Performance Analysis of Nonlinear Systems Combining Integral Quadratic Constraints and Sum-of-Squares Techniques

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by

Melissa Erin Summers

A dissertation submitted in partial satisfaction of the
requirements for the degree of
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in

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in
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in the

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Committee in charge:

Professor Andrew Packard, Co-chair
Professor Murat Arcak, Co-chair
Professor Laurent El Ghaoui
Professor Roberto Horowitz

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Abstract

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This thesis investigates performance analysis for nonlinear systems, which consist of both known and unknown dynamics and may only be defined locally. We apply combinations of integral quadratic constraints (IQCs), developed by Megretski and Rantzer, and sum-of-squares (SOS) techniques for the analysis.

In this context, analysis of stability and input-output properties is performed in three ways. If the known portion of the dynamics is linear, the stability test from Megretski and Rantzer, which generalize early frequency-domain based theorems of robust control (Zames, Safonov, Doyle, and others), are well suited. If the known portion of the dynamics is nonlinear, frequency domain methods are not directly applicable. SOS methods using polynomial storage functions to satisfy dissipation inequalities are used to certify the stability and performance characteristics. However, if the known dynamics are high dimensional, then this approach to the analysis is (currently) intractable. An alternate approach is proposed here to address this dimensionality issue. The known portion is decomposed into a linear interconnection of smaller, nonlinear systems. We derive IQCs satisfied by the nonlinear subsystems. This is computationally feasible. With this library of IQCs coarsely describing the subsystems' behaviors, we apply the techniques from Megretski and Rantzer to the interconnection description involving the known linear part and all of the individual subsystems.

Traditionally, IQCs have been used to cover unknown portions of the dynamics. Our approach is novel in that we cover known nonlinear dynamics with IQCs, by employing SOS methods including novel techniques for estimating the input-output gain of a system. This perspective is a step towards reducing the dimensionality of the analysis of large, interconnected nonlinear systems.

The IQC stability analysis by Megretski and Rantzer is only applicable for systems that are well-posed in the large. This thesis makes contributions towards extending this analysis for with more limited notions of well-posedness. We define the notion of a local or "conditional" IQC, and develop a new test to verify stability and performance criteria.

We also study a specific class of interconnected, passive subsystems. If the subsystems also exhibit gain roll-off at high frequencies, one would expect improved analysis results. In fact, we characterized the gain roll-off property as an integral quadratic constraint, and achieved an improved bound on the performance with respect to the allowable time delay in order for the interconnected system to remain stable. In the case where the interconnection is cyclic, we derive an analytical condition for stability.

To Kat

For teaching me to cherish
the joy of the inhale,
the peace of the exhale,
and the endless song of infinity in between.

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Acknowledgments

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Chapter 1

Introduction

This thesis investigates performance analysis for nonlinear systems, which consist of both known and unknown dynamics and may only be defined locally. We apply combinations of integral quadratic constraints (IQCs), developed by Megretski and Rantzer, and sum-of-squares (SOS) techniques for the analysis.

In this context, analysis of stability and input-output properties is performed in three ways:

1. If the known portion of the dynamics is linear, the stability test from Megretski and Rantzer, which generalize early frequency-domain based theorems of robust control (Zames, Safonov, Doyle, and others), are well suited;

2. If the known portion of the dynamics is nonlinear, frequency domain methods are not directly applicable. SOS methods using polynomial storage functions to satisfy dissipation inequalities are used to certify the stability and performance characteristics. However, if the known dynamics are high dimensional, then this approach to the analysis is (currently) intractable;

3. an alternate approach is proposed here to address this dimensionality issue. The known portion is decomposed into a linear interconnection of smaller, nonlinear systems. We derive IQCs satisfied by the nonlinear subsystems. This is computationally feasible. With this library of IQCs coarsely describing the subsystems' behaviors, we apply the techniques from Megretski and Rantzer to the interconnection description involving the known linear part and all of the individual subsystems.

