DESIGN OF PRECOMPENSATOR FOR DECOUPLING PROBLEM

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Abstract-For a class of linear time-invariant multivariable systems which can not be decoupled by state variable feedback, but which are invertible, we propose an algorithm of designing a precompensating dynamic system which results in a new system that can be decoupled by state variable feedback.

Consider a linear time-invariant multivariable system representation,

$$\dot{x} = Ax + Bu$$
$$y = Cx \tag{1}$$

where x is an n-vector, u is an m-vector, y is an m-vector, and A, B and C are n x n, n x m and m x n constant matrices, respectively.

We consider the control law

$$u = Fx + Gw \tag{2}$$

where F is an m x n constant matrix, G is an m x m nonsingular constant matrix, and w is the input of the overall system.

<u>Theorem 1</u> (Falb and Wolovich[1])

A system with representation (1) can be decoupled by using the control law in (2), if and only if the m x m matrix

$$B^* = \begin{bmatrix} c_1 A^{d_1} B \\ c_2 A^{d_2} B \\ \vdots \\ c_m A^{d_m} B \end{bmatrix} \tag{3}$$

is nonsingular, where c_i is the i-th row of C; and

$$\begin{aligned} d_i &= \min \{k: \; c_i A^k B \neq 0, \; k = 0,1,\ldots, n-1\} \\ \text{or} \quad d_i &= n-1 \text{ if } c_i A^k B = 0 \text{ for all integers } k \varepsilon [0, n-1] \end{aligned} \tag{4}$$

In particular, we may pick $G = (B^*)^{-1}$

$$\text{and} \quad F = -(B^*)^{-1} \begin{bmatrix} c_1 A^{d_1+1} \\ \vdots \\ c_m A^{d_m+1} \end{bmatrix} .$$

For a precise definition of decoupling refer to Ref. 1; for an alternate treatment see Ref. 2.

<u>Comment</u>

From the Laurent expansion of the matrix transfer function

$H(s) \triangleq C(sI-A)^{-1}B = \sum_{j=0}^{\infty} \frac{CA^j B}{s^{j+1}}$, it is easy to check that

$c_i A^{d_i} B = \lim_{s \to \infty} s^{d_i+1} h_i(s)$, where $h_i(s)$ is the i-th row of $H(s)$, so B^* can be computed directly from the matrix transfer function: B^* is completely determined by the input-output properties of (1), and is independent of the state representation. This fact will be used repeatedly in the following.

Definition 1 (Gilbert[3])

Let $H(s) = C(sI-A)^{-1} B$ be the transfer function of (1), then H(s) is said to be weakly coupled if and only if

$$\begin{cases} 1. & \det H(s) \neq 0 \text{ a.e.} \\ 2. & \det B^{*} = 0 \end{cases} \tag{5}$$

where B^{*} is given by (3)

We state and prove the following theorem which was suggested by Gilbert[3].

Theorem 2

Given a system with a weakly coupled transfer function, we can always decouple it by the insertion of a precompensating dynamic system at the input terminals, then apply the feedback law specified in Theorem 1.

Proof.

The proof of this theorem is given by an algorithm discussed later.

Lemma 1

Suppose that $H(s) = C\,(sI - A)^{-1}\,B$ is an m x m matrix transfer function of (1), in which each element is a proper rational function of s;

let $d = \min \{k: \lim_{s\to\infty} s^{k} \det H(s) \neq 0$, k is a positive integer$\}$, (6)

and for i = 1,2,...,m let

$$d_i = \min \{j: \; c_i A^j B \neq 0, \; j = 0, \ldots, n-1\}$$

$$\text{or } d_i = n-1 \quad \text{if } c_i A^j B = 0 \quad \text{for all integers } j \in [0, n-1] \qquad (4')$$

then

(a) $d \geq m + \sum_{i=1}^{m} d_i$;

(b) $d = m + \sum_{i=1}^{m} d_i$ if and only if B^* is nonsingular;

(c) if A is an nxn matrix, then det H(s) $\neq 0$ a.e. implies that $n \geq d$.

Proof.

