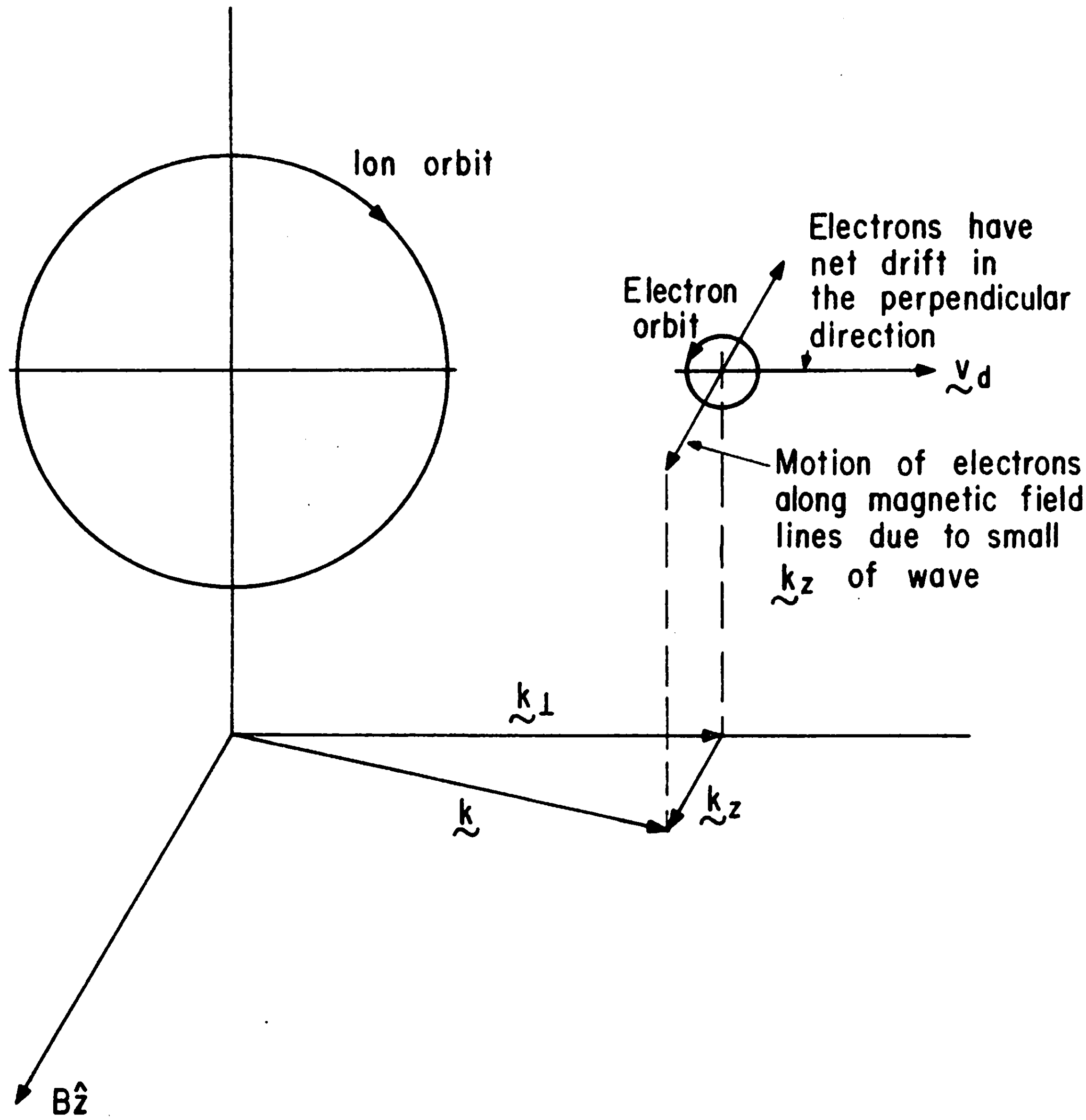

Ion orbit
Electrons have net drift in the perpendicular direction
Electron orbit
$\underset{\sim}{v}_d$
Motion of electrons along magnetic field lines due to small $\underset{\sim}{k}_z$ of wave
$\underset{\sim}{k}_\perp$
$\underset{\sim}{k}$
$\underset{\sim}{k}_z$
$B\hat{z}$

Fig. 1

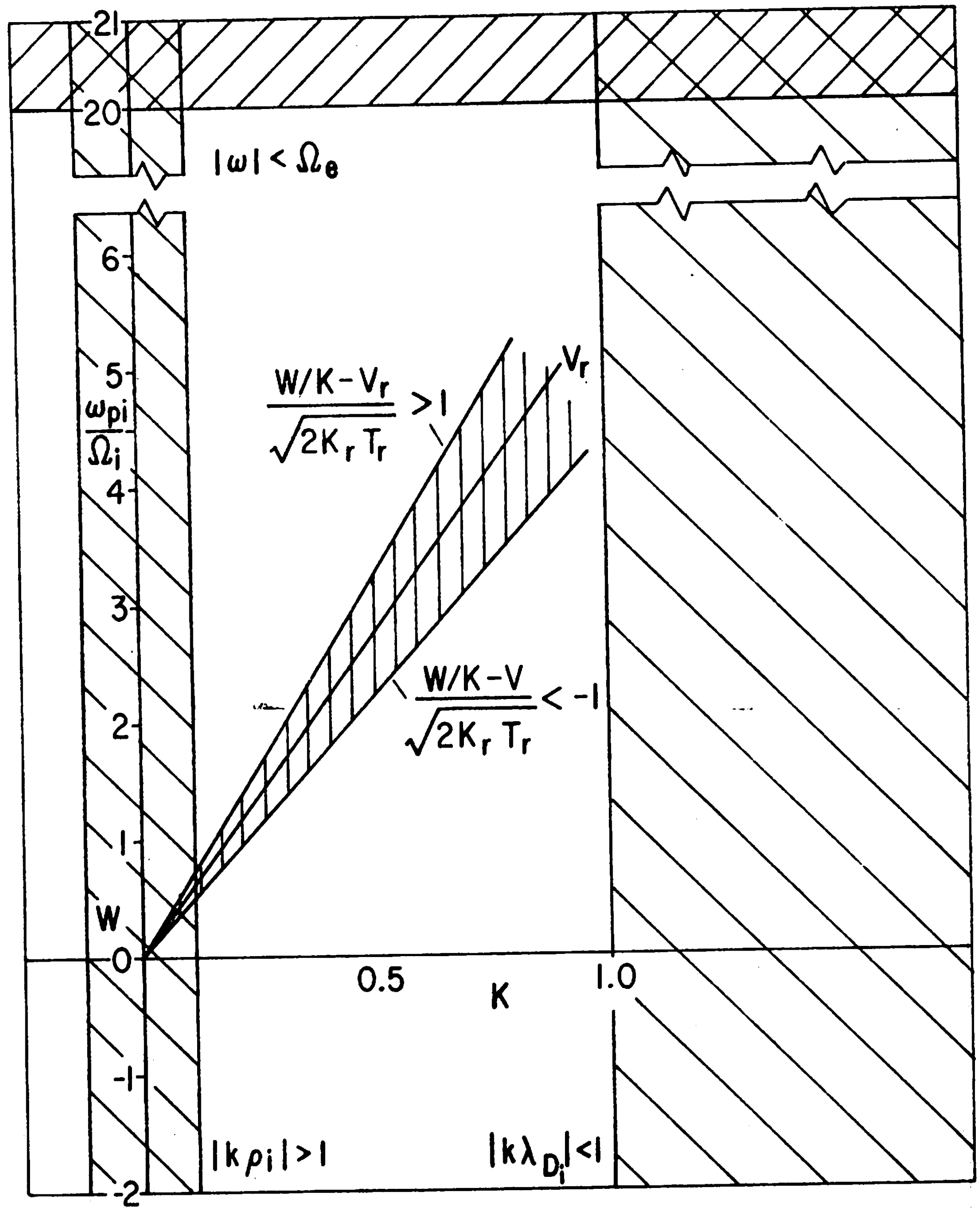

21
20
|ω| < Ωe
6
5
ωpi/Ωi
4
3
2
1
W
0
-1
-2
0.5
K
1.0
(W/K − Vr)/√(2Kr Tr) > 1
Vr
(W/K − V)/√(2Kr Tr) < −1
|kρi| > 1
|kλDi| < 1