Traditionally, IQCs have been used to cover unknown portions of the dynamics. This third approach is novel in that we cover known nonlinear dynamics with IQCs. We employ SOS methods to generate IQCs to cover the known dynamics. In particular, the IQCs established in this thesis involve estimating the $\mathbf{L}_2 \to \mathbf{L}_2$ norm of a system with a linear offset and a weight. Many weights and linear offsets can be chosen, which in turn establishes many local IQCs for the system. Ultimately, we envision the possibility of creating large libraries (with hundreds of thousands of entries) of small (1-3 state) nonlinear system models, each with an associated (long) list of IQCs which the model satisfies (locally). A large system can be decomposed into and interconnection, and the interconnection can be quickly analyzed, using the library. If the analysis is inconclusive, an alternative decomposition can be proposed, and the analysis repeated. This approach is a step towards reducing the dimensionality of the analysis of large, interconnected nonlinear systems.

The IQC stability analysis by Megretski and Rantzer is only applicable for systems that are defined on all of $\mathbf{L}_{2e}$ and bounded on $\mathbf{L}_2$. We extend the stability test to include interconnections of many locally stable operators which satisfy many local IQCs. This thesis makes large contributions towards extending this analysis for local systems. We define the notion of a local or "conditional" IQC. For systems which satisfy local IQCs, we develop a new test to verify stability and performance criteria.

In Chapter 2 we develop theoretical and numerical tools for quantitative local analysis of nonlinear systems. Specifically, sufficient conditions are provided for bounds on the reachable set and $\mathbf{L}_2$ gain of the nonlinear system subject to norm bounded disturbance inputs. The main theoretical results are extensions of classical dissipation inequalities but enforced only on local regions of the state and input space. Computational algorithms are derived from these local results by restricting to polynomial systems, using convex relaxations, e.g. the S-procedure, and applying sum-of-squares optimizations. Several pedagogical and realistic examples are provided to illustrate the proposed approach.

In Chapter 3 we first recall the definitions of IQCs, stability theorems [59], and performance analysis techniques using IQCs [4]. We introduce the definition of a local IQC for a bounded, causal operator which is defined locally. We outline a procedure for establishing local IQCs using linear offsets, linear weights and estimates of local $\mathbf{L}_2$ gains. We show how SOS methods can be used to generate these IQCs and present and example demonstrating the technique. The stability criteria for [59] rely on the operators being defined on all of $\mathbf{L}_{2e}$ and bounded on $\mathbf{L}_2$. We present conditions in which a global extension, which meets the criteria of [59], of a local operator exists and satisfies the same IQC on all of $\mathbf{L}_2$. If an operator satisfies many IQCs, then it also satisfies a positive combination of those IQCs. We show if a local operator satisfies a special class of IQCs, then an extension which also satisfies an IQC in that class exists. For a special class of parameterized systems, we show how IQCs for one system in the paramter space map to IQCs for another system with a different choice of parameters using input-output and time-scaling.

In Chapter 4 we tie together concepts addressed in 3 to establish performance analysis for interconnections of systems who satisfy local IQCs. First, we develop a frequency domain performance analysis test for a system with a single operator which satisfies a single, local IQC. Next, we extend frequency domain results for a single operator which satisfy many local IQCs. Finally, we address techniques for evaluating performance criteria for an interconnection of many operators, each with satisfy many local IQCs. For this most general case, a state-space condition is presented for the case when the performance metric is an $\mathbf{L}_2 \to \mathbf{L}_2$ gain criteria. An example of an interconnection of three systems, which each satisfy many IQCs is presented. We compare the frequency domain IQC analysis with a direct SOS approach and a worst-case input simulation.

In Chapter 5 we consider networks of passive systems with time delays in the interconnections, and present a stability analysis technique with the help of the integral quadratic constraint (IQC) framework. Unlike the classical passivity approach that fails to characterize delay robustness, and the small-gain approach that conservatively accounts for arbitrarily large delays, the new technique gives sharp stability estimates that depend on the duration of delay. Since the effect of delay depends on its duration relative to the time scales of the system, we make use of a "roll-off" IQC that captures magnitude roll-off at high frequencies, thus, providing the critical time-scale information. We then combine this roll-off IQC with an output strict passivity IQC that incorporates gain and phase information, and demonstrate the benefit of this combined IQC approach on a cyclic interconnection structure with delay. Finally, we develop a technique to verify these IQCs for classes of nonlinear state-space models and present an example from Internet congestion control.

Chapter 6 presents the conclusions and suggestions for future research.

The Appendix lists notation, facts, lemmas and proofs concerning functional analysis, Lipschitz extensions, polynomials, sum-of-squares, and the s-procedure.

