ON THE EQUIVALENCE OF MINTY'S PAINTING THEOREM AND TELLEGEN'S THEOREM

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ABSTRACT

We show in this paper that Minty's Painting Theorem and Tellegen's Theorem are equivalent. We also present a generalization of Minty's Theorem to vector spaces over the real field and a new proof of the theorem.

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1. Introduction

Tellegen's Theorem and Minty's painting Theorem are widely recognized as two of the most basic results in network theory [1], [2]. Since they are both purely topological it is of some interest to explore their relationship to each other. We show in this paper that they are essentially equivalent. It is however rather difficult to give a convincing proof of this fact by simply using one of these results to prove the other, for, in the process, we could also be using other properties of graphs implicitly. We therefore proceed as follows: we state Minty's painting condition for two different graphs and show that if two graphs satisfy this condition their respective coboundary and cycle spaces must be complementary orthogonal. Next we generalize Minty's Painting Conditions to vector spaces over real fields and show that if two spaces $\mathcal{V}_1$, $\mathcal{V}_2$ are complementary orthogonal they must satisfy Minty's Painting Condition.

2. Preliminaries

We deal with finite sets throughout. If S is a set then $|S|$ is the cardinality of S. A ***vector*** $\mathbf{f}$ on S over the real field $\mathbb{R}$ is a function $\mathbf{f}$: $S \rightarrow \mathbb{R}$. *Addition, scalar multiplication* and *linear combination* are defined as usual for vectors on the same set. A collection of vectors is said to be a ***vector space*** if it is closed under addition and scalar multiplicaiton. *Rank* of a vector space $\mathcal{V}$ is denoted $\mathbf{r}(\mathcal{V})$. *Support* of a vector $\mathbf{f}$ is denoted $\|\mathbf{f}\|$ and is the set of all elements on which $\mathbf{f}$ takes nonzero values. A vector $\mathbf{f}$ is said to be *minimal* in $\mathcal{V}$ iff no $\mathbf{g} \in \mathcal{V}$ such that $\|\mathbf{g}\|$ is properly contained in $\|\mathbf{f}\|$. If vectors $\mathbf{f}$, $\mathbf{g}$ are on S then $\langle \mathbf{f},\mathbf{g} \rangle \equiv \sum_{e \in S} f(e)g(e)$. $\mathbf{f}, g$ are said to be *orthogonal* iff $\langle \mathbf{f},\mathbf{g} \rangle = 0$. The collection of all vectors orthogonal to every vector in a vector space $\mathcal{V}$ on S forms another vector space. We denote it by $\mathcal{V}^*$. $\mathcal{V}, \mathcal{V}^*$ are said to be complementary orthogonal. Let $\mathcal{V}$ be a vector space on S. Then

$\nu \cdot T \equiv \{\mathbf{f}_T$: there exists $\mathbf{f}_s \in \nu$ such that $\mathbf{f}_T = \mathbf{f}_S / T\}$

$\nu \times T \equiv \{\mathbf{f}_T$: there exists $\mathbf{f}_S \in \nu$ such that $\mathbf{f}_T = \mathbf{f}_S / T$ and $\mathbf{f}_S(e) = 0, e \in S - T\}$

We will use the following simple results without proof (See [3]).

Theorem P1. Let ν be a vector space on S. Then

(a) $(\nu^*)^a = \nu$

(b) $(\nu) + r(\nu^*) = |S|$

Theorem P2. Let ν be a vector space S. Let $T \subseteq S$. Then

(a) $(\nu \cdot T)^* = \nu^* \times T$

(b) $(\nu \times T)^* = \nu^* \cdot T$

We assume familiarity with the usual definitions of directed graphs, circuits, cutsets, forests, coforests, fundamental circuit matrix, fundamental cutset matrix, circuit vector, cutset vector, etc. We denote the fundamental circuit of a forest T of graph G with respect to the edge e outside T as $L(G, e, T)$ (L for 'loop') and the fundamental cutset of a coforest $\overline{T}$ of graph G with respect to the edge e outside $\overline{T}$ as $C(G, e, \overline{T})$. The space of vectors generated by the rows of a fundamental circuit (cutset) matrix of G is called the space of cycles (coboundaries) of G and is denoted by $\nu_{cy}(G)(\nu_{cob}(G))$.