Fig. 2

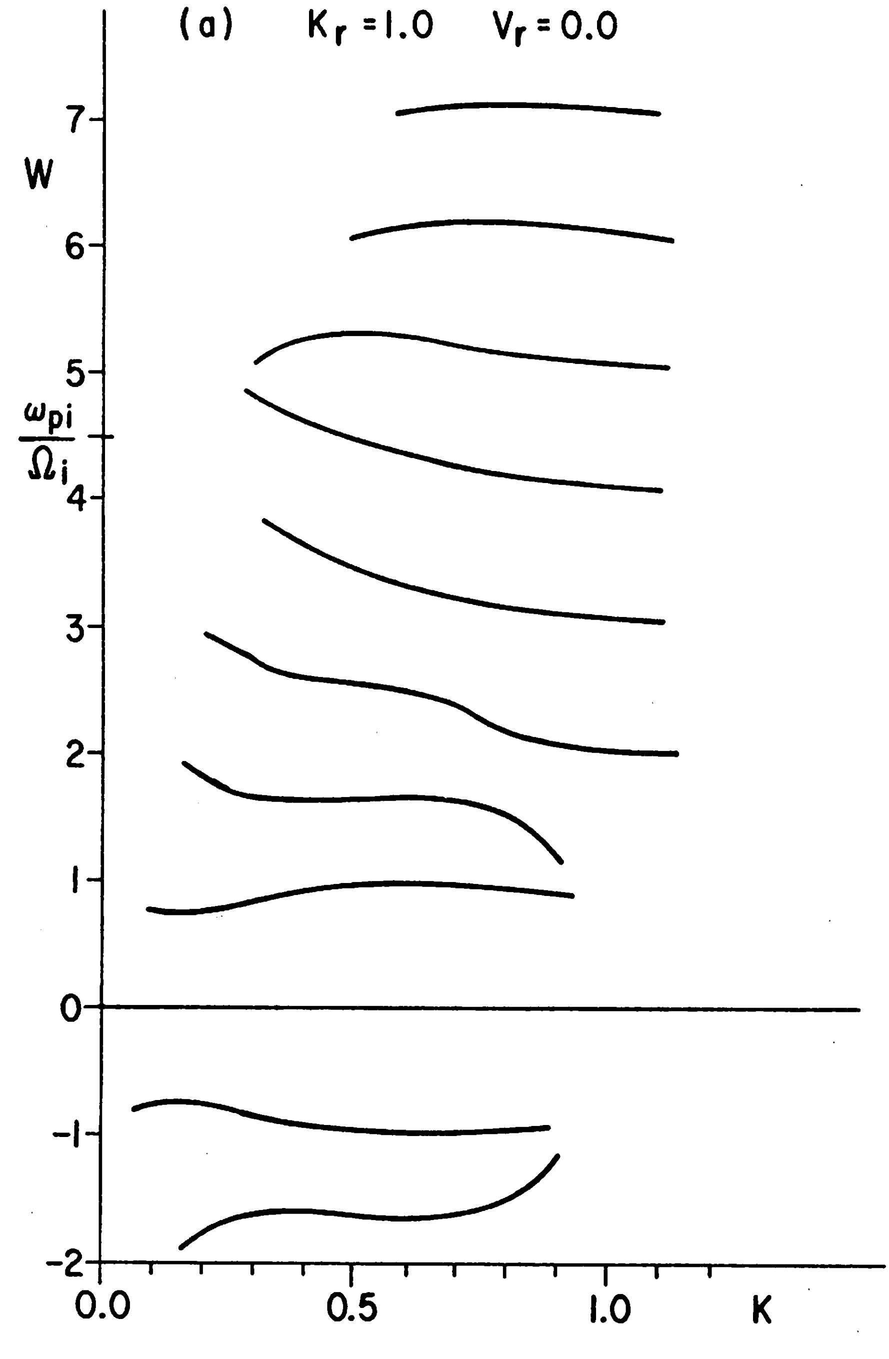


Fig. 3a

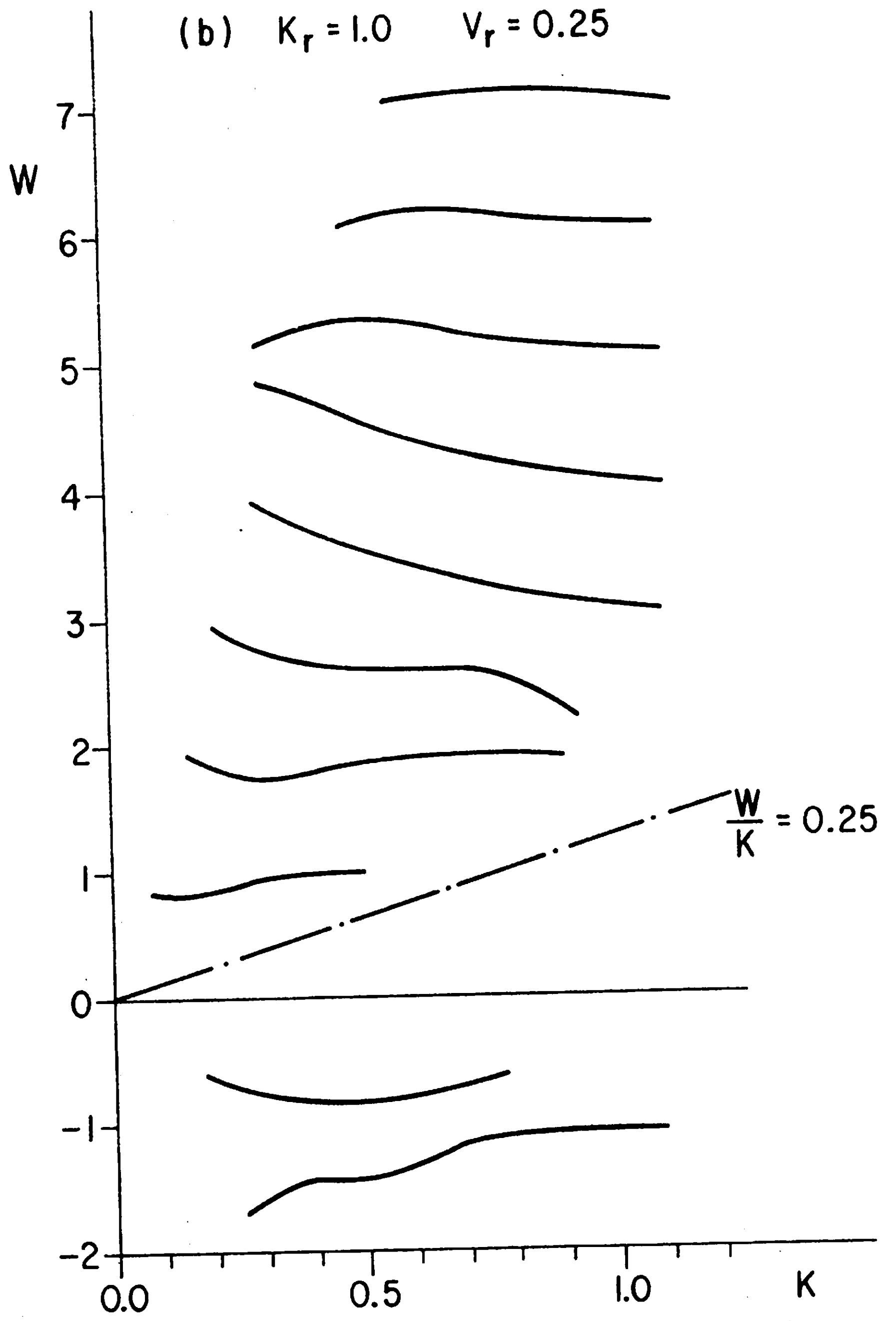


Fig. 3b

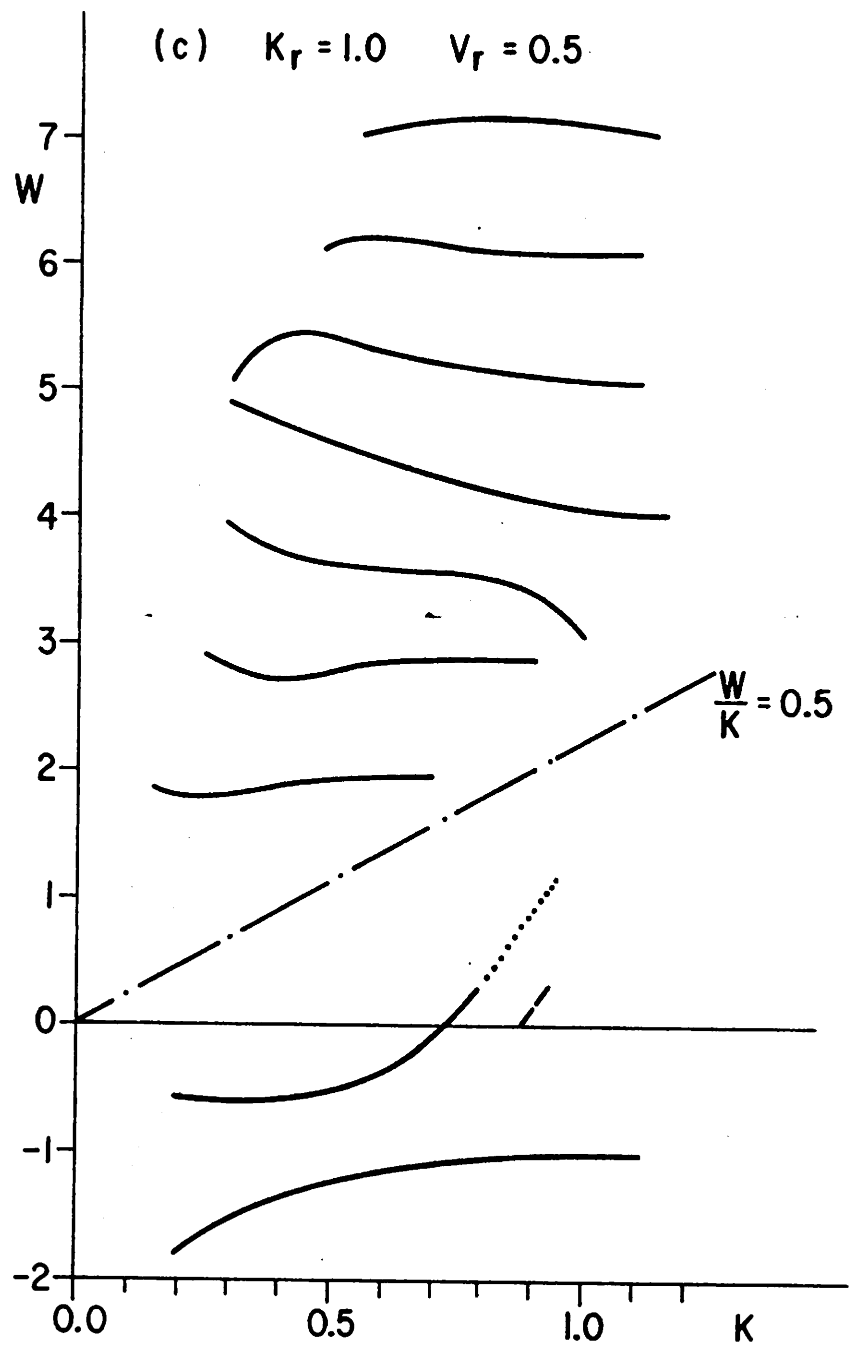


Fig. 3c

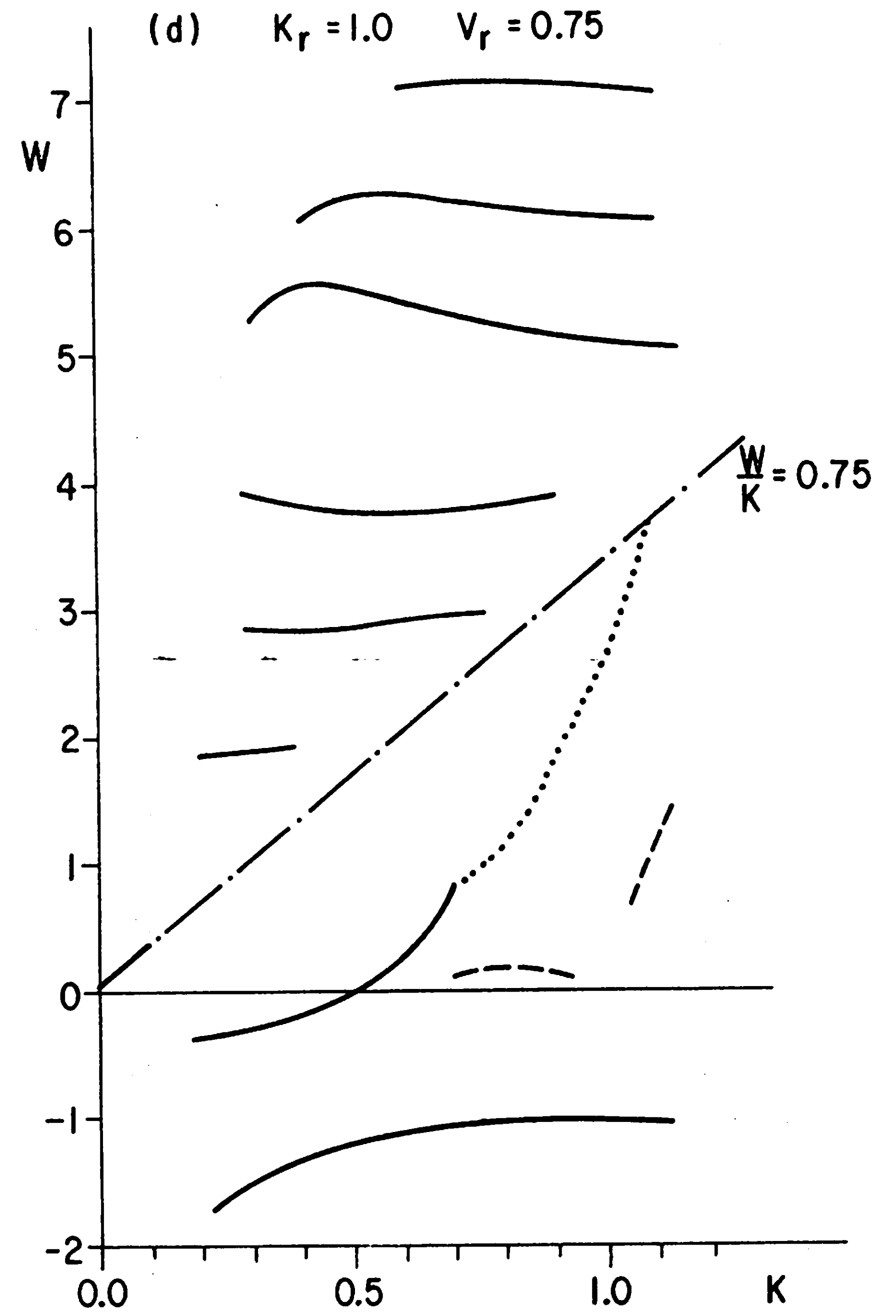