Chapter 2

Performance Characterizations

2.1 Context and Acknowledgements

We rely on the notions of the performance characterizations for $\mathbf{L}_2$ gain and reachability, and the sum-of-squares (SOS) techniques used to estimate the performance characterizations in subsequent chapters. This material is to appear in the International Journal of Robust and Nonlinear Control, co-authored by Abhijit Chakraborty, Weehong Tan, Ufuk Topcu, Peter Seiler, Gary Balas, and Andrew Packard entitled "Quantitative Local $\mathbf{L}_2$-gain and Reachability Analysis for Nonlinear Systems".

2.2 Introduction

We focus on dynamical systems governed by differential equations of the form

$$\dot{x}(t) = f(x(t)) + g(x(t))w(t), \tag{2.1}$$

$$z(t) = h(x(t)), \tag{2.2}$$

where $t \in \mathbb{R}$, $x(0) = x_0 \in \mathbb{R}^n$, $x(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^p$, $w(t) \in \mathbb{R}^m$. The functions $f : \mathbb{R}^n \to \mathbb{R}^n, g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are assumed to be Lipschitz continuous, or locally Lipschitz continuous, depending on the situation. If f and g are not Lipschitz continuous (as in the case of polynomial f and g, for example), then the differential equation may exhibit finite escape times in the presence of bounded inputs and/or initial conditions.

There is a large literature on input/output gain of nonlinear dynamical systems described by ordinary differential equations (ODEs) [76, 39]. Disturbance rejection and noise insensitivity are critical metrics of performance in a closed-loop control system, and being able to quantify such metrics allows one to discriminate among competing designs. The importance of the gain, and other general properties of dissipativeness (e.g., passivity, or more general forms) is realized in hierarchical interconnection theorems, such as small-gain, passivity theorems, and integral quadratic constraints, where coarse input/output properties of a collection of individual subsystems can be used to infer properties of specific interconnections of these components, [103, 101, 27, 59, 99, 5].

The overarching goal of the research, reported here and in related papers, [83, 89, 91, 90, 82] is quantitative, local analysis of nonlinear dynamical systems. By "quantitative" we mean algorithms and sufficient conditions which lead to concrete guarantees about a particular system's response. By "local" we refer to guarantees about the reachability and/or system gain which are predicated on assumptions concerning the magnitude of initial conditions and input signals. We extensively use, without further citation, the basic, fundamental ideas from dissipative systems theory [100], [40], barrier functions and reachability [97], [71], and nonlinear optimal control [76], [39]. Specifically, we employ inequalities involving the Lie derivative of a scalar function, the *storage* function, that hold throughout regions of the state and input space, which when integrated over trajectories of the system, give certificates of input/output properties of the system. The necessity of the existence of such storage functions to prove input/output properties, which leads to the most elegant results of the above mentioned works, is actually not used. Our computational approach is based on polynomial storage functions of fixed degree which can be viewed as extensions of known linear matrix inequality conditions to compute reachable sets and input/output gains for linear systems [12]. Due to the restriction to polynomial storage functions our results typically do not approach the theoretical optimal storage functions. Current theoretical work is addressing the necessity of polynomial storage functions for systems with polynomial vector fields, and is of deep theoretical importance for our work. Some results, [69] are positive, while others are negative, [1, 2].

A number of recent publications have used sum-of-squares relaxations for polynomial optimization in the analysis of dynamical systems or design of feedback control laws. Reference [29] derives sufficient conditions based on dissipation inequalities for a number of interesting questions, including analysis of minimum phase behavior and design of synchronizing feedback. The formulations are not local, as the sufficient conditions are imposed throughout the entire state space. Similar techniques are used in [63] for global reachability and input-output gain analysis.

Reference [72] studies a rich set of system models, encompassing hybrid dynamics with polynomial vector fields for the continuous evolution. This and related work, [73], derives sufficient conditions, based on barrier certificates, for the verification of a set of temporal properties, including safety, reachability, and eventuality. An alternative approach on quantitative analysis of dynamical systems is based on the computation of reachable sets in the state space as the solution of certain Hamilton-Jacobi-Isaacs partial differential equations [88]. The toolbox in [60] provides an implementation of this method using level set methods.

