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COMPUTER GENERATION OF EQUIVALENT NETWORKS

by

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Summary

The theory of continuously equivalent networks is extended to the state equations and to include a scaling option. The theory is then applied to the classical problem of the design of low- and band-pass filters with loss in inductors only. The alignment problem in active filters is also approached from an equivalence viewpoint.

Introduction

Although the subject of equivalent networks has long been recognized to be of academic interest, it is only recently that a conscious attempt has been made to utilize the theory in practical design problems. This paper extends some recent work in this subject and then considers some examples of use of the computer in generation of equivalent networks.

The Approach

A differential approach to the equivalent network problem was first proposed by Schoeffler[1] and has been investigated by other authors.[2, 3] It will be found useful to work with the A matrix rather than the nodal or branch admittance matrices, so that the development of Schoeffler must be altered.

Assume that the state equations of the circuit are of the form (with no mutual inductance, capacitor loops, etc.)

$$-\begin{bmatrix} L_1 & & & & & 0 \\ & \ddots & & & & \\ & & L_n & & & \\ & & & C_1 & & \\ & & & & \ddots & \\ 0 & & & & & C_m \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_{L_1} \\ \vdots \\ v_{c_m} \end{bmatrix} = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} i_{L_1} \\ \vdots \\ v_{c_m} \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k\, v_{in} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (1)$$

or

$$-[M]\ [\dot{y}] = [A]\ [y] - [B]\ v_{in} \quad (2)$$

Assume v_{in} appears only in the r^{th} equation as shown and further assume that one of the state variables (the g^{th}) is the output voltage. Both of these conditions can be satisfied by placing an inductor of zero-value in series with the source and a capacitor of zero value across any pair of nodes regarded as the output. Further, assume the equations are arranged so that $r = 1$, $g = p = m + n$.

Taking the Laplace Transform of both sides yields (for zero initial conditions)

$$[B]\ V_{in} = \{ s\ [M] + [A] \}\ [Y] \quad (3)$$

where

$$[Y] = [Y(s)] = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} \qquad V_{in} = V_{in}(s) \quad (4)$$

Now consider the problem of adjustment of the network elements while maintaining constant the ratio of the output voltage to the input voltage. Letting each element be a function of some dummy variable x, then one may postulate a differential change in the element values that produce the new state variables

$$[Y(x + \Delta x)] = \frac{\{[U] + \Delta x\ [a]\}\ [Y(x)]}{1 + \Delta x\, a_{pp}} \quad (5)$$

where

$$[a] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{p-1,1} & & & \vdots \\ 0 & 0 & \cdots & a_{pp} \end{bmatrix}, \quad [Y(x)] = [Y(x, s)] \quad (6)$$

so that $y_p(x + \Delta x) = y_p(x)$ and equivalence is maintained. Eq. (3) then becomes

$$[B]\, V_{in} = \{ s\, [M(x+\Delta x)] + [A(x+\Delta x)] \} \{ Y(x+\Delta x) \} \tag{7}$$

$$= \frac{\{ s\,[M(x+\Delta x)] + [A(x+\Delta x)]\}\{[U] + \Delta x\,[a]\}\,[Y(x)]}{1 + \Delta x\, a_{pp}} \tag{8}$$

Taking note that

1) $[B(x)]$ has only one nonzero entry
2) $V_{in}(x + \Delta x) = V_{in}(x)$ since V_{in} is a source.

then

$$[B(x+\Delta x)]\; V_{in}(x+\Delta x) = \frac{\{[U] + \Delta x\,[b]\}\,[B(x)]\, V_{in}(x)}{1 + \Delta x\; b_{11}} \tag{9}$$

if

$$[b] = \begin{bmatrix} b_{11} & b_{12} \cdots & b_{1p} \\ 0 & b_{22} \cdots & b_{2p} \\ & \cdot & \cdot \\ 0 & \quad \cdot & \cdot \\ \cdot & \qquad \cdot & \cdot \\ \cdot & & \cdot \\ 0 & b_{p2} \cdots & b_{pp} \end{bmatrix} \tag{10}$$

Multiplying both sides of Eq. (8) by $\{[U] + \Delta x\; [b]\}\,/(1 + \Delta x\, b_{11})$ gives

$$[B]\; v_{in} = \frac{\{[U] + \Delta x\,[b]\}\;\{ s\,[M(x+\Delta x)] + A[(x+\Delta x)]\}}{(1 + \Delta x\; a_{pp})\;\; (1 + \Delta x\;\; b_{11})} \cdot \{[U] + \Delta x\,[a]\}\; [Y(x)] \tag{11}$$

$$= \{ s\; [M(x)] + [A(x)]\}\; [Y(x)] \tag{12}$$

Thus

$$\{ s\; [M(x)] + [A(x)]\} = \frac{\{[U] + \Delta x\,[b]\}\{ s\,[M(x+\Delta x)] + [A(x+\Delta x)]\}\{[U] + \Delta x\; [a]\}}{(1 + \Delta x\; a_{pp})\;\, (1 + \Delta x\; b_{11})} \tag{13}$$