Fig. 3d

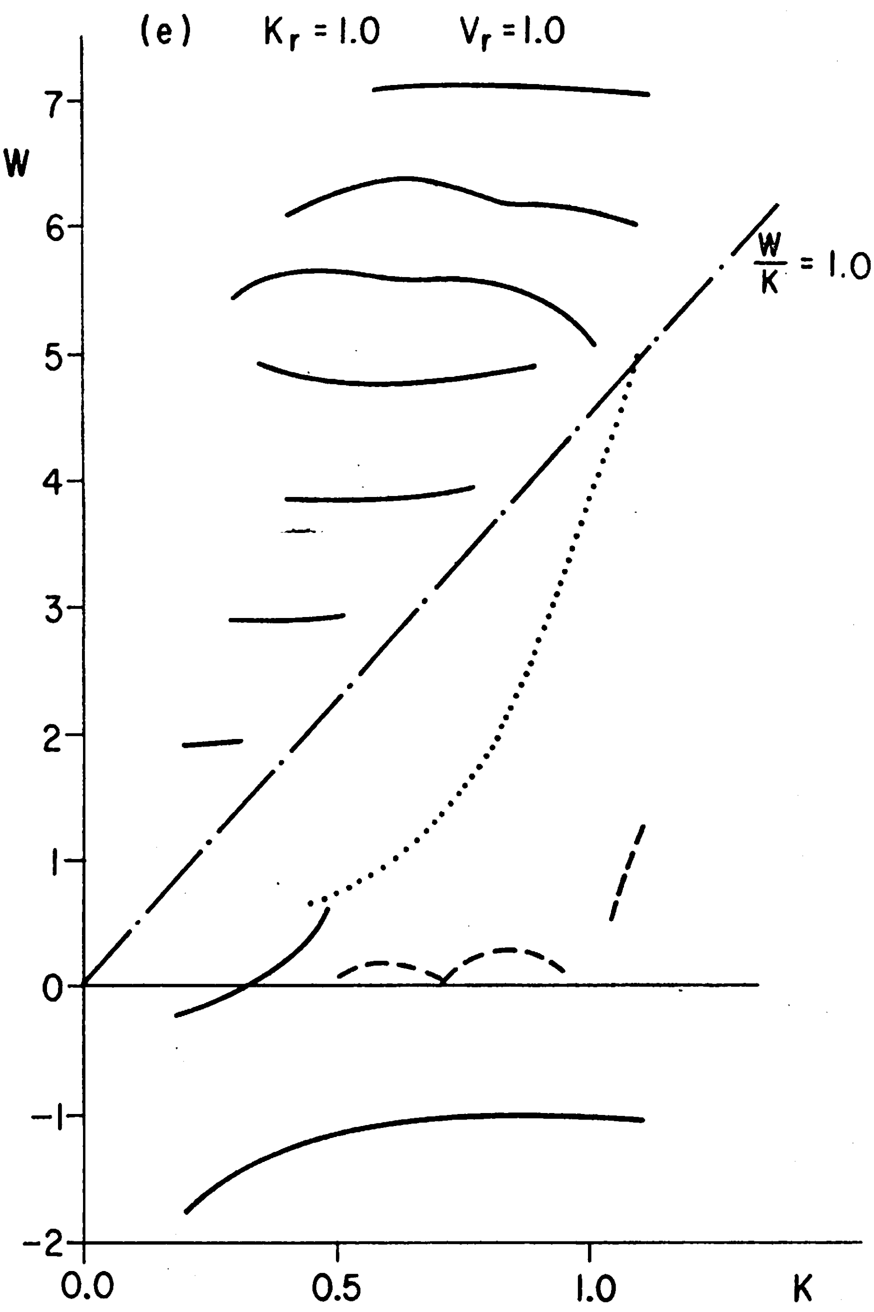


Fig. 3e

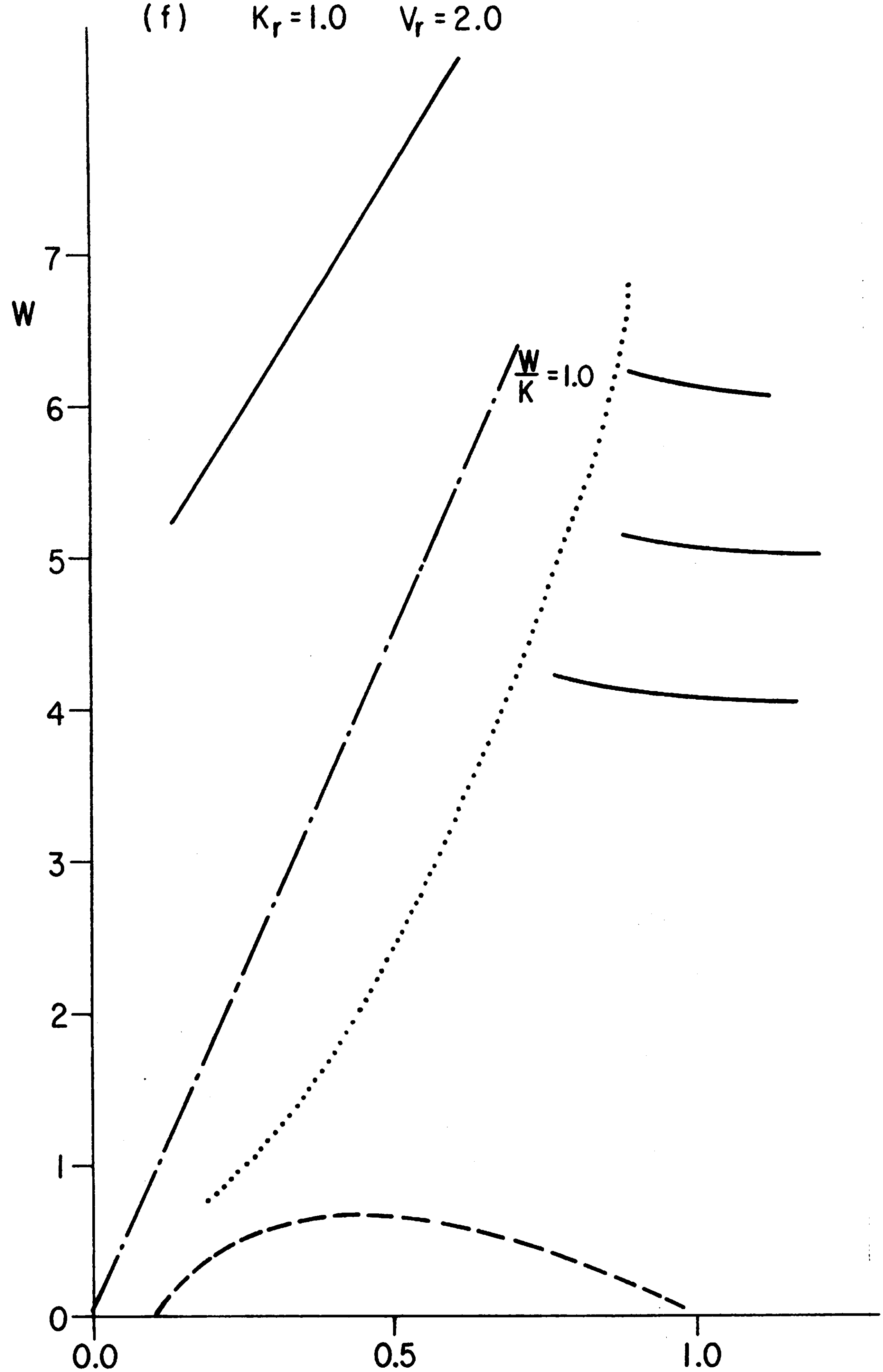


Fig. 3f

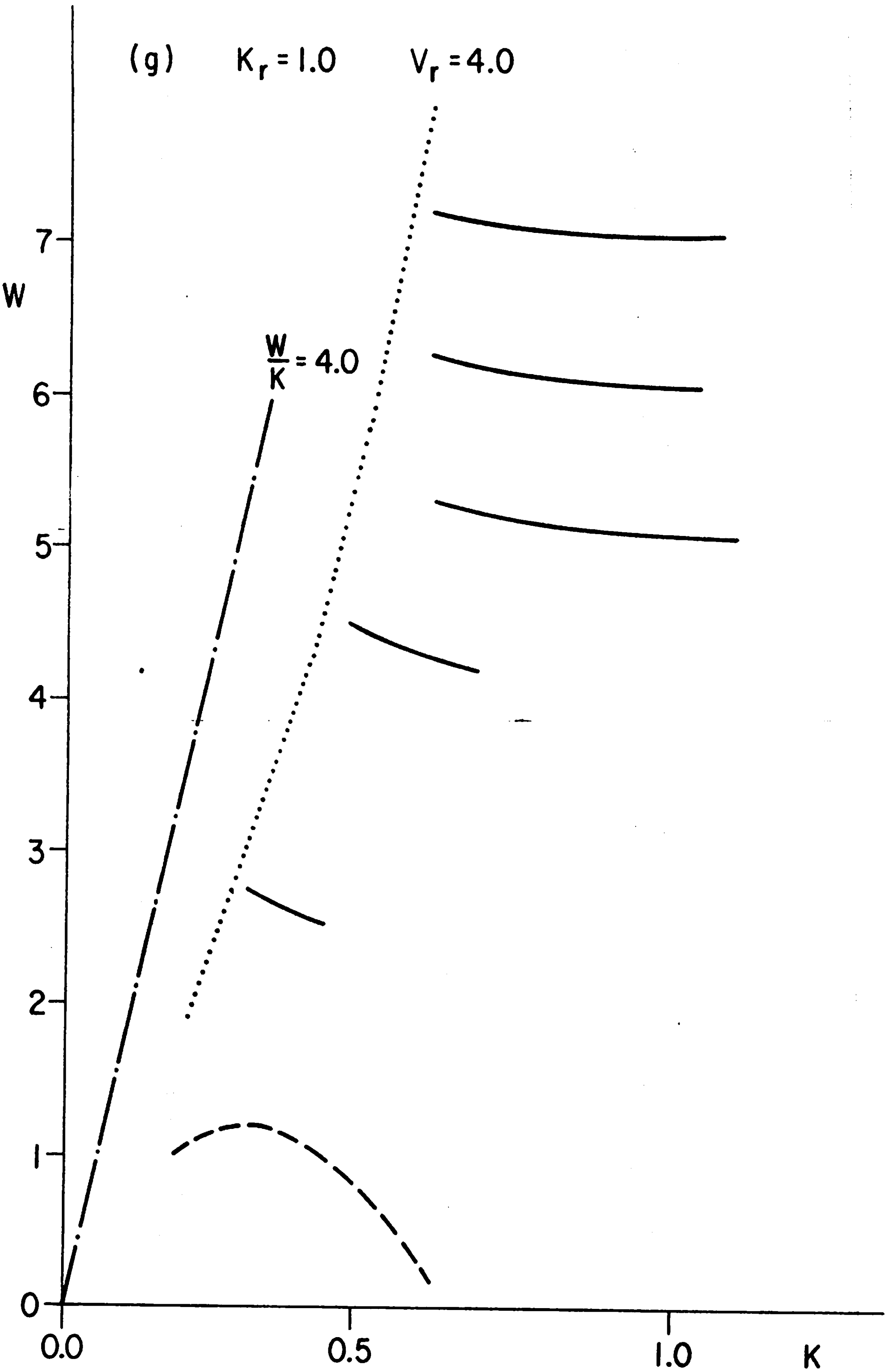


Fig. 3g

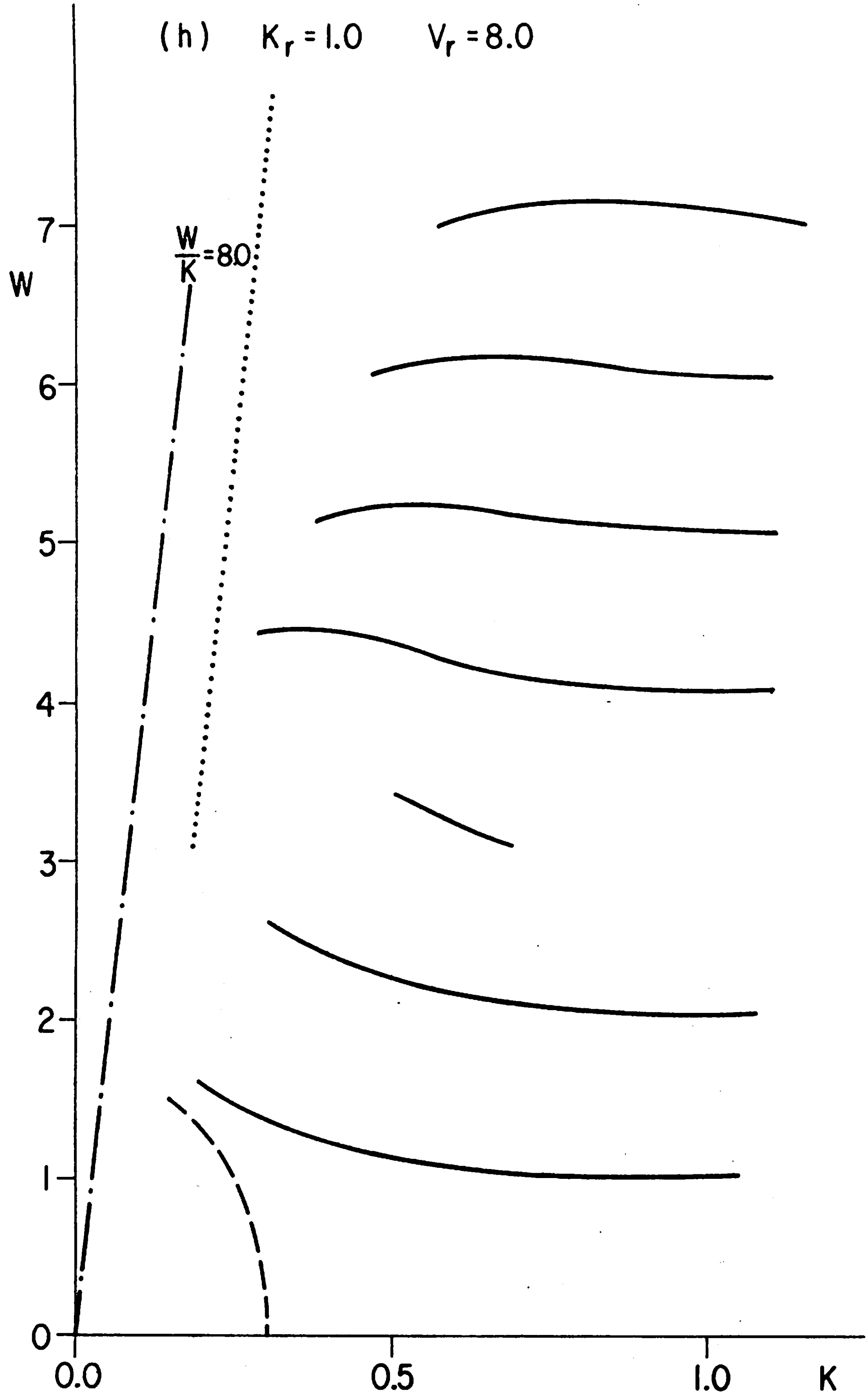