Two recent works that are very similar in spirit to this research are [24] and [105]. Reference [24] uses local dissipation inequalities (similar to those used here but with a restricted class of supply rates) to characterize input-output gain properties of nonlinear systems. We assume the disturbance inputs are such that the state remains within a specified region. For systems with vector fields rational in the states, it provides semidefinite programming based methods to search for polynomial storage functions that satisfy these dissipation inequalities.

Reference [105] introduces a nonlinear $\mathbf{L}_2$ gain function that bounds the output $\mathbf{L}_2$ norm as a function of the input $\mathbf{L}_2$ norm. The gain function is characterized in terms of a dissipation property of an augmented system with storage functions that are solutions of certain partial differential inequalities.

Computing the exact region of attraction (ROA) for an equilibrium point of a nonlinear system is a related problem, loosely corresponding to a special choice of supply rate. Most research focuses on constructing invariant subsets of the ROA, computing a Lyapunov function and a sublevel set of this function that is a provably invariant subset of the ROA [26, 33, 94, 22, 21, 63, 36, 85, 86, 67, 23, 20, 18, 19, 104, 65, 17, 84]. Much of the recent work uses sum-of-squares (SOS) relaxation methods to compute polynomial Lyapunov functions for systems described by polynomial or rational dynamics. Exciting new approaches to treat high-dimensional systems based on large-scale decomposition techniques are now available, [3].

2.3 Reachability

In this section, we establish conditions which guarantee invariance of certain sets under $\mathbf{L}_2$ and pointwise-in-time ($\mathbf{L}_\infty$-like) constraints on w. These are subsequently referred to as "reachability" results, since the conclusions yield outer bounds on the set of reachable states. In that vein, w is interpreted as a disturbance, whose worst-case effect on the state x is being quantified. We obtain bounds on x that are tightly linked with the assumed bounds on w and x_0, and specifically allow for systems which are not well-defined on all input signals (finite escape times). Computational approaches based on the S-procedure and sum-of-squares are introduced in Section 2.9.

A known set $\mathcal{W} \subseteq \mathbb{R}^m$ is used to express any $\mathbf{L}_\infty$-like, pointwise-in-time bound on the signal w, namely $w(t) \in \mathcal{W}$ for all t. Setting $\mathcal{W} = \mathbb{R}^m$ is equivalent to the absence of known, pointwise-in-time bounds on w.

Theorem 2.1. *Suppose $\mathcal{W} \subseteq \mathbb{R}^m$. Assume that f and g in (2.1) are Lipschitz continuous on $\mathbb{R}^n$. Suppose $\tau > 0$, and a differentiable $Q : \mathbb{R}^n \to \mathbb{R}$ satisfies $Q(0) < \tau^2$ and*

$$\Omega^{cc,0}_{Q,\tau^2} \times \mathcal{W} \subseteq \left\{(x, w) \in \mathbb{R}^n \times \mathbb{R}^m : \nabla Q(x) \cdot [f(x) + g(x)w] \leq w^T w\right\}. \tag{2.3}$$

Consider $x_0 \in \Omega^{cc,0}_{Q,\tau^2}$ with $Q(x_0) < \tau^2$ and $w \in \mathbf{L}_2^m$ with $w(t) \in \mathcal{W}$ for all t. If $\|w\|_2^2 < \tau^2 - Q(x_0)$, the solution to (2.1) with $x(0) = x_0$ satisfies $Q(x(t)) < \tau^2$ for all t, and hence $x(t) \in \Omega^{cc,0}_{Q,\tau^2}$ for all t.

Proof. Suppose not, and define $T > 0$ such that $Q(x(t)) < \tau^2 \ \forall\, t \in [0, T)$ and $Q(x(T)) = \tau^2$. Indeed, such a T exists since x and Q are continuous and $Q(x_0) < \tau^2$. Hence, on $[0, T]$, $x(t) \in \Omega^{cc,0}_{Q,\tau^2}$. Since Q is differentiable and x is absolutely continuous and $w(t) \in \mathcal{W}$ for all t, integrating the dissipation inequality in equation (2.3) over the interval $[0, T]$ gives

$$Q(x(T)) \leq Q(x_0) + \|w\|^2_{2,T}\,.$$

Recall $\tau^2 = Q(x(T))$, so $\|w\|_2^2 \geq \|w\|_{2,T}^2 \geq \tau^2 - Q(x_0)$, establishing the result by contradiction. $\square$