Let $H(s) = C\,(sI - A)^{-1}\,B$

$$= \begin{bmatrix} \sum_{j=0}^{\infty} \frac{c_1 A^j B}{s^{j+1}} \\ \hline \sum_{j=0}^{\infty} \frac{c_2 A^j B}{s^{j+1}} \\ \hline \vdots \\ \hline \sum_{j=0}^{\infty} \frac{c_m A^j B}{s^{j+1}} \end{bmatrix} = \begin{bmatrix} \frac{b_1^*}{s^{d_1+1}} + \sum_{j=d_1+1}^{\infty} \frac{c_1 A^j B}{s^{j+1}} \\ \hline \frac{b_2^*}{s^{d_2+1}} + \sum_{j=d_2+1}^{\infty} \frac{c_2 A^j B}{s^{j+1}} \\ \hline \vdots \\ \hline \frac{b_m^*}{s^{d_m+1}} + \sum_{j=d_m+1}^{\infty} \frac{c_m A^j B}{s^{j+1}} \end{bmatrix} \qquad (7)$$

i.e. we expand each row of H(s) in the Laurent expansion about zero, refer to the comment following Theorem 1, and an alternative definition of B^* in eq (9) below, where b_j^* denotes j-th row of B^*, we get the above expression for H(s). It is easy to see that

$$\det H(s) = \frac{\det B^*}{s^{d_1 + d_2 + \ldots + d_m + m}} + (\text{lower order terms})$$

So det $B^* \neq 0$ if and only if that $d = m + \sum_{i=1}^{m} d_i$.

This proves (b). The above reasoning also proves (a).

From Proposition 2, p. 53, in Ref. 3, det H(s) can be written as $\frac{h(s)}{\det(sI-A)}$, where h(s) is a polynomial in s of degree not greater than n-m. So if det $H(s) \neq 0$ a.e., it is easy to see that $n \geq d$.

ALGORITHM FOR THE DESIGNING OF PRECOMPENSATOR

Given a weakly coupled transfer function H(s), the following algorithm gives a precompensator which results in a new system that can be decoupled by state feedback as discussed in Theorem 1.

Step 1. Calculate det H(s), where det H(s) is a proper rational function of s.

If det H(s) = 0 for all s in the complex plane, stop! (This is not a weakly coupled system, this algorithm isn't applicable.)

If det $H(s) \neq 0$ a.e., calculate $d_j = \min \{k: \lim_{s\to\infty} s^{k+1} h_j(s) \neq 0, k = 0, 1, \ldots\}$ (8)

where $h_j(s)$ is the j-th row of H(s).

Note that det $H(s) \neq 0$ a.e. $\Rightarrow h_j(s) \neq 0$ a.e. $\Rightarrow d_j < \infty$ $j=1,2,\ldots,m$

Step 2. Construct the m x m constant matrix B^* as follows,

$$b_j^* = \lim_{s\to\infty} s^{d_j+1} h_j(s) \quad j = 1,2,\ldots,m \tag{9}$$

where b_j^* is the j-th row of B^*, and $h_j(s)$ is the j-th row of H(s).

Note that the definition of B^* and of d_j, $j=1,2,\ldots,m$ in (8) and (9) is equivalent to that in (3) and (4). Note also that $b_j^* \neq 0$, $j=1,2,\ldots,m$.

Step 3. The assumption that the given transfer function is <u>weakly coupled</u> implies that rank $(B^*) \triangleq p < m$. Perform elementary <u>column</u> operations on H(s), using constant multipliers, (this corresponds to linear recombinations of <u>input</u> terminals of the given system), in order to get a new transfer function, say $H_1(s)$, so that its corresponding B_1^* has its last m-p columns identically zero; the first p columns of B_1^* are linearly independent. Moreover, the process can be carried out so that there are p rows, say r_1, $r_2, \ldots, r_p$), whose only nonzero elements form a p x p <u>nonsingular diagonal</u> matrix.

i.e.

$$B_1^* = \left[\begin{array}{c|c} \overbrace{\begin{matrix} x & \otimes & \otimes & \otimes & x \\ b_1 & 0 & 0 & 0 & 0 \\ x & \otimes & \otimes & \otimes & x \\ 0 & b_2 & 0 & 0 & 0 \\ 0 & 0 & b_3 & 0 & 0 \\ \otimes & \otimes & \otimes & \otimes & \otimes \\ \otimes & \otimes & \otimes & \otimes & \otimes \\ 0 & 0 & 0 & 0 & b_p \end{matrix}}^{p} & \overbrace{\quad \bigcirc \quad}^{m-p} \end{array}\right]$$

where the b_i's are non-zero constants; the b_i's are the diagonal elements of the p x p nonsingular diagonal matrix.