Separating powers of s yields

$$\frac{[M(x + \Delta x)] - [M(x)]}{\Delta x} =$$

$$[b]\,[M(x+\Delta x)] + [M(x+\Delta x)]\; [a] - (b_{11} + a_{pp})\,[M(x+\Delta x)] + 0(\Delta x) \tag{14}$$

$$\frac{[A(x + \Delta x)] - [A(x)]}{\Delta x} =$$

$$[b]\,[A(x+\Delta x)] + [A(x+\Delta x)]\; [a] - (b_{11} + a_{pp})\; [A(x+\Delta x)] + 0(\Delta x) \tag{15}$$

which in the limit become

$$\frac{d\,[M]}{d\;\, x} = [b]\; [M] + [M]\; [a] - (b_{11} + a_{pp})\; [M] \tag{16}$$

$$\frac{d\,[A]}{d\;\, x} = [b]\; [A] + [A]\; [a] - (b_{11} + a_{pp})\; [A] \; . \tag{17}$$

The form of this equation is somewhat more general than Schoeffler's.[1]

If the right-hand side of Eq. (9) is not divided by $1 + \Delta x\; b_{11}$, then Eqs. (16) and (17) become

$$\frac{d\,[M]}{d\; x} = [b]\,[M] + [M]\,[a] - a_{pp}\; [M] \tag{18}$$

$$\frac{d\,[A]}{d\; x} = [b]\,[A] + [A]\,[a] - a_{pp}\; [A] \tag{19}$$

Further, Eq. (9) may be solved in part to yield

$$V_{in}(x + \Delta x) = (1 + \Delta x\, b_{11})\; V_{in}(x) \qquad (20)$$

so that taking the limit as before gives

$$V_{in}(x) = V_{in}(0) \quad \exp\{b_{11}x\} \qquad (21)$$

Hence eliminating the term $1 + \Delta x \;\; b_{11}$ results in a scaling of the voltage transfer function by an amount $\exp\{-b_{11}x\}$.* The scaling option will be found useful in the following examples:

Application 1: Nonuniformly Lossy Ladders

The object of the following examples will be to use the previous theory to generate lossy resistively terminated networks from lossless ones. Thus, the state equations are a natural approach to the problem since the lossless elements are contained in $[M]$, whereas the loss components are contained in $[A]$.

Example 1: Singly-terminated Ladders: Geffe[4] has given exact formulae for design of singly-terminated ladders with lossy inductors. A simple example is the network of Fig. 1. As an exercise to test the mettle of this equivalence viewpoint, one could attempt to obtain similar design formulae for the simple case. The matrices of interest are

$$[M] = \begin{bmatrix} C_1/G_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & C_2 \end{bmatrix} [A] = \begin{bmatrix} 1 & R_1 & 0 \\ -1 & R_2 & 1 \\ 0 & -1 & G_3 \end{bmatrix} B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad (22)$$

$$[b] = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{bmatrix} [a] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad (23)$$

where $G_3 = 0$. Inserting the above in Eqs. (18) and (19) results (in part) in trivial constraints which require $[b]$ and $[a]$ to be of the form

$$\left.\begin{aligned} [b] &= \begin{bmatrix} b_{11} & -a_{12}(C_1/L_2G_1) & -a_{13}(C_1/C_2G_1) \\ 0 & -a_{11} & -a_{23}(L_2/C_2) \\ 0 & 0 & -a_{22} \end{bmatrix} \\ [a] &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}\right\} \qquad (24)$$

Further, since the voltage transfer function cannot be sealed by growing a resistor in series with L_1, $b_{11} = 0$.

The remaining constraints generate the differential equations

$$\left(\frac{C_1}{G_1}\right)' = a_{11}\; C_1/G_1 \qquad (25)$$

$$L_1' = (a_{22} - a_{11})\, L_1 \qquad (26)$$

$$C_2' = (-a_{22}\; C_2) \qquad (27)$$

$$R_1' = a_{11}\, R_2 + a_{12} + a_{13}\,(C_1/C_2\, G_1) + a_{22}\, R_1 \qquad (28)$$

$$R_2' = (a_{11} - a_{22})\; R_2 - a_{12} \qquad (29)$$

where $a_{12} = -a_{11} \;\; L_1(G_1/C_1) \quad a_{13} = -a_{11}$. (30)

These equations are readily solved to yield

$$C_1/G_1 = C_1^o/G_1^o \quad e^{a_{11}x} \qquad (31)$$

$$L_1 = L_1^o \quad e^{(a_{22} - a_{11})x} \qquad (32)$$

$$C_2 = C_2^o \quad e^{-a_{22}x} \qquad (33)$$

$$R_2 = (L_1^o\, G_1^o/C_1^o)\, \exp(a_{22} - a_{11})x\,[1 - \exp(-a_{11})x]$$
$$+ \;\; R_2^o \;\; \exp \;\; (a_{22} - a_{11}) \qquad (34)$$

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*A similar statement applies if $1 + \Delta\, x\, a_{pp}$ is eliminated in Eq. (5).

$$R_1 = -R_2 + \left[R_1^o + R_2^o + (C_1^o/G_1^o C_2^o)(1 - \exp a_{11} x)\right] \exp\{a_{22} x\} \quad (35)$$

thus achieving exact design formulae for any choice of a_{11} and a_{22}.*

It has not been found possible to extend the above analysis to include Geffe's general case; however, the correct matrix manipulation may yield a closed form result.