Remark 2.1. *Without loss of generality, Q in Theorem 2.1 can be taken to be zero at $x = 0$. For instance, define $\tilde{Q}(x) := Q(x) - Q(0)$ and $\tilde{\tau}^2 := \tau^2 - Q(0)$. The conditions of Theorem 2.1 hold with $\tilde{Q}$ replacing Q, and the same norm bound (i.e. reachable set) is obtained.*

Remark 2.2. *Condition* (2.3) *can be equivalently expressed as*

$$\Omega^{cc,0}_{Q,\tau^2} \subseteq \left\{ x \in \mathbb{R}^n : \max_{w \in \mathcal{W}} \nabla Q(x) \cdot [f(x) + g(x)w] - w^T w \leq 0 \right\}. \tag{2.4}$$

The next theorem relaxes the assumption that f and g are globally Lipschitz continuous in exchange for assuming boundedness of $\Omega^{cc,0}_{Q,\tau^2}$.

Theorem 2.2. *Suppose f and g in* (2.1) *are locally Lipschitz continuous, and hence Lipschitz continuous on any bounded set. Assume all the other conditions of Theorem 2.1 are satisfied. If, in addition,*

$$\Omega^{cc,0}_{Q,\tau^2} \quad \textit{is} \quad \textit{bounded}, \tag{2.5}$$

then the conclusion of Theorem 2.1 remains true.

Proof. By Lemma 7.2, since $\Omega^{cc,0}_{Q,\tau^2}$ is bounded, f and g can be extended to globally Lipschitz continuous functions, $\tilde{f}$ and $\tilde{g}$ such that $f(x) = \tilde{f}(x)$ and $g(x) = \tilde{g}(x)$ for all $x \in \Omega^{cc,0}_{Q,\tau^2}$. The conditions of Theorem 2.1 hold for $\tilde{f}$ and $\tilde{g}$, and hence the conclusions apply to solutions of

$$\dot{x}(t) = \tilde{f}(x(t)) + \tilde{g}(x(t))w(t). \tag{2.6}$$

Consequently, for all x_0 with $Q(x_0) < \tau^2$ and $w \in \mathbf{L}_2^m$ with $\|w\|_2^2 < \tau^2 - Q(x_0)$, the solution of (2.6) satisfies $x(t) \in \Omega^{cc,0}_{Q,\tau^2}$ for all t. Since the solution remains in the region where $f = \tilde{f}$ and $g = \tilde{g}$, it must be that the solution to (2.1) is the same function, and has the properties as claimed. $\square$

2.4 Reachability Refinement

The sufficient conditions presented thus far consider any differentiable Q as a barrier function. In the computational approach we pursue later, polynomials play a key role, and the choice of Q will be restricted to polynomials of a given degree. This restriction limits the expressiveness of Q, and may introduce "slack" in the differential inequality (DIE) (2.3), meaning that the maximum of the DIE in (2.4) is 0 for <u>some</u>, but not all values of x. Here, following [83], we partially remove the slack, yielding a function M whose sublevel sets are the same as those of Q. The reachability bound certified by M is generally an improvement of the bound guaranteed by Q.

Theorem 2.3. *Suppose $\tau > 0$ and $k : \mathbb{R} \to \mathbb{R}$ is piecewise continuous, with $0 < k(\xi) \leq 1$ for all $\xi \in [0,\ \tau^2]$. Assume that f and g in (2.1) are Lipschitz continuous, and a differentiable $Q : \mathbb{R}^n \to \mathbb{R}$ satisfies $Q(0) = 0$ and*

$$\Omega^{cc,0}_{Q,\tau^2} \subseteq \left\{ x \in \mathbb{R}^n : \max_{w \in \mathcal{W}} \nabla Q(x) \cdot [f(x) + g(x)w] - k(Q(x))w^T w \leq 0 \right\}. \tag{2.7}$$

Then, for all $x_0 \in \Omega^{cc,0}_{Q,\tau^2}$, $T > 0$, with $Q(x_0) < \tau^2$ and $w \in \mathbf{L}_2^m$ with $w(t) \in \mathcal{W}$ for all t and

$$\|w\|^2_{2,T} < \int_{Q(x_0)}^{\tau^2} k^{-1}(\xi) d\xi, \tag{2.8}$$

the solution to (2.1) satisfies $x(t) \in \Omega^{cc,0}_{Q,\tau^2}$ for all $t \in [0, T]$.