Fig. 3h

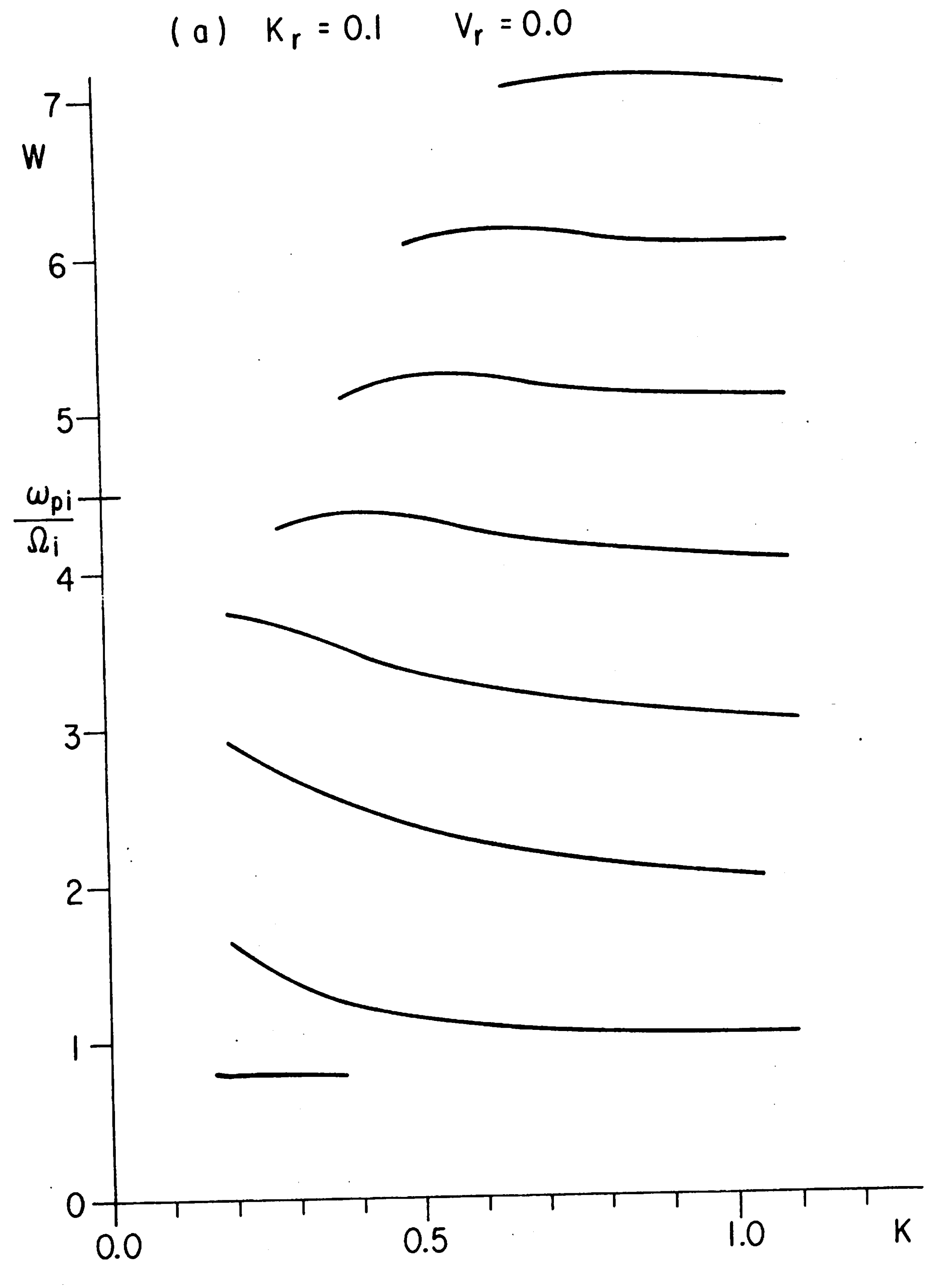


Fig. 4a

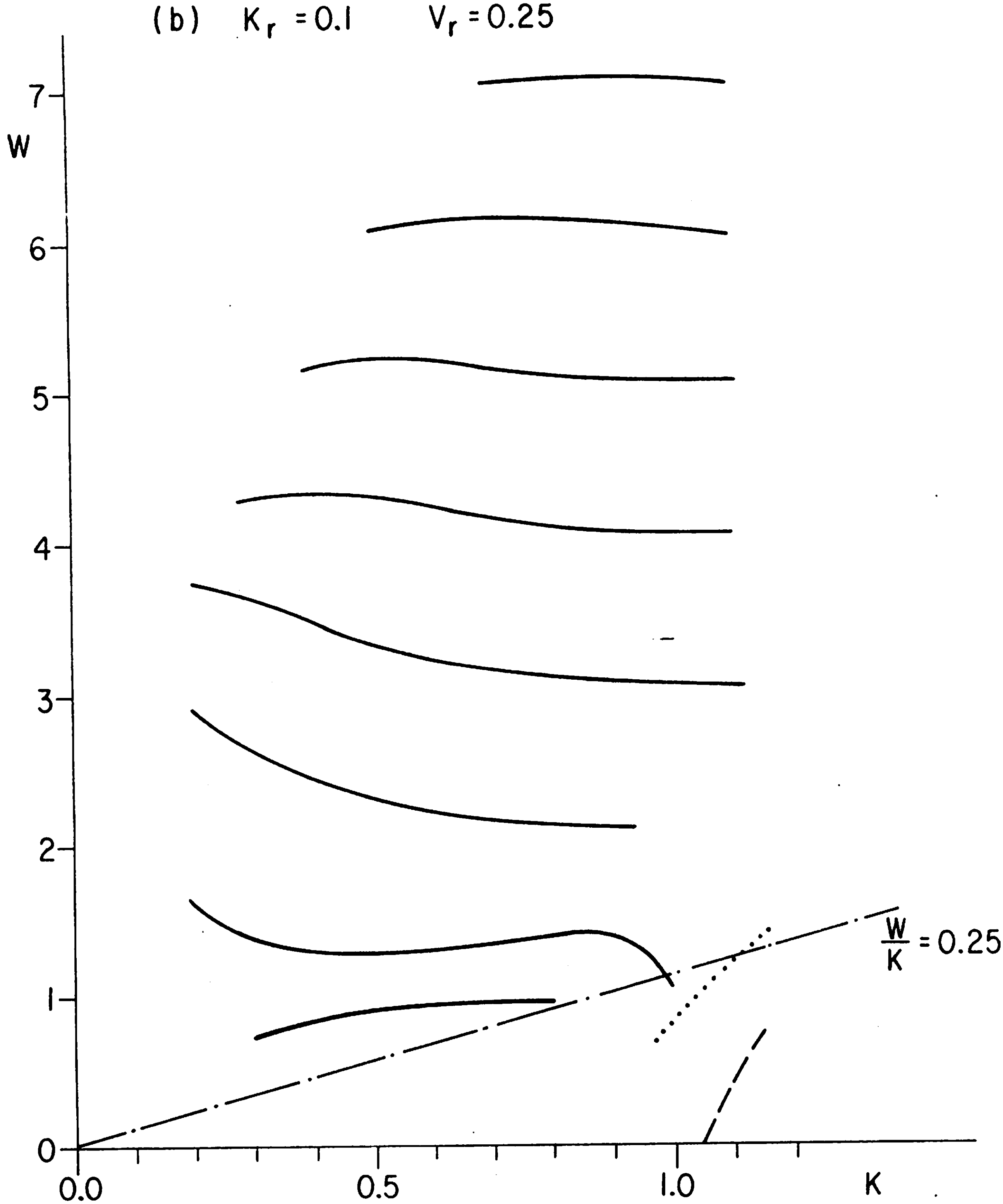


Fig. 4b

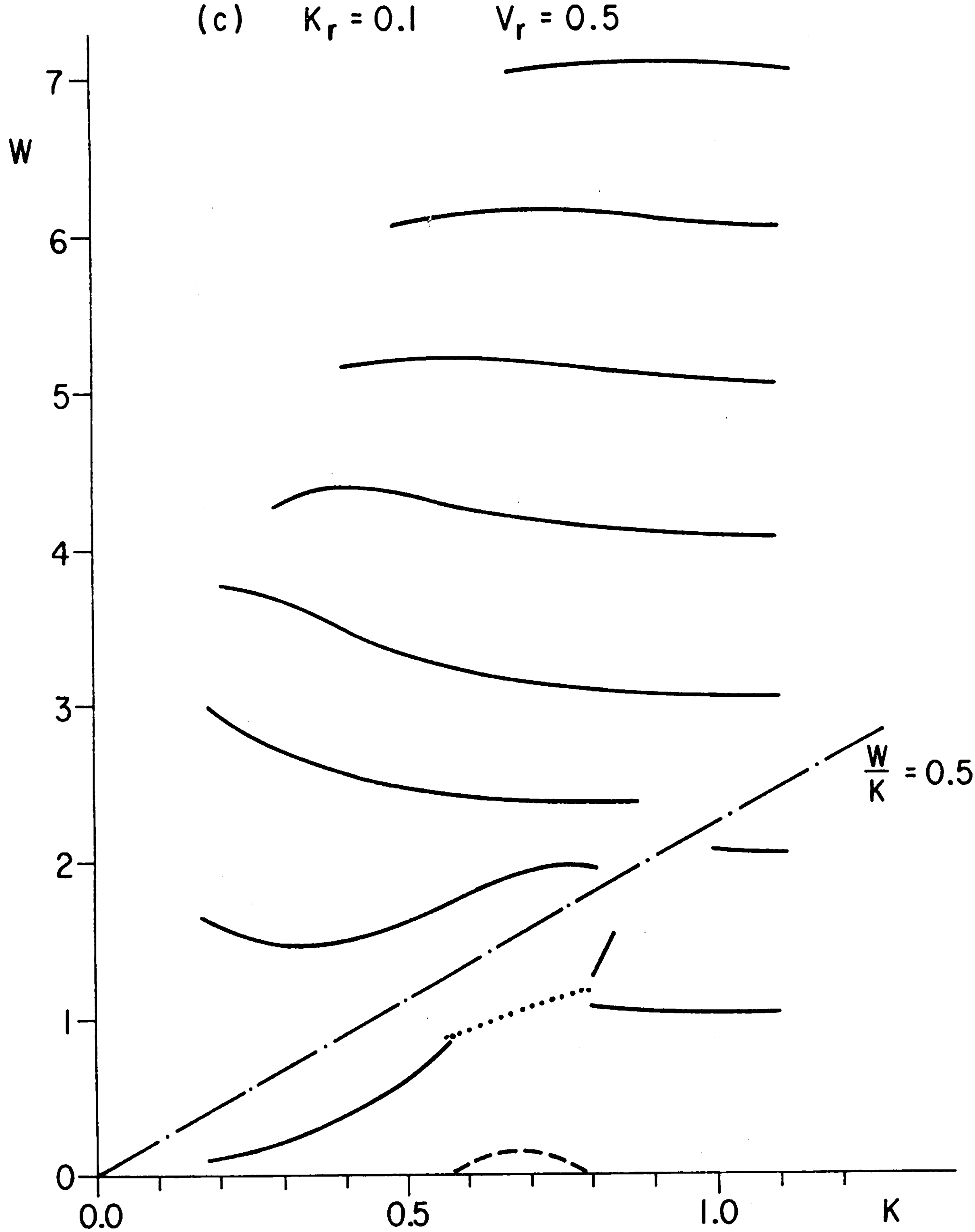


Fig. 4c