3. Minty's Painting Theorem Implies Tellegem's Theorem

Definition 3.1. Let G be a directed graph on S. Let S_p, S_q be disjoint subsets of S. We say that a cutset (circuit) T is *pq-directed* iff $T \subseteq S_p \cup S_q$ and all the edges of $T \cap S_q$ are similarly oriented in the cutset (circuit). We say T *meets* e iff $e \in T$.

We now state Minty's painting condition for graphs (MPCG).

Definition 3.2: MPCG: Let G_1, G_2 be directed graphs on S. We say that the circuits of G_1 and the cutsets of G_2 satisfy MPCG iff *exactly one* of (a), (b) below is true for every partition of S into sets S_r ('red'), S_b ('blue'), S_g ('green') with e^* (dark green) belonging to S_g.

(a) there exists an rg-directed circuit in G_1 that meets e^*

(b) there exists a bg-directed cutset in G_2 that meets e^*

We now show that if circuits of G_1 and cutsets of G_2 satisfy MPCG then $\nu_{cob}(G_2)$, $\nu_{cy}(G_1)$ are complementary orthogonal. Note that this is essentially the same as saying that Minty's painting theorem (which states that circuits of G_1 and cutsets of G_2 satisfy MPCG) implies Tellegen's theorem. We need the following simple Lemmas.

Lemma 3.1. Let G_1, G_2 be directed graphs on S. Let circuits of G_1 and cutsets of G_2 satisfy MPCG. Then no coforest of G_1 contains a cutset of G_2 i.e., every coforest of G_1 is contained in some coforest of G_2.

Proof. Let coforest $\bar{T}_1$ of G_1 contain cutset C_2 of G_2. Let $e \in C_2$ choose $(\bar{T}_1 - e)$ as S_b (blue), forest T_1 as S_r (red) and $\{e\}$ as S_g (green). Let e be chosen as e^*. Let $L_1 = L(G_1, e, T_1)$. Observe that the simultaneous existence of L_1 and C_2 constitutes a violation of MPCG for circuits of G_1 and cutsets of G_2. Hence $\bar{T}_1$ contains no cutset of G_2.

Lemma 3.2. Let G_1, G_2 be directed graphs on S. Let circuits of G_1 and cutsets of G_2 satisfy MPCG. Let $\bar{T}_1$, $\bar{T}_2$ be coforests of G_1, G_2 respectively such that $\bar{T}_1 \subseteq \bar{T}_2$. Then $\bar{T}_1 = \bar{T}_2$.

Proof. Suppose $\bar{T}_1 \subseteq \bar{T}_2$ but not equal to it. Let $e \in \bar{T}_2 - \bar{T}_1$. Let the corresponding forests of G_1, G_2 be T_1, T_2 respectively. Choose $\bar{T}_2 - e$ as S_b ("blue"), e as e^* ("dark green") and $\{e\}$ as S_g ("green") and T_2 as S_r ("red"). Since $e \cup S_r$ contains no circuit of G_1 and $e \cup S_b$ contains no cutset of G_2 it follows that circuits of G_1 and cutsets of G_2 cannot satisfy MPCG. We conclude that $\bar{T}_2 - \bar{T}_1 = \varphi$. Hence $\bar{T}_1 = \bar{T}_2$.