Example 2: Doubly-terminated Ladders:

An attempt has been made to find a closed form solution for the simplest doubly-terminated ladders (Fig. 2a and 2b). The matrices for the network of Fig. 2a are identical to those of Example 1, except that now $G_3 \neq 0$, $b_{11} \neq 0$, $G_1' = 0$, and $G_3' = 0$. If there is a closed-form solution, it is not apparent from the differential equations and so a computer solution was necessary. It may readily be shown that there are four remaining linear constraints on the five unknowns of Eq. (24), leaving one unknown to be selected arbitrarily. Selecting b_{11} to be this constant and letting $b_{11} = 1$,** then the voltage transfer function will be scaled down, as indeed it must if resistance is to be inserted in series with the inductors. Further, the flat loss introduced is readily shown to be (20x/2.3) db. The flow chart is shown in Fig. 3.

In transforming the network of Fig. 2b, seven equations in eight unknowns are generated, so again selection of $b_{11} = 1$ generates a unique set of constraints. It should be noted that, to keep identical dissipation factors in each inductor, a constraint of the form

$$\frac{d}{dx}\left(\frac{R_1}{L_1}\right) = \frac{d}{dx}\left(\frac{R_2}{L_2}\right) \quad (36)$$

must be imposed as the integration proceeds.

Element values for the fourth order Butterworth filter are given in Table 1.† For r = 1 (equal termination), it was found impossible to generate any equivalent nonuniformly lossy networks. For $0 < r < 1$, either of two lossless realizations may be used as initial networks in the transformation process; the first is found in Ref. 5, the second is the dual and impedance scaled version of the first. The dual network was found to allow for significantly more loss, except for r = 0 when no dual exists. The maximum inductor loss was found in every case to strongly depend on the termination ratio, and the largest d is approximately twice that allowable for uniformly-lossy networks.

The nonuniformly lossy band-pass ladder (Fig. 4) may, of course, also be approached from the equivalence viewpoint. In this case, true equivalence cannot be maintained since any inductor loss will shift a transmission zero from the origin to the negative real axis. However, if one considers the output to be the inductor current of L_2 rather than the capacitor (output) voltage C_2, then

$$I_L(s) = \frac{V_c(s)}{R_2 + s L_2}$$

the bothersome transmission zero disappears, and only a small error is made in keeping the actual frequency response of the filter invariant (if $d = R_2/L_2$ is small).* It now happens that the dual network allows less inductor loss (Table 2). Further the maximum value of d allowed for any value of Q was approximately twice the normalized real part of the pole closest to the imaginary axis and so is again twice that expected from uniform predistortion arguments. Finally, calculation shows the filters with the largest inductor loss are overall somewhat less sensitive than the lossless variety. The price paid is the larger element value spread in the former.

Application 2: Active-RC Networks

One of the difficulties in the design of active-RC filters is the sensitivity problem. Classically, this has been approached from an a priori viewpoint, i.e., how to minimize the effects of changes in the active element on the network response. Viewing a posteriori, the problem changes to that of alignment or "how to restore the original transfer function" or, formally, the equivalent network problem. We may therefore consider the application of the computer in establishing an alignment procedure for such networks.

Example 1: Consider the low-pass filter of Fig. 5,[6] but normalized to a source resistance of 1Ω. The matrices of interest are

$$M = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \quad A = \begin{bmatrix} Y_A + Y_B + 1 & -Y_B \\ Y_M - Y_B & Y_B + Y_C \end{bmatrix} \quad (38)$$

$$a = \begin{bmatrix} a_{11} + a_{12} \\ 0 & a_{22} \end{bmatrix} \quad b = \begin{bmatrix} b_{11} & -a_{12}(C_1/C_2) \\ 0 & b_{22} \end{bmatrix} \quad (39)$$

*The network could be normalized to unit source resistance by using the expression for R_1 as an impedance scaling factor.

**Note that the a's and b's may be scaled arbitrarily, since this is equivalent to scaling x.

†Typical computing time was two minutes on an IBM 7090.

*The flat loss is now (20x/2.3) - 20 $\log_{10} L_2(x)/L_2(0)$; for the dual band-pass network, it happens b_{11} must now be normalized to -1 to grow positive resistors.

There are six elements and five a's and b's to be chosen. In solving for the elemental derivatives as in Eqs. (25)-(29), it happens that a_{11} and a_{22} appear only as the difference $a_{11} - a_{22}$, so that only four a's and b's are distinct. Thus, constraining three elements to remain unchanged (Y_c, C_1, and C_2 being the most different to change in lumped circuits), b_{11} can be chosen to be the only arbitrary constant. Interpreting Y_A, Y_B, etc. in terms of transistor parameters, $R_e(x)$ and $R_f(x)$ may be obtained to maintain equivalence for a given change in β ($\beta(x)$ Fig. 6). It is noteworthy that

1. a minimum value of β exists below which the transfer function cannot be maintained (without a change in Y_c, C_1, or C_2).