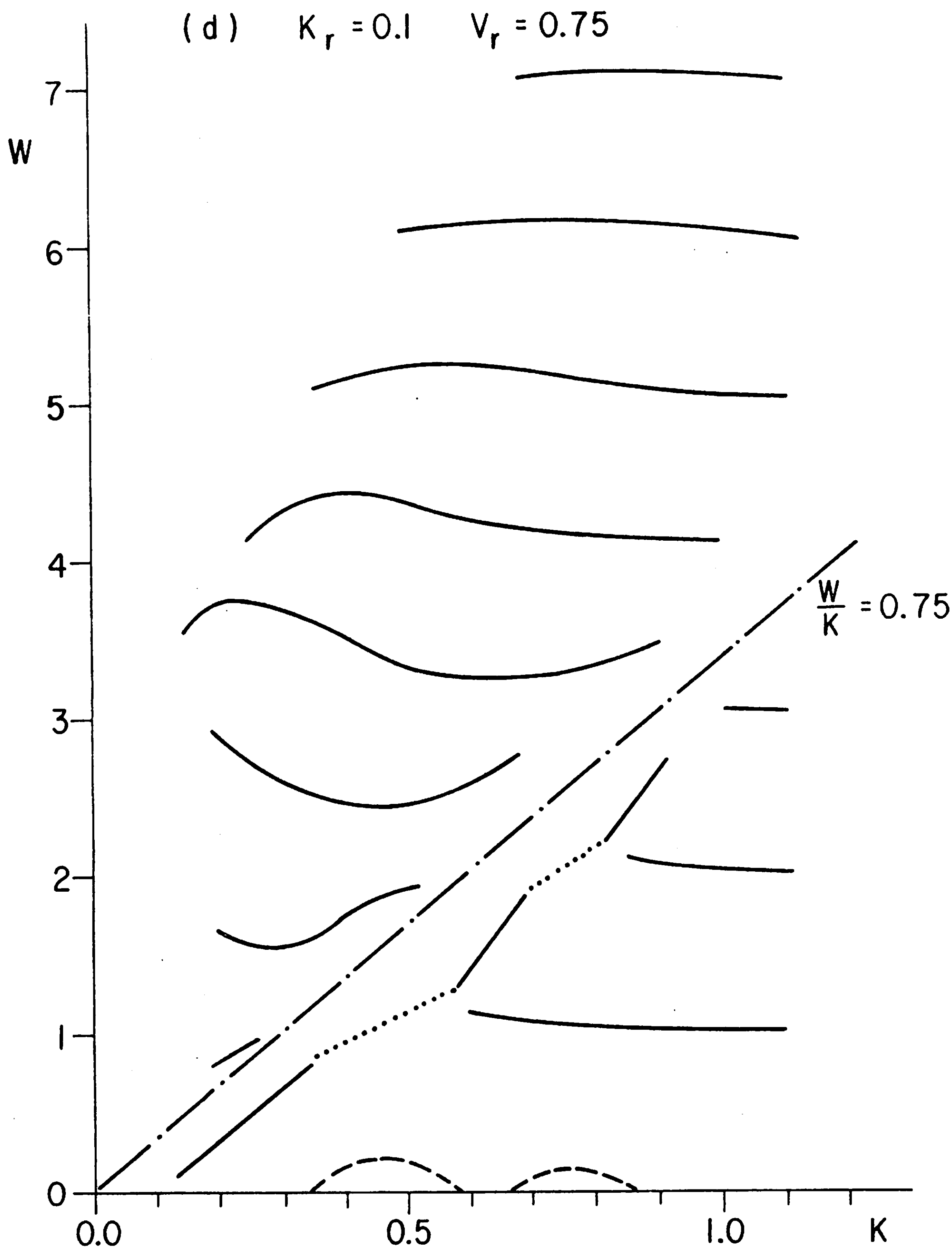


Fig. 4d

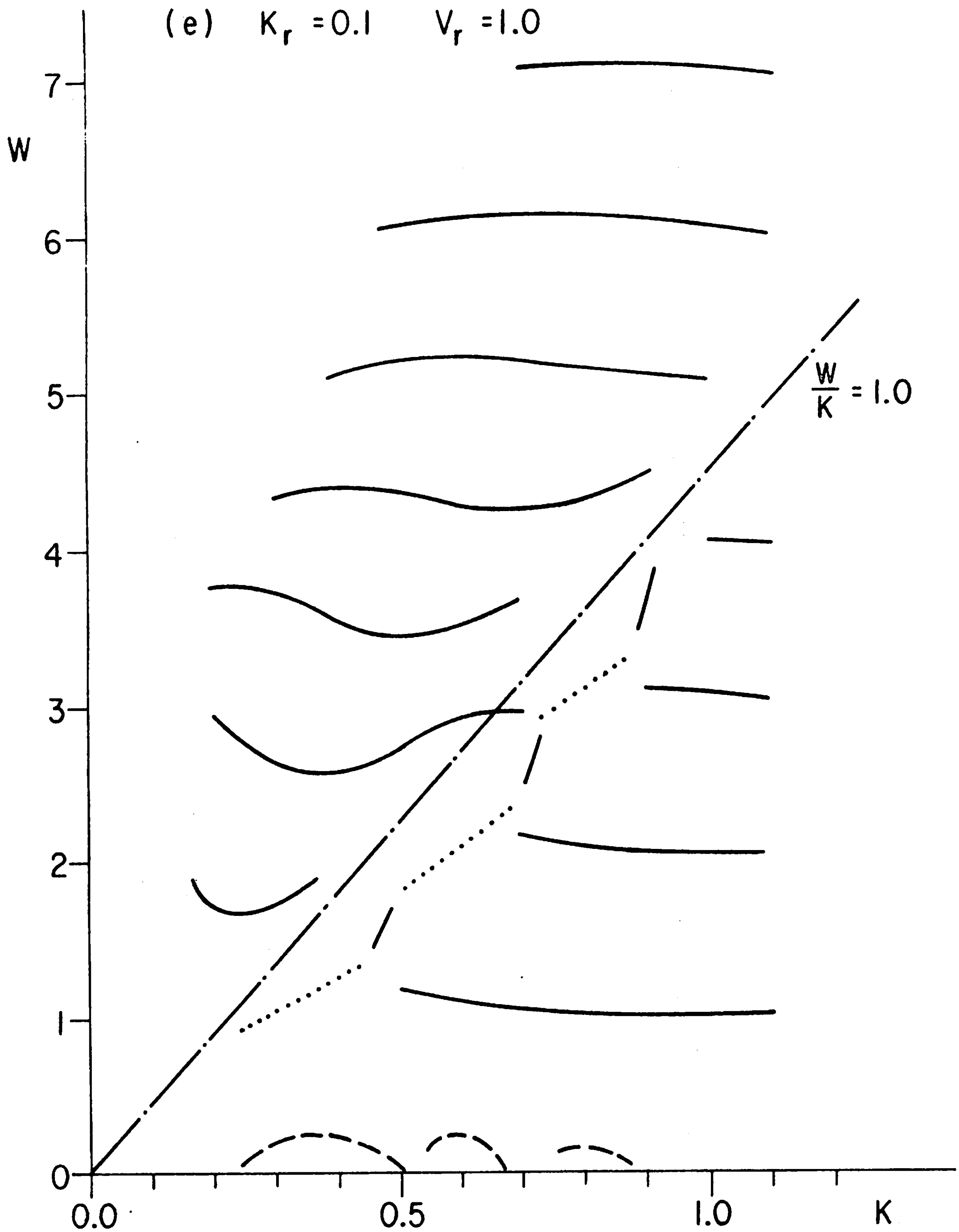


Fig. 4e

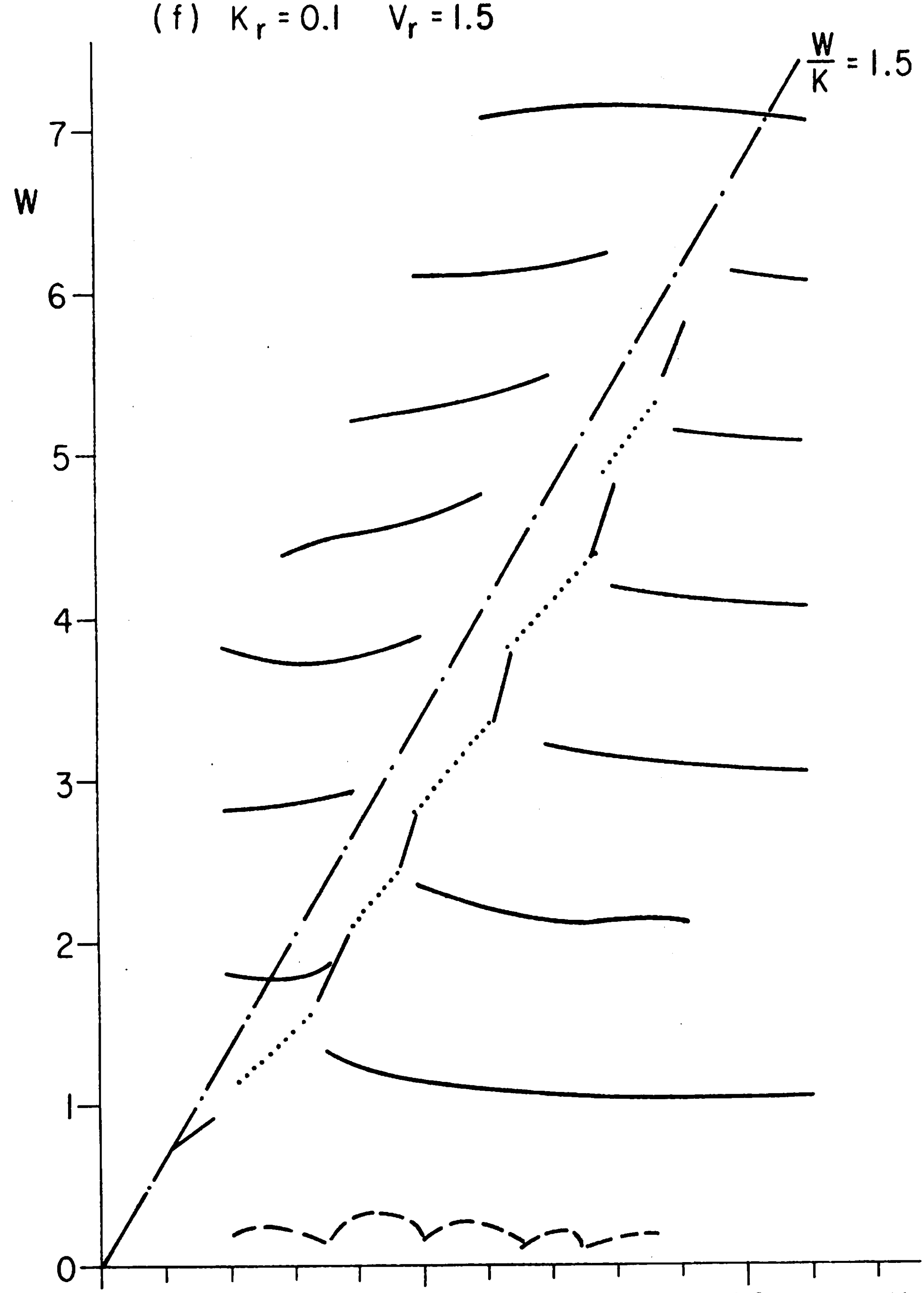


Fig. 4f

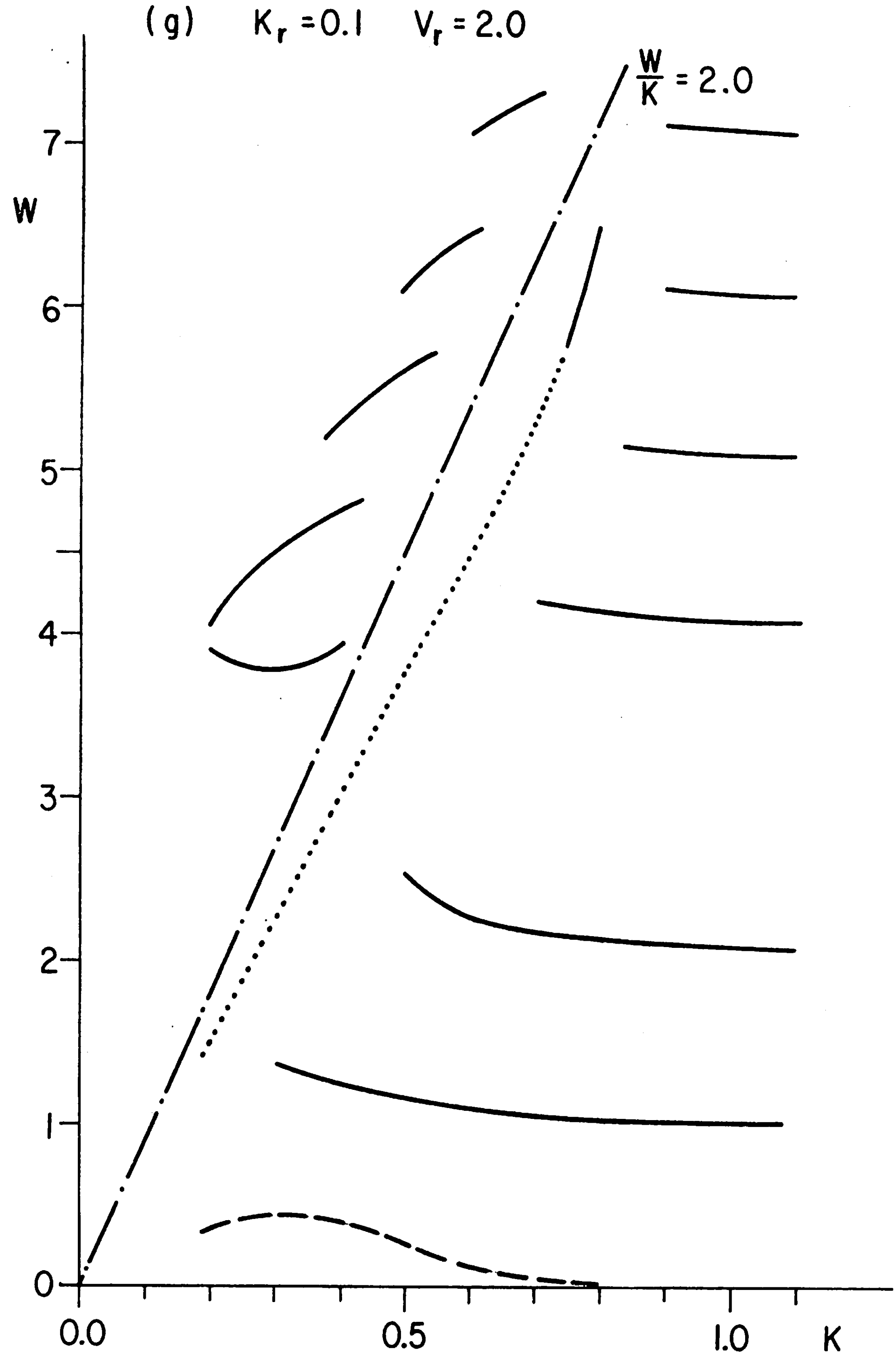


Fig. 4g