Proof. Define $M(x) := \int_0^{Q(x)} \frac{1}{k(\xi)} d\xi$ and $\tau_e^2 := \int_0^{\tau^2} \frac{1}{k(\xi)} d\xi > 0$. Note that $\tau_e \geq \tau$ and M is a differentiable function satisfying $M(0) = 0$ and

$$\nabla M(x) = \frac{1}{k(Q(x))} \nabla Q(x).$$

It follows from the definitions of $M(x)$ and τ_e^2 that $M(x) \leq \tau_e^2$ if and only if $Q(x) \leq \tau^2$. Therefore

$$\Omega^{cc,0}_{Q,\tau^2} = \Omega^{cc,0}_{M,\tau_e^2} \subseteq \left\{ x \in \mathbb{R}^n : \max_{w \in \mathcal{W}} \nabla M(x) \cdot [f(x) + g(x)w] \leq w^T w \right\}. \tag{2.9}$$

By Theorem 2.1, for all $x_0 \in \Omega^{cc,0}_{M,\tau_e}$ with $M(x_0) < \tau_e^2$, $T > 0$ and $w \in \mathbf{L}_2^m$ with $w(t) \in \mathcal{W}$ for all t and $\|w\|^2_{2,T} < \tau_e^2 - M(x_0)$, the solution to (2.1) satisfies $x(t) \in \Omega^{cc,0}_{M,\tau_e^2}$ for all $t \in [0, T]$. Recalling (from (2.9)) that $\Omega^{cc,0}_{Q,\tau^2} = \Omega^{cc,0}_{M,\tau_e^2}$ completes the proof. □

Remark 2.3. *The only difference between (2.4) and (2.7) is that $w^T w$ is replaced by $k(Q(x))w^T w$. Consequently, if $k(\xi) < 1$ for some ξ, then (2.7) is a stronger condition than (2.4) and, consequently, the new allowable bound on $\|w\|_2$ in (2.8) is larger than the original bound of $\tau^2 - Q(x_0)$.*

2.5 L$_2$ Gain

In this section, we establish conditions which bound the $\mathbf{L}_2$ gain of (2.1) under $\mathbf{L}_2$ and pointwise-in-time ($\mathbf{L}_\infty$-like) constraints on w. The results are local, in that the obtained bounds on the gain depend on bounds on w and x_0. For the system in (2.1), recall the following definition from [76].

Definition 2.1. *The system* (2.1) *is said to have finite* $\mathbf{L}_2$ *gain if there exist a finite constant* $\rho > 0$ *and for every initial condition* x_0*, a finite constant* $\phi(x_0) \geq 0$ *such that solutions of* (2.1) *satisfy*

$$\|z\|_{2,T} \leq \phi(x_0) + \rho\|w\|_{2,T} \tag{2.10}$$

for all $w \in \mathbf{L}_2$ *and for all* $T \geq 0$. □

Theorem 2.4. *Suppose* $\mathcal{W} \subseteq \mathbb{R}^m$*. Assume that* f*,* g *and* h *in* (2.1) *are Lipschitz continuous. Suppose* $\gamma > 0$*,* $R > 0$*, and a differentiable* $V : \mathbb{R}^n \to \mathbb{R}$ *satisfies* $V(0) = 0$*, and*

$$\Omega^{cc,0}_{V,R^2} \setminus 0 \subseteq \{x \in \mathbb{R}^n : V(x) > 0\}, \tag{2.11}$$

$$\Omega^{cc,0}_{V,R^2} \times \mathcal{W} \subseteq \left\{(x,w) \in \mathbb{R}^n \times \mathbb{R}^m : \nabla V(x) \cdot [f(x) + g(x)w] \leq w^T w - \frac{1}{\gamma^2} h^T(x)h(x)\right\}. \tag{2.12}$$