Theorem 3.1. Let G_1, G_2 be directed graphs on S such that the circuits of G_1 and the cutsets of G_2 satisfy MPCG. Then

$$(\nu_{cy}(G_1))^* = \nu_{cob}(G_2)$$

Proof. By Lemmas 3.1, 3.2 every forest of G_1 is also a forest of G_2. Let T be a

of G_1 and G_2. $B^1 = \begin{matrix} \bar{T} & T_1 \\ [u & T] \end{matrix}$ be a fundamental circuit matrix of G_1 and

let $Q^2 = \begin{matrix} \bar{T} & T \\ [Q_{11}^2 & U] \end{matrix}$ be a fundamental cutset matrix of G_2 with respect to this

forest. We now show that any row of B^1 is orthogonal to any row of Q^2. Let $L = L(G_1, e_1, T)$ and let $C = C(G_2, e_2, \bar{T})$. Then $C \cap L$ has precisely two elements, namely e_1 and e_2 with $e_1 \in T$ and $e_2 \in \bar{T}$. Consider the rows i_L *of* B^1 and v_C of Q^2. Now choose $\{e_1, e_2\}$ as S_g (green), e_1 as e^* (dark green), $\bar{T} - e_1$ as S_b (blue) and $T - e_2$ as S_r (red). If there is a bg-directed cutset of G_2 that meets e_1 it can only be C and if there is an rg-directed circuit of G_1 that meets e_1 it can only be L since the fundamental circuit of T with respect to e_2 in G_1 is unique and the fundamental cutset of $\bar{T}$ with respect to e_1 in G_2 is unique. By MPCG there exists a bg-directed cutset of G_2 that meets e_1 *or* there exists an rg-directed circuit of G_1 that meets e_1 but *not both*. We therefore conclude that $i_L(e_1) \cdot v_c(e_1) = -i_L(e_2) \cdot v_c(e_2)$. Hence $\langle i_L, v_C \rangle = 0$. Now $r(\nu_{cob}(G_2)) + r(\nu_{cy}(G_1)) = |S|$ since number of rows B^1 + number of rows of

$Q^2 = |S|$. Hence $(\nu_{cy}(G_1))^* = \nu_{cob}(G_2)$. Q.E.D.

4. Tellegen's Theorem Implies Minty's Painting Theorem

In this section we generalize Minty's Painting condition to vector spaces over the real field. We then show that two vector spaces ν_1, ν_2 on S are orthogonal only if they satisfy Minty's Painting condition. It is then easy to see that Tellegen's theorem implies Minty's Painting Theorem.

Definition 4.1. Let ν be a vector space on S. Let S_p, S_q be disjoint subsets of S. We say that a vector **f** in ν is pq-directed iff $\|\mathbf{f}\| \subseteq S_p \cup S_q$ and $f(e_1), f(e_2)$ have the same sign whenever they are nonzero and $e_1, e_2 \in S_q$. We say that **f** meets e^* iff $\mathbf{f}(e^*) \neq 0$.

Definition 4.2. (Minty's painting conditions for vector spaces (WMPCV and MPCV stand for weaker and stronger forms)). Let ν_1, ν_2 be vector spaces on S over $\mathbb{R}$. (ν_1, ν_2) satisfy (MPCV) WMPCV iff exactly one of (a), (b) below is true for every partition of S into sets S_r, S_b, S_g with $e^* \in S_g$.

(a) There exists a (minimal) vector in ν_1, that meets e^* and is rg-directed;

(b) There exists a (minimal) vector in ν_2 that meets e^* and is bg-directed.

Theorem 4.1. Let ν be a vector space on S over $\mathbb{R}$. Then (ν, ν^*) satisfy WMPCV.

Proof. We first show that both (a) and (b) of WMPCV cannot hold simultaneously. Suppose $\mathbf{i} \in \nu$ and satisfies (a) and $\mathbf{v} \in \nu^*$ and satisfies (b). Then clearly $\mathbf{i}(e) \cdot \mathbf{v}(e) \neq 0$ only if $e \in S_g$. It follows that if $\mathbf{i}(e) \cdot \mathbf{v}(e)$ is not equal to zero it always has the same sign. Hence $\langle \mathbf{i}, \mathbf{v} \rangle \neq 0$ since $\mathbf{i}(e^*), \mathbf{v}(e^*)$ are nonzero. But this contradicts the fact that $\mathbf{i}, \mathbf{v}$ belong to ν, ν^* respectively. We conclude that (a), (b) of WMPCV cannot be simultaneously satisfied.