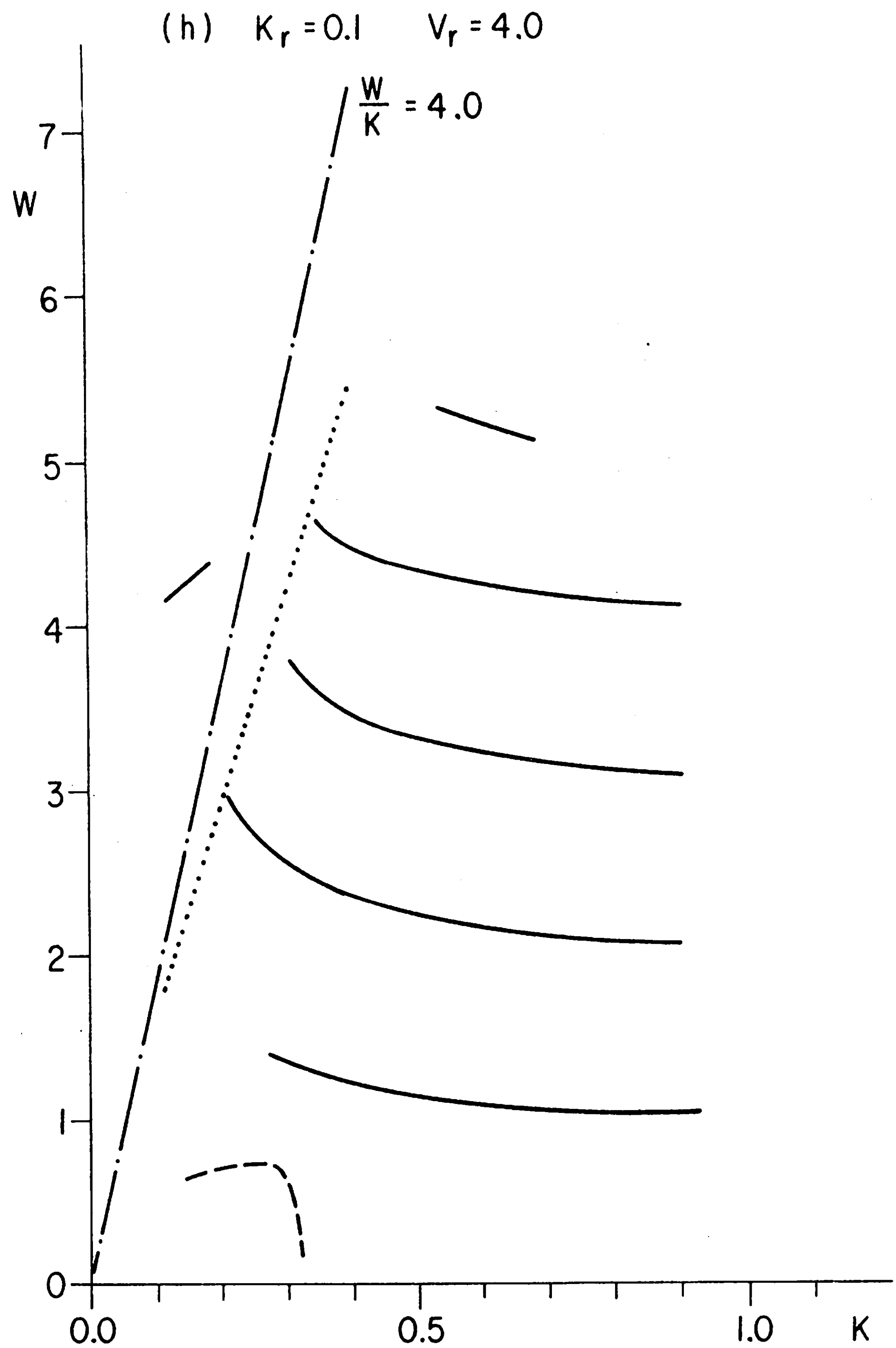


Fig. 4h

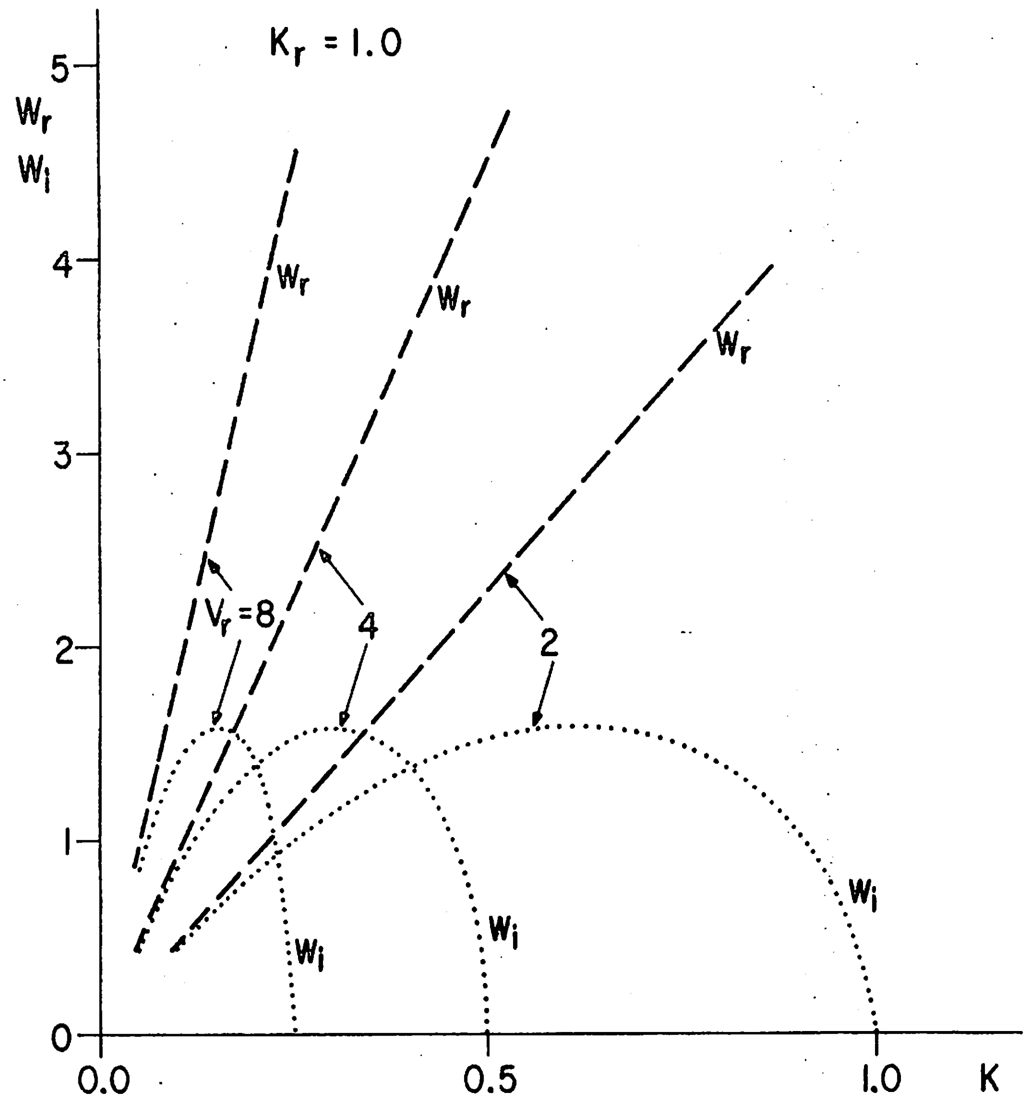
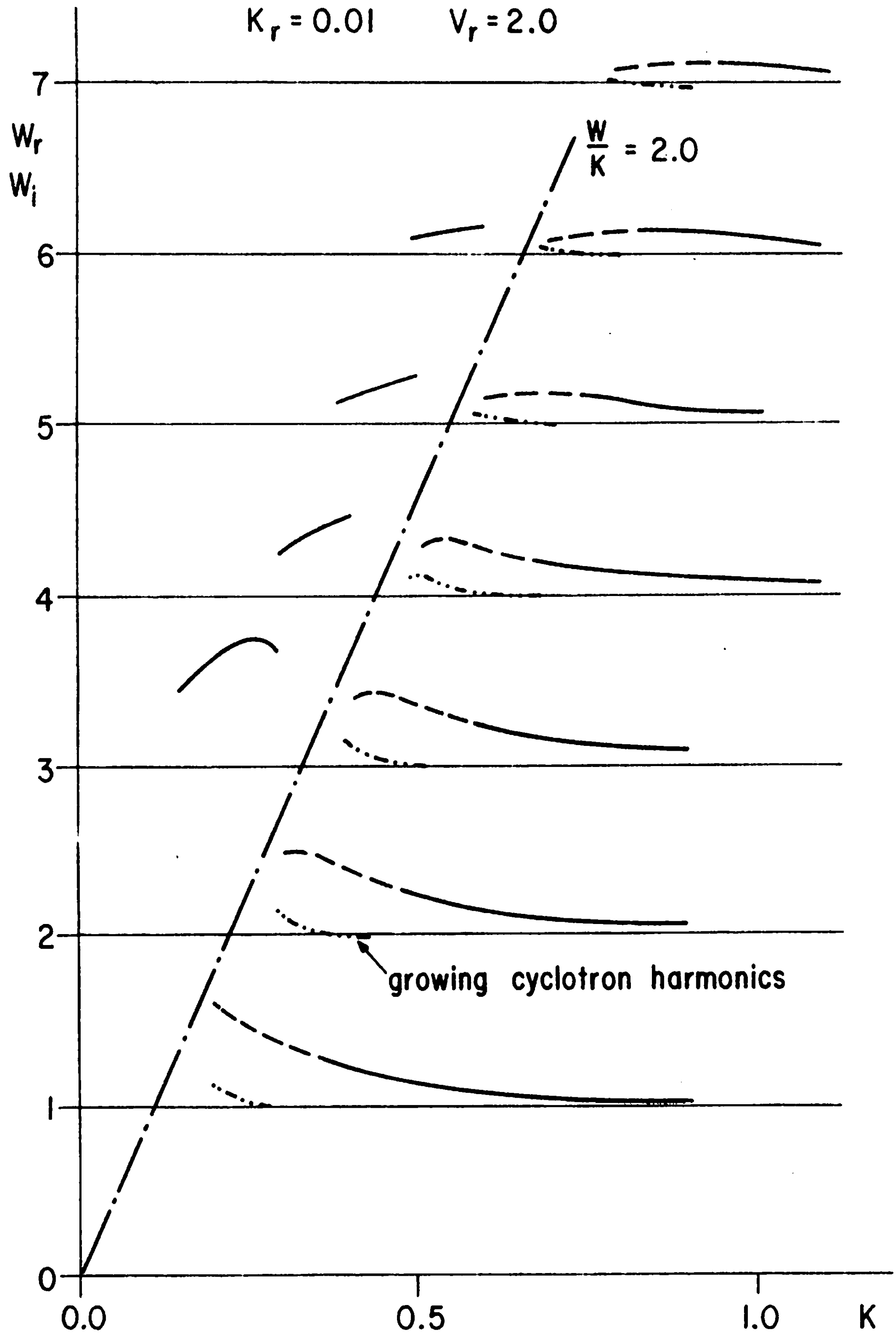
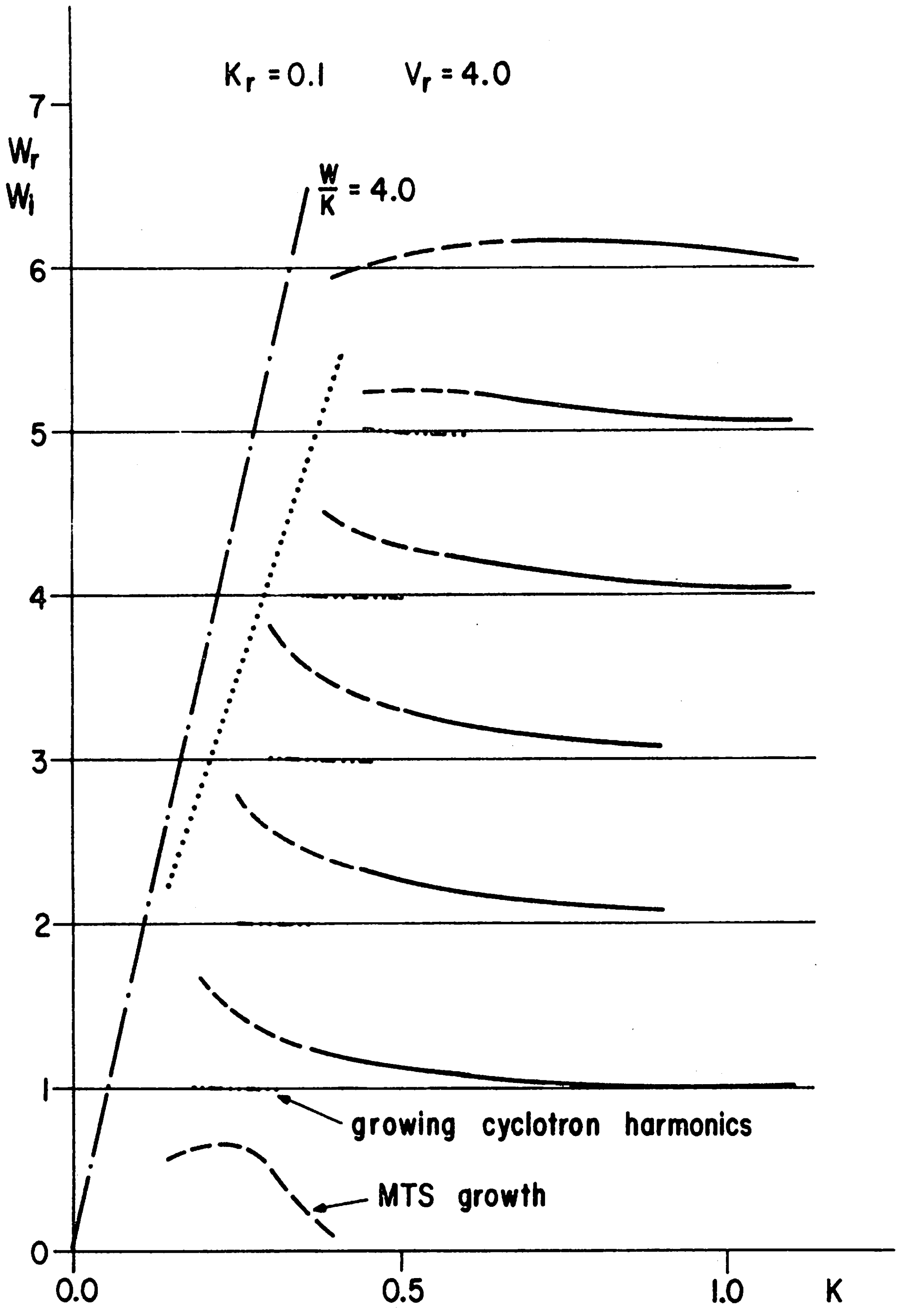


Fig. 5



$K_r = 0.01$ $V_r = 2.0$