2. a region ($3 \leq x \leq 7$) exists over which $\beta(x)$ is relatively constant. If the circuit should operate in the region then, a small change in β would have to be compensated by a large change in both R_f and R_e — this in spite of the low sensitivity in this region. There does exist a simple relationship between the sensitivity functions and the curves of Fig. 7: Taking the derivative of the transfer function T(s), one has

$$\frac{dT}{dx} = 0$$

$$= \frac{\partial T}{\partial \beta}\frac{d\beta}{dx} + \frac{\partial T}{\partial R_e}\frac{dR_e}{dx} + \frac{\partial T}{\partial R_f}\frac{dR_f}{dx} \tag{40}$$

so that the sensitivities ($\partial T/\partial \beta$, $\partial T/\partial R_e$, $\partial T/\partial R_f$) and the slopes of the curves of Fig. 7 ($d\beta/dx$, dR_e/dx, dR_f/dx) are orthogonal. Eq. (40), however, gives no hint to the relative magnitude of each set.

Conclusions

There are several means of extending the results given for nonuniformly lossy filters:

1. Generate tables of element values for all low-order Butterworth, Tchebycheff and maximally-flat time delay networks.
2. Explore the possibility of using the degrees of freedom available in the interaction process to obtain selected element values. For example, if the constraint of Eq. (36) is not imposed, one of the a's may be selected arbitrarily and the problem of using this freedom properly is posed.
3. Search for exact formulae for each element value, either using the classical approach of Takahasi[7] or, using Takahasi's results as initial conditions, generate, by means of matrix manipulation in Eqs. (18)-(19), elemental values as functions of x as in Eqs. (31)-(35).

Concerning active filter design, it would be of interest to

1. Develop design graphs for active filters (particularly band-pass) which permits a choice in the characteristics of the active element (similar to Fig. 7).
2. Investigate more thoroughly the possible degeneracies which arise in the constraint equations, so that one could predict if possible, the number of "alignment" elements he must be prepared to vary to accommodate for an uncontrolled variation in some other element. With the advent of integrated circuits, which can be designed with resistive elements that vary in a controlled fashion, such graphs could also be used to determine the variation required to maintain a response characteristic over wide environmental changes.

Acknowledgment

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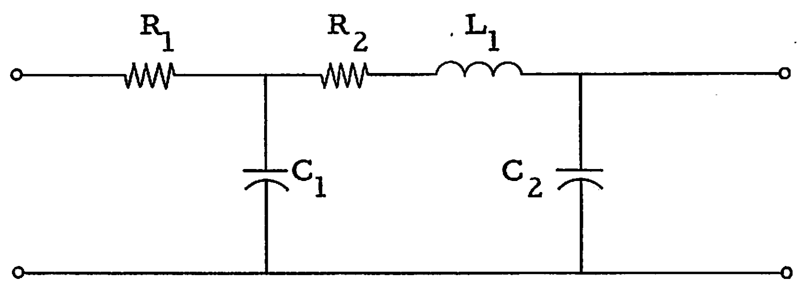


Fig. 1. Singly-terminated ladder.

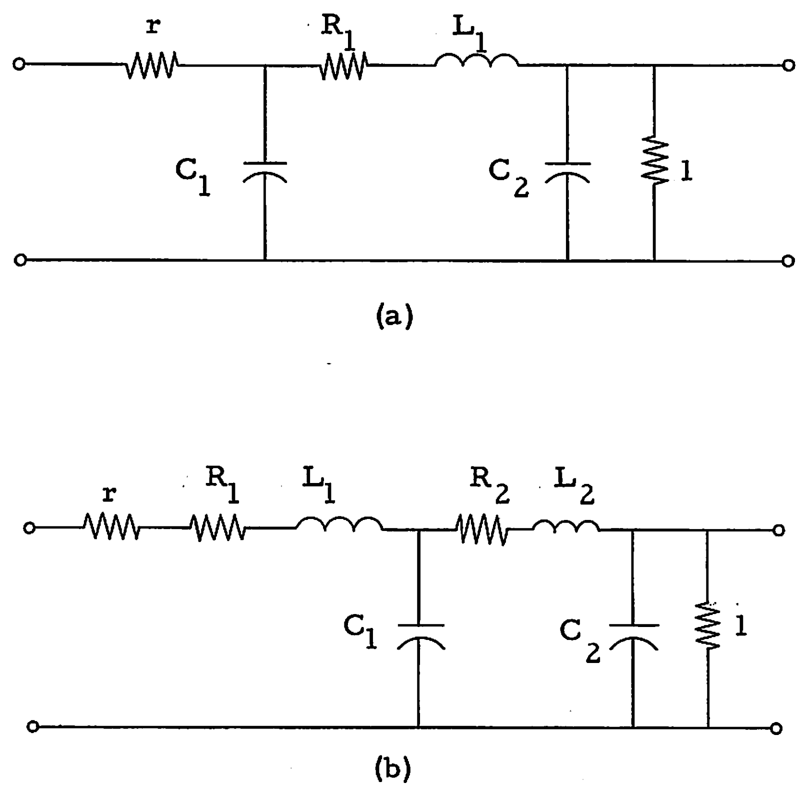


Fig. 2. Doubly-terminated ladder.