Step 4. Now we have p columns in B_1^* with nonzero elements, and among these p columns we have p' columns (say $i_1, i_2, \ldots, i_{p'}$) with two or more nonzero elements, where $1 \leq i_1 \leq i_2 < \ldots < i_{p'} \leq p$, we claim that $1 \leq p' \leq p$, since if $p' = 0$, then some rows in

B_1^* are identically zero, this contradicts with the definition of B_1^*.

Multiply $i_1, i_2, \ldots, i_{p'}$ -th column in $H_1(s)$ by $\frac{1}{s}$, (this corresponds to putting an integrator in series with the corresponding input terminal). Call the resulting transfer function $\tilde{H}(s)$.

Step 5. With respect to $\tilde{H}(s)$, calculate $\tilde{d}_j$, $j = 1,2,\ldots,m$ and $\tilde{B}^*$ in the same way as in (8), (9). If $\det \tilde{B}^* \neq 0$, then apply the feedback law in Theorem 1 to decouple $\tilde{H}(s)$. Otherwise repeat step 3, 4 until we get that $\det \tilde{B}^* \neq 0$.

Proof. of Theorem 2.

We are going to show that using the above algorithm, in a finite number of iterations of step 3, 4 and 5, we come up to $\det \tilde{B}^* \neq 0$.

In step 1, with respect to $H(s)$, we calculate d, d_j, $j = 1,2,\ldots,m$ using eq (6) and (8). Similarly, in step 4, with respect to $\tilde{H}(s)$, we calculate $\tilde{d}$, $\tilde{d}_j$, $j = 1,2,\ldots,m$.

Furthermore, they are related in the following way

$$\tilde{d} = d + p' \tag{10}$$

$$\tilde{d}_j = d_j + 1 \quad \text{if } j \in \{1,2,\ldots,m\} \setminus \{r_1, r_2, \ldots, r_p\} \tag{11}$$

where " \" denotes set difference.

$$\tilde{d}_j = d_j + 1 \text{ if } j \in \left\{r_k : k \in \{i_1, i_2, \ldots, i_{p'}\}\right\} \tag{12}$$

$$\tilde{d}_j = d_j \quad \text{if } j \in \left\{r_k : k \in \{1,2,\ldots,p\} \setminus \{i_1, i_2, \ldots, i_{p'}\}\right\} \tag{13}$$

Add eq (11) - (13), we have

$$\sum_{j=1}^{m} \tilde{d}_j = m - p + p' + \sum_{j=1}^{m} d_j$$

or $\tilde{d} - m - \sum_{i=1}^{m} \tilde{d}_i = d - m - \sum_{i=1}^{m} d_i - m + p$ (14)

Since in step 3, we have $\det(B^*) = 0$, i.e. $m > p$,

so $(\tilde{d} - m - \sum_{i=1}^{m} \tilde{d}_i) < (d - m - \sum_{i=1}^{m} d_i)$,

i.e. the difference between d and $m + \sum_{i=1}^{m} d_i$ is reduced after we perform step 3 and 4.

It is clear that in a finite number of iterations of steps 3, 4, and 5, we obtain $\tilde{d} = m + \sum_{i=1}^{m} \tilde{d}_i$, and by Lemma 1, this is equivalent to $\det \tilde{B}^* \neq 0$.

Q.E.D.