W_r
W_i
$\frac{W}{K} = 2.0$
growing cyclotron harmonics
7
6
5
4
3
2
1
0
0.0
0.5
1.0
K

Fig. 6



K_r = 0.1
V_r = 4.0
W_r
W_i
W/K = 4.0
7
6
5
4
3
2
1
0
0.0
0.5
1.0
K
growing cyclotron harmonics
MTS growth

Fig. 7

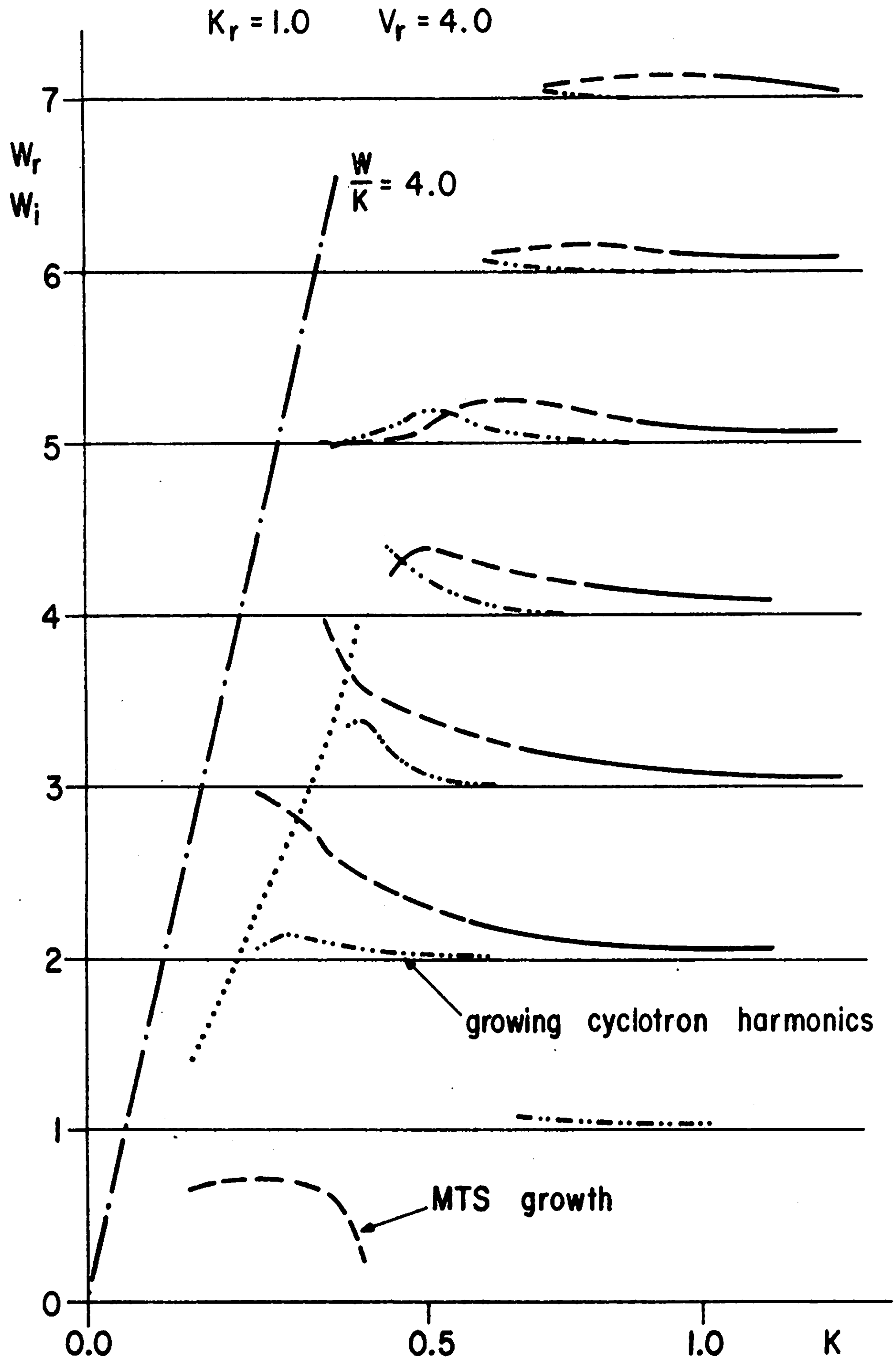

$K_r = 1.0$ $V_r = 4.0$
W_r
W_i
$\frac{W}{K} = 4.0$
7
6
5
4
3
2
1
0
growing cyclotron harmonics
MTS growth
0.0
0.5
1.0
K

Fig. 8