Element Values for Fourth Order Nonuniformly Lossy Butterworth Lowpass Filter (Figure 2)

($d = R_i/L_i$)

r = 0 (not dual)

d	L_1	C_1	L_2	C_2	Flat loss (db)
0.00	1.531	1.577	1.082	0.3827	0.00
0.05	1.641	1.518	1.150	0.3979	1.13
0.1	1.757	1.458	1.225	0.4144	2.27
0.15	1.877	1.390	1.310	0.4323	3.39
0.2	2.002	1.322	1.407	0.4519	4.52
0.25	2.128	1.251	1.517	0.4733	5.65
0.3	2.254	1.178	1.645	0.4968	6.73
0.35	2.375	1.104	1.794	0.5228	7.81
0.4	2.486	1.030	1.970	0.5516	8.91
0.45	2.577	0.9575	2.181	0.5838	9.95
0.5	2.633	0.8886	2.437	0.6200	11.0
0.55	2.629	0.8271	2.758	0.6610	11.9
0.6	2.525	0.7801	3.168	0.7077	12.9
0.65	2.250	0.7656	3.720	0.7617	13.8
0.68	1.955	0.7952	4.155	0.7982	14.3
0.70	1.679	0.8545	4.507	0.8244	14.6
0.71	1.511	0.9082	4.706	0.8381	14.7
0.72	1.310	0.9970	4.936	0.8529	14.8
0.73	1.092	1.137	5.177	0.8674	14.9
0.736	0.945	1.272	5.335	0.8764	15.0
0.74	0.8373	1.403	5.449	0.8827	15.1
0.745	0.7000	1.632	5.593	0.8903	15.1
0.748	0.6074	1.846	5.690	0.8952	15.1
0.75	0.5449	2.032	5.754	0.8985	15.2
0.753	0.4476	2.527	5.854	0.9033	15.2
0.755	0.3807	2.818	5.922	0.9066	15.2
0.757	0.3124	3.389	5.991	0.9099	15.2

r = 1/8 (dual)

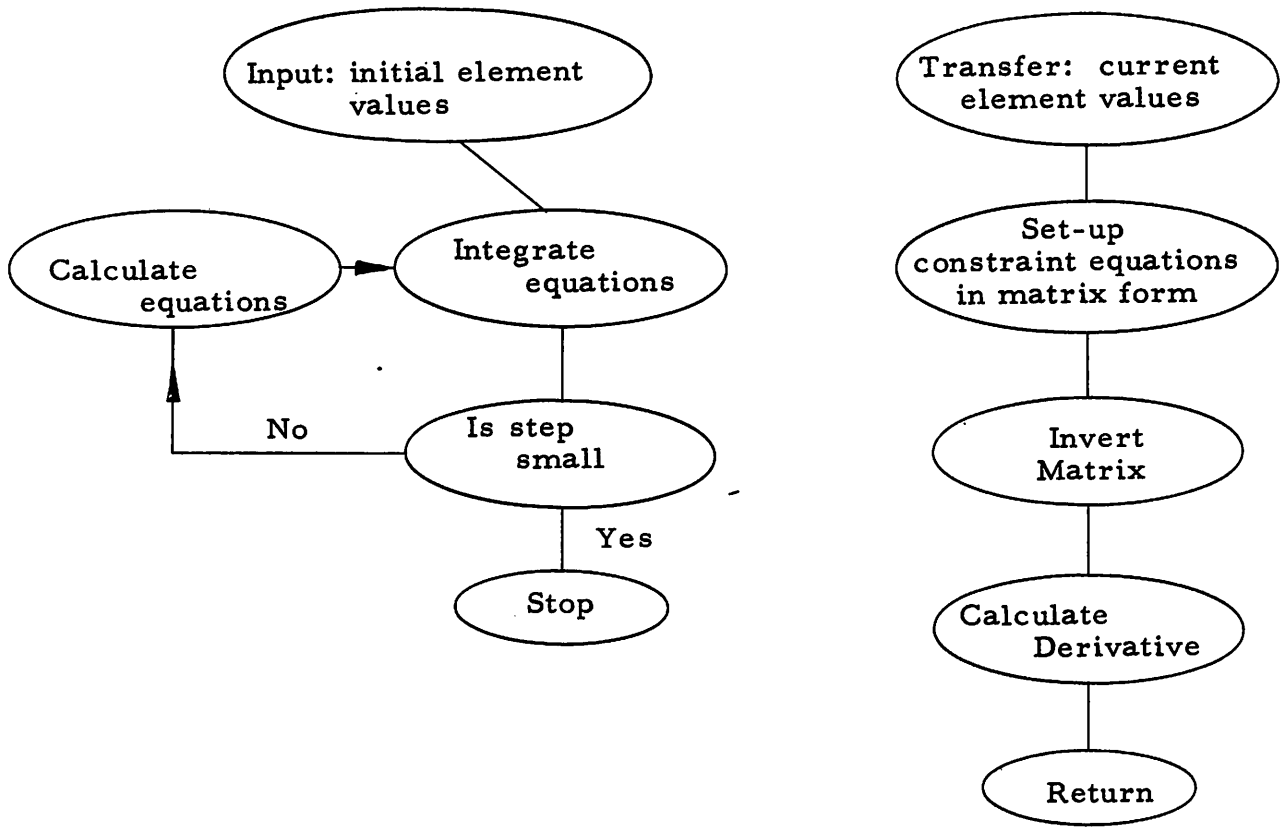
d	L_1	C_1	L_2	C_2	Flat loss (db)
0.00	1.571	1.626	1.116	0.3944	0.00
0.03	1.632	1.586	1.157	0.4035	0.624
0.06	1.695	1.545	1.200	0.4130	1.25
0.09	1.759	1.503	1.247	0.4230	1.87
0.12	1.825	1.461	1.297	0.4335	2.50
0.15	1.891	1.418	1.351	0.4446	3.12
0.18	1.959	1.374	1.409	0.4564	3.75
0.21	2.027	1.329	1.472	0.4687	4.37
0.24	2.094	1.283	1.541	0.4818	4.99
0.27	2.161	1.238	1.616	0.4957	5.61
0.30	2.227	1.191	1.699	0.5105	6.23
0.33	2.289	1.145	1.790	0.5262	6.84
0.36	2.348	1.096	1.890	0.5430	7.45
0.39	2.399	1.054	2.002	0.5609	8.06
0.42	2.442	1.010	2.127	0.5801	8.66
0.45	2.472	0.9666	2.269	0.6008	9.25
0.48	2.484	0.9258	2.430	0.6232	9.83
0.51	2.470	0.8884	2.616	0.6475	10.4
0.54	2.419	0.8565	2.836	0.6742	10.9
0.57	2.311	0.8341	3.102	0.7038	11.5
0.60	2.104	0.8327	3.445	0.7377	11.9
0.61	1.995	0.8427	3.590	0.7505	12.1
0.62	1.843	0.8664	3.767	0.7650	12.3
0.6244	1.581	0.9340	4.020	0.7830	12.4