We will now show that at least one of (a), (b) must be true. This is obviously so if $|S| = 1$. Suppose this is so for $|S| = n-1$. Let $|S| = n$. Let S be partitioned into S_r, S_b, S_g and $e^* \in S_g$. We now consider a number of cases.

Case 1. $S_b \neq \varphi$.

Let $e \in S_b$. Consider $\nu \cdot (S-e)$. By the inductive assumption WMPCV holds for $(\nu \times (S-e), (\nu \times (S-e))^*)$ with respect to the partition $(S_r, S_b - e, S_g)$ and the element e^*, i.e., there exists a bg-directed vector $\mathbf{v}$ that meets e^* in $\nu^* \cdot (S-e)$. Or there exists an rg-directed vector $\mathbf{i}$ that meets e^* in $\nu \times (S-e)$, since by Theorem P2, $(\nu \times (S-e))^* = \nu^* \cdot (S-e)$. Since $e \in S_b$ it follows that there must exist a bg-directed vector, that meets e^*, in ν^* or an rg-directed vector, that meets e^*, in ν.

Case 2. $S_b \cup (S_g - e^*) = \varphi$.

We have, $r(\nu^* \times e^*) + r(\nu \cdot e^*) = 1$ by Theorem P1. It follows that we have either a vector $\mathbf{v}$ in ν^* with $\|v\| = e^*$ or we have an rg-directed vector in ν. Thus the theorem holds.

Case 3. $S_b = \varphi, S_g - e^* \neq \varphi$.

Let $e \in S_G - e^*$. By the inductive assumption the theorem holds for $\nu \times (S-e), \nu^* \cdot (S-e))$ and $(\nu \cdot (S-e), \nu^* \times (S-e))$, with respect to the partition $(S_r, S_b, S_g - e)$ and the element e^*. Suppose there exists a bg-directed vector $\mathbf{v}$ that meets e^* in $\nu^* \times (S-e)$ or an rg-directed vector $\mathbf{i}$ that meets e^* in $\nu \times (S-e)$. Clearly these vectors can be extended to appropriate vectors which take zero value on e, meet e^*, and are bg-directed in ν^* or rg-directed in ν as the case may be. So the theorem holds in this case. Let us therefore assume the $\nu^* \times (S-e)$ does not have vectors that meet e^* and are bg-directed and $\nu \times (S-e)$ does not have vectors that meet e^* and are rg-directed. By the inductive assumption it follows that $\nu \cdot (S-e)$ has an

rg —directed vector $\mathbf{i}'$ that meets e^* and $\nu^* \cdot (S-e)$ has a bg —directed vector $\mathbf{v}'$ that meets e^*. Let us without loss of generality assume that $\mathbf{i}'(e^*)$, $\mathbf{v}'(e^*)$ are positive. Now there exists vectors $\mathbf{v}_S'$, $\mathbf{i}_S'$ belonging respectively to ν^*, ν such that $\mathbf{v}' = \mathbf{V}'_s/(S-e)$ and $\mathbf{i}' = \mathbf{i}'_s/(S-e)$. We have $\langle \mathbf{v}'_s, \mathbf{i}'_s \rangle = 0$. But this means $\mathbf{v}_s(e) \cdot \mathbf{i}_s(e) = -\langle \mathbf{v}', \mathbf{i}' \rangle$. The right side is negative since $\mathbf{v}'$ is bg —directed and $\mathbf{i}'$ is rg —directed and $\mathbf{v}'(e^*)$, $\mathbf{i}'(e^*)$ are positive. Hence $\mathbf{v}'_s(e), \mathbf{i}'_s(e)$ have opposite signs. Hence $\mathbf{v}'_s$ is bg —directed and meets e^* or $\mathbf{i}'_s$ is rg —directed and meets e^*. Thus the theorem holds for ν, ν^*.