Example

Consider the following matrix transfer function,

$$H(s) = \begin{bmatrix} \frac{2s+3}{s^2+3s+2} & \frac{6}{s+3} & \frac{1}{s+2} \\ 0 & \frac{4}{s^2+5s+6} & \frac{1}{s^2+4s+3} \\ \frac{1}{s+1} & \frac{3(s+5)}{s^2+4s+3} & \frac{1}{2(s+3)} \end{bmatrix}$$

From step 1, we obtain that $d_1 = 0$, $d_2 = 1$, $d_3 = 0$ and from step 2,

$$B^* = \begin{bmatrix} 2 & 6 & 1 \\ 0 & 4 & 1 \\ 1 & 3 & \frac{1}{2} \end{bmatrix}$$

Note that B^* is singular and $\det H(s) \neq 0$ a.e., i.e. this is a weakly coupled system.

Following step 3; i.e. performing elementary column operations on H(s), using constant multipliers: (a) add to the third column of H(s) the product of the first column of H(s) by $\frac{1}{4}$ and the product of the second column of H(s) by $-\frac{1}{4}$, (b) add to the second column of H(s) the product of the first column of H(s) by - 3.

The resulting transfer function is $H_1(s)$ and

$$H_1(s) = \begin{bmatrix} \frac{2s+3}{s^2+3s+2} & \frac{-9s-15}{s^3+6s^2+11s+6} & \frac{\frac{7}{4}s+\frac{9}{4}}{s^3+6s^2+11s+6} \\ 0 & \frac{4}{s^2+5s+6} & \frac{1}{s^3+6s^2+11s+6} \\ \frac{1}{s+1} & \frac{6}{s^2+4s+3} & \frac{-\frac{5}{2}}{s^2+4s+3} \end{bmatrix}$$

The corresponding $B_1^* = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Refer to step 4; multiply the first column of $H_1(s)$ by $\frac{1}{s}$, we get $\tilde{H}(s)$, the corresponding $\tilde{B}^* = \begin{bmatrix} 2 & -9 & \frac{7}{4} \\ 0 & 4 & 0 \\ 1 & 6 & -\frac{5}{2} \end{bmatrix}$

Since $\tilde{B}^*$ is nonsingular, so we can apply the control law in Theorem 1 to decouple $\tilde{H}(s)$. If we use the following realization of $\tilde{H}(s)$, as in Ref. 6,

$$\tilde{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{4} \\ -\frac{1}{2} & -3 & \frac{5}{4} \\ 0 & 0 & -1 \\ 0 & 6 & -\frac{3}{2} \\ 0 & -4 & \frac{1}{2} \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} \frac{3}{2} & -1 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & -5 & 0 & 0 & -\frac{7}{6} & -1 \end{bmatrix}$$

Then $$G = (\tilde{B}^*)^{-1} = \begin{bmatrix} \frac{10}{27} & \frac{4}{9} & \frac{7}{27} \\ 0 & \frac{1}{4} & 0 \\ \frac{4}{27} & \frac{7}{9} & \frac{-8}{27} \end{bmatrix} \tag{15}$$

$$\text{and} \quad F = -(\tilde{B}^*)^{-1} \begin{bmatrix} \tilde{c}_1 \tilde{A}^2 \\ \tilde{c}_2 \tilde{A}^2 \\ \tilde{c}_3 \tilde{A}^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{17}{27} & \frac{1}{9} & \frac{1}{27} & \frac{-40}{27} & \frac{-16}{9} & \frac{-11}{18} & -\frac{5}{3} \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 1 & 0 & -\frac{9}{4} \\ 0 & \frac{-4}{27} & \frac{4}{9} & -\frac{86}{27} & \frac{16}{27} & -4 & \frac{-40}{9} & -\frac{29}{3} \end{bmatrix} \tag{16}$$

Figure 1 shows the interconnection among the given system H(s), the precompensator and the state variable feedback.

Remark

Instead of putting an integrator in series with the input terminal, we may use $\frac{1}{s+\alpha}$ as the transfer function of the building block of the precompensator.

Conclusion

Given an m x m transfer function matrix H(s), if it is weakly coupled, we may apply the algorithm in this letter to design a precompensator, then together with the feedback law specified in Theorem 1, we can always decouple it. Morse and Wonham[4] have found minimal order precompensator for this purpose by a geometric approach, but they propose no algorithm suitable for computation. Silverman[5] has a different way of designing a precompensator based on an algorithm for inverting a system.