r = 1/4 (dual)

d	L_1	C_1	L_2	C_2	Flat loss (db)
0.00	1.596	1.672	1.151	0.4072	0.00
0.03	1.652	1.628	1.192	0.4165	0.574
0.06	1.710	1.583	1.237	0.4262	1.15
0.09	1.768	1.537	1.285	0.4365	1.72
0.12	1.828	1.491	1.337	0.4473	2.31
0.15	1.887	1.444	1.393	0.4587	2.88
0.18	1.950	1.397	1.454	0.4708	3.47
0.21	2.007	1.349	1.520	0.4836	4.05
0.24	2.066	1.302	1.592	0.4972	4.62
0.27	2.123	1.254	1.671	0.5116	5.20
0.30	2.176	1.206	1.758	0.5270	5.78
0.33	2.225	1.158	1.854	0.5435	6.35
0.36	2.266	1.112	1.961	0.5611	6.92
0.39	2.298	1.066	2.081	0.5802	7.49
0.42	2.315	1.023	2.217	0.6008	8.05
0.45	2.311	0.9819	2.375	0.6233	8.59
0.48	2.274	0.9459	2.561	0.6483	9.13
0.51	2.183	0.9191	2.790	0.6767	9.64
0.53	2.064	0.9123	2.984	0.6987	9.95
0.54	1.968	0.9173	3.107	0.7115	10.1
0.55	1.791	0.9432	3.284	0.7284	10.2

r = 1/2 (dual)

d	L_1	C_1	L_2	C_2	Flat loss (db)
0.00	1.593	1.765	1.226	0.4350	0.00
0.02	1.623	1.730	1.256	0.4416	0.328
0.04	1.654	1.694	1.288	0.4484	0.657
0.06	1.684	1.659	1.321	0.4554	0.987
0.08	1.714	1.624	1.356	0.4628	1.32
0.10	1.744	1.588	1.392	0.4704	1.65
0.12	1.774	1.553	1.431	0.4783	1.98
0.14	1.803	1.517	1.472	0.4866	2.32
0.16	1.831	1.482	1.515	0.4952	2.65
0.18	1.857	1.447	1.561	0.5042	2.98
0.20	1.883	1.412	1.611	0.5136	3.32
0.22	1.906	1.377	1.664	0.5235	3.66
0.24	1.927	1.343	1.720	0.5339	4.00
0.26	1.944	1.309	1.781	0.5449	4.33
0.28	1.956	1.276	1.847	0.5565	4.66
0.30	1.963	1.244	1.919	0.5689	4.99
0.32	1.961	1.213	1.999	0.5822	5.32
0.34	1.947	1.184	2.088	0.5967	5.65
0.36	1.912	1.159	2.192	0.6131	5.96
0.38	1.835	1.142	2.324	0.6328	6.25
0.39	1.743	1.144	2.421	0.6469	6.38



(a) main program.

(b) derivative subroutine

Fig. 3. Flow chart.