Lemma 4.1 is needed for the proof of Theorem 4.2.

Lemma 4.1. Let ν be a vector space on S over $\mathbb{R}$. Let $\mathbf{v} \in \nu$ and let $e \in \|\mathbf{v}\|$. If $\mathbf{v}$ is not minimal there exists a vector $\mathbf{v}'$ such that $e \not\subseteq \|\mathbf{v}'\|$ and $\|\mathbf{v}'\| \subseteq \|\mathbf{v}\|$.

Proof. There exists a minimal vector v'' such that $\|\mathbf{v}''\| \subseteq \|\mathbf{v}\|$. If $e \not\subseteq \mathbf{v}''$, we are done. Otherwise consider minimal $\mathbf{v} - \left[\frac{\mathbf{v}(e)}{\mathbf{v}''(e)}\right] \cdot \mathbf{v}''$. This vector satisfies the required conditions.

Theorem 4.2. Let ν be a vector space on S. Let S_p, S_q be disjoint subsets of S. Let $e \in S_q$. Let $\mathbf{v}$ be a pq —directed vector of ν that meets e. Then there exists a minimal vector in ν, that meets e, is pq —directed, and has its support contained in the support of $\mathbf{v}$.

Proof. The theorem clearly holds when the cardinality of $\|\mathbf{v}\| = 1$. Assume it holds when $\|\mathbf{v}\| < n$. Let $\|\mathbf{v}\| = n$. If $\mathbf{v}$ is minimal there is nothing to prove. If $\mathbf{v}$ is not minimal we know by Lemma 4.1 that there exists a vector $\mathbf{v}'$ that does not meet e and satisfies $\|\mathbf{v}'\| \subseteq \|\mathbf{v}\|$. Let $x \in \|\mathbf{v}'\|$ be such that $\frac{|\mathbf{v}(x)|}{|\mathbf{v}'(x)|} = \min \frac{|\mathbf{v}(y)|}{|\mathbf{v}'(y)|}$. Consider the vector $\left[\mathbf{v} - \left[\frac{\mathbf{v}(x)}{\mathbf{v}'(x)}\right]\mathbf{v}'\right]$. This is pq —directed, meets e and has cardinality less than n. It therefore contains a minimal vector belonging to ν that is pq −directed and meets e.

Theorems 4.1 and 4.2 imply

Theorem 4.3. (ν,ν^*) satisfy MPCV.

We state the following simple result from graph theory without proof.

Lemma 4.3. Let G be a directed graph. If $\mathbf{f}$ is a minimal coboundary (cycle) of G there exists a cutset (circuit) T of G such that the cutset (circuit) vector $\mathbf{f}_T$ corresponding to T satisfies $\mathbf{f} = \lambda \mathbf{f}_T$ for some scalar λ.

Remark. Although we have generalized Minty's Theorem to vector spaces over the real field it must be pointed out that the result is useful primarily when all nonzero entries of a minimal vector can be taken to be of the same magnitude. This happens only where the vector space is generated by a unimodular matrix. For such vector spaces essentially every property of graphs, that is provable by Minty's Theorem, would be true.

Tellegen's Theorem, Theorem 4.3 and Lemma 4.3 imply

Theorem 4.4. (Minty's Painting Theorem for Graphs). The circuits of a directed graph G and the cutsets of G satisfy MPCG.

Conclusion

We have shown in this paper that Tellegen's Theorem and Minty's Painting Theorem for graphs are equivalent. We have in the process generalized Minty's Painting Theorem to vector spaces over the real field and also have given a new proof for it.

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