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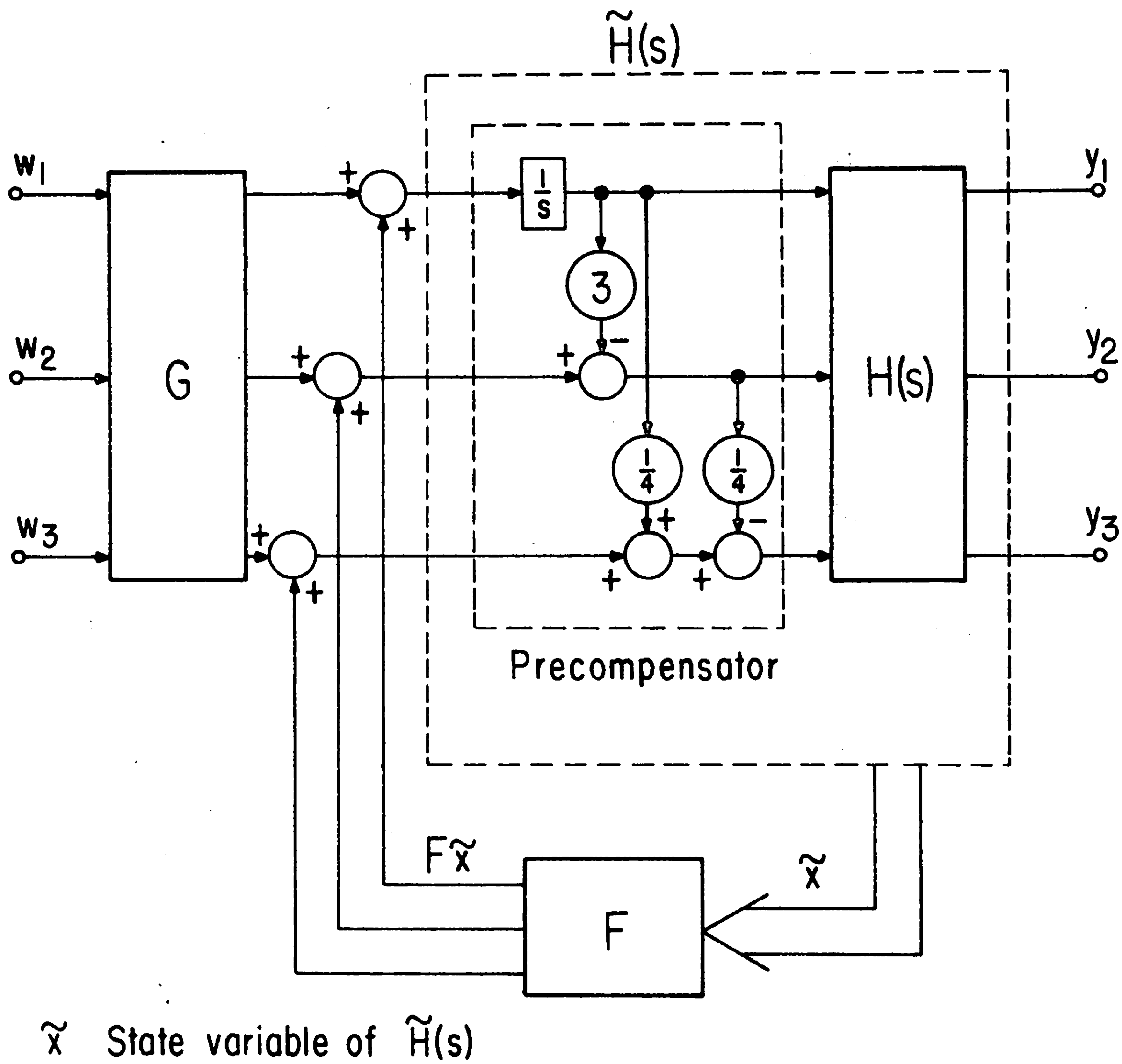

Fig. 1. The block H(s) is the given system. The precompensator is designed according to the algorithm. The block F consists of adders and multipliers; its input is $\tilde{x}$ and its output is $F\tilde{x}$, the matrix F is given by eq (16).
Similarly the block G has w as input and Gw as output , the G matrix is given by (15). The new overall system with input w and output y is decoupled.