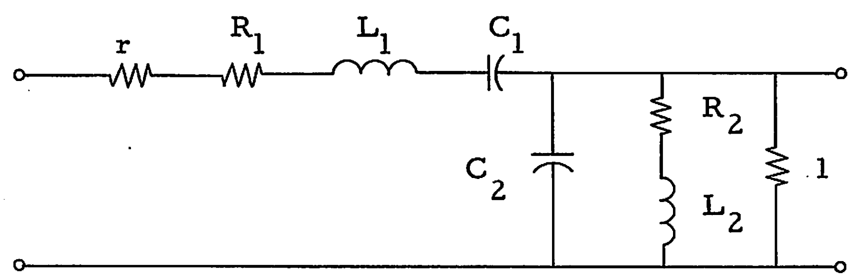
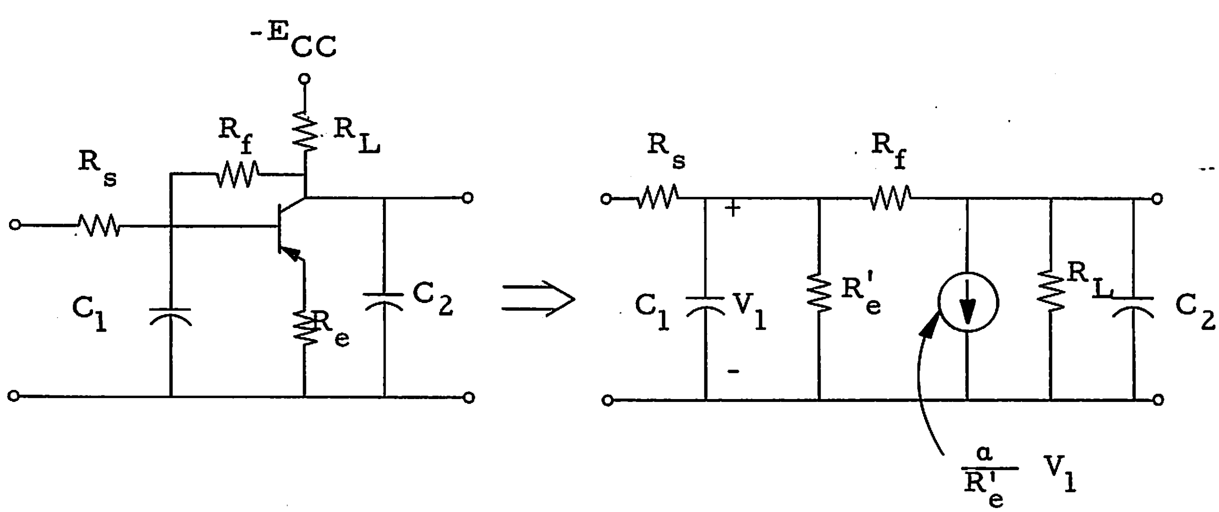


Fig. 4. Band-pass filter.



$$R'_e = R_e + r_e + r_b(1 - a)$$

Fig. 5. Active RC low pass filter.

Element Values for Nonuniformly Lossy Four-Pole Butterworth Bandpass Filters (Figure 4)
($Q = \omega_0/BW$, $\omega_0 = 1$, $d = R_i/L_i$)

r = 1/4, Q = 5, (not dual)

d	L_1	C_1	L_2	C_2	Flat loss (db)
0.00	0.9960	1.004	0.03188	31.37	0.00
0.01	1.084	0.9208	0.03240	30.92	0.580
0.02	1.192	0.8364	0.03325	30.19	1.25
0.03	1.326	0.7502	0.03458	29.10	1.87
0.04	1.501	0.6613	0.03663	27.55	2.42
0.05	1.742	0.5678	0.03988	25.41	3.03
0.06	2.111	0.4654	0.04537	22.50	3.67
0.07	2.742	0.3507	0.05397	19.34	4.62
0.08	3.357	0.2761	0.05245	20.65	7.00
0.09	3.963	0.2282	0.04418	25.12	10.1
0.10	4.764	0.1867	0.03430	32.89	14.1
0.105	5.290	0.1671	0.02908	39.03	16.4
0.11	5.943	0.1479	0.02374	48.04	19.2
0.115	6.776	0.1292	0.01830	62.56	22.6
0.12	7.876	0.1108	0.01279	89.78	27.1
0.125	9.395	0.09261	0.00721	159.5	33.6
0.13	11.63	0.07468	0.00159	725.5	48.6

r = 1/2, Q = 5, (not dual)

d	L_1	C_1	L_2	C_2	Flat loss (db)
0.00	2.2415	0.4461	0.05977	16.73	0.00
0.01	2.487	0.4014	0.06194	16.18	0.621
0.02	2.814	0.3539	0.06550	15.35	1.19
0.03	3.294	0.3012	0.07176	14.08	1.86
0.04	4.151	0.2355	0.08457	12.14	2.62
0.05	4.879	0.1929	0.08574	12.44	4.24
0.06	5.436	0.1691	0.07743	14.11	6.22
0.07	6.100	0.1482	0.06762	16.43	8.49
0.08	6.495	0.1383	0.06247	17.90	11.2
0.09	8.076	0.1095	0.04645	24.44	14.5
0.10	9.651	0.09100	0.03546	32.23	18.4
0.105	10.70	0.08186	0.02989	38.33	20.8
0.110	12.00	0.07280	0.02430	47.26	23.6
0.115	13.66	0.06380	0.01867	61.61	27.2
0.120	15.86	0.05487	0.01301	88.47	31.5

r = 1/2, Q = 5, (dual)

d	L_1	C_1	L_2	C_2	Flat loss (db)
0.00	8.365	0.1195	0.2231	4.483	0.00
0.005	8.243	6.1214	0.2122	4.713	0.320
0.01	8.092	0.1238	0.2011	4.974	0.634
0.015	7.905	0.1269	0.1860	5.275	1.08
0.02	7.675	0.1308	0.1776	5.628	1.25
0.025	7.392	0.1361	0.1649	6.054	1.53
0.03	7.039	0.1432	0.1513	6.588	1.85
0.035	6.589	0.1536	0.1359	7.302	2.16
0.04	6.067	0.1686	0.1183	8.304	2.57
0.044	5.933	0.1753	0.1068	9.042	3.13

r = 1/2, Q = 10, (not dual)

d	L_1	C_1	L_2	C_2	Flat loss (db)
0.00	4.483	0.2231	0.02989	33.96	0.00
0.005	4.974	0.2009	0.03092	32.36	0.639
0.01	5.630	0.1775	0.03264	30.68	1.22
0.015	6.597	0.1513	0.03571	28.07	1.87
0.02	8.540	0.1163	0.04318	23.33	2.51
0.025	10.31	0.09421	0.04424	23.29	4.10
0.03	11.53	0.08318	0.03973	26.26	6.11
0.035	13.04	0.07296	0.03481	30.23	8.43
0.04	15.00	0.06315	0.02973	35.61	11.0
0.045	17.65	0.05331	0.02457	43.30	14.2
0.05	21.46	0.04369	0.01934	55.21	18.0
0.055	27.37	0.03416	0.01408	76.04	23.1
0.06	37.78	0.02470	0.00878	122.2	29.7
0.065	60.93	0.01530	0.00345	311.3	42.1

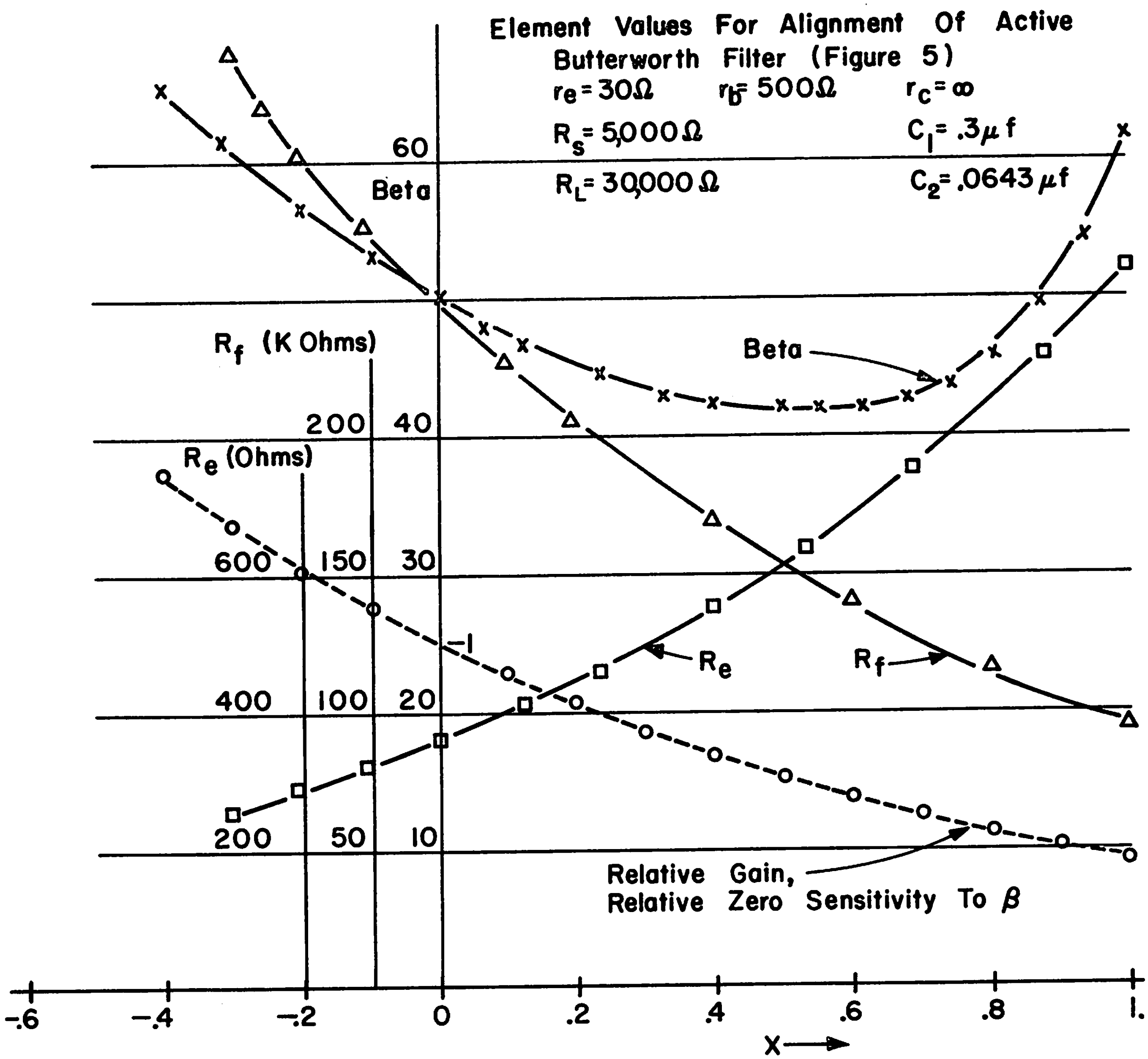


Fig. 6. Element Values For Alignment of Active Butterworth Filter