Consider $x_0 \in \Omega^{cc,0}_{V,R^2}$ *with* $V(x_0) < R^2$*,* $T > 0$ *and* $w \in \mathbf{L}_2^m$ *with* $w(t) \in \mathcal{W}$ *for all* t*. If* $\|w\|^2_{2,T} \leq R^2 - V(x_0)$*, the solution to* (2.1) *with* $x_0 = x(0)$ *satisfies* $x(t) \in \Omega^{cc,0}_{V,R^2}$ *for all* $t \in [0,T]$ *and*

$$\|z\|_{2,T} \leq \gamma\sqrt{V(x_0)} + \gamma \|w\|_{2,T}\,. \tag{2.13}$$

Moreover, if conditions (2.11) *and* (2.12) *hold, then any constraint on* w *and* x_0 *that ensures* $x(t) \in \Omega^{cc,0}_{V,R^2}$ *for the solutions will also yield the gain bound in* (2.13)

Proof. The conditions in (2.12) are stricter than the reachability conditions in (2.3), hence the norm-bound on w ensures that the trajectories remain in $x(t) \in \Omega^{cc,0}_{V,R^2}$. Hence (2.12) can be integrated over the solution on $[0,T]$, giving

$$\gamma^2 V(x(T)) + \|z\|^2_{2,T} \leq \gamma^2 V(x_0) + \gamma^2 \|w\|^2_{2,T}\,.$$

The additional assumption that V is nonnegative on $\Omega^{cc,0}_{V,R^2}$ implies

$$\|z\|_{2,T} \leq \gamma\sqrt{V(x_0)} + \gamma \|w\|_{2,T} \tag{2.14}$$

as claimed (via completion-of-squares). Finally, it is clear that the bound is true for any $w \in \mathbf{L}_2^m$ and x_0 under the condition that the state trajectories remain in $\Omega^{cc,0}_{V,R^2}$. □

Remark 2.4. *The* $\mathbf{L}_2$ *gain supply rate in* (2.12) *(i.e.,* $w^T w - \frac{1}{\gamma^2} h^T(x)h(x)$*) can be replaced by a more general supply rate* $r(w, h(x))$*. If* (2.11) *and the modification to* (2.12) *hold, then for combinations of inputs* w *and initial conditions* x_0 *which lead to* $x(t) \in \Omega^{cc,0}_{V,R^2}$ *for all* t*, dissipativity with respect to the supply rate* $r(w,z)$ *is established, namely* $0 \leq V(x(T)) \leq V(x(0)) + \int_0^T r(w(t), z(t))\, dt$*. Then, a separate analysis, employing the results from Section 2.6 leads to explicit bounds on* w *and* x_0 *which render* $x(t) \in \Omega^{cc,0}_{V,R^2}$ *for all* t*, and hence guarantee the dissipativeness.*

Corollary 2.1. *Suppose* f*,* g *and* h *are locally Lipschitz continuous, the conditions of Theorem 2.4 hold and, in addition,* $\Omega^{cc,0}_{V,R^2}$ *is bounded. The conclusion of Theorem 2.4 holds.*

Proof. The proof follows by applying Lemma 7.2 to f, g and h. □

2.6 Combining reachability bounds with $\mathbf{L}_2$ gain estimates

As noted in the proof of Theorem 2.4, the bound (2.13) holds for any constraint on w and x_0 which ensures that $x(t)$ remains in $\Omega^{cc,0}_{V,R^2}$. In Theorem 2.4, one such condition is $\|w\|^2_{2,T} < R^2 - V(x_0)$. However, it is advantageous to make a separate reachability analysis (using a new storage function) to ascertain bounds on w which keep $x(t) \in \Omega^{cc,0}_{V,R^2}$. Theorem 2.5 below clarifies this process.

Theorem 2.5. *Assume the conditions of Theorem 2.1 and Theorem 2.4 hold. If, in addition,*

$$\Omega^{cc,0}_{Q,\tau^2} \subseteq \Omega^{cc,0}_{V,R^2}, \tag{2.15}$$

then, for all $x_0 \in \Omega^{cc,0}_{Q,\tau^2}$ with $Q(x_0) < \tau^2$, $T > 0$, and $w \in \mathbf{L}_2^m$ with $w(t) \in \mathcal{W}$ for all t and $\|w\|^2_{2,T} \leq \tau^2 - Q(x_0)$, the solution to (2.1) satisfies $x(t) \in \Omega^{cc,0}_{Q,\tau^2}$ for all $t \in [0,T]$ and $\|z\|^2_{2,T} \leq \gamma^2 V(x_0) + \gamma^2 \|w\|^2_{2,T}$.

Proof. The solution satisfies $x(t) \in \Omega^{cc,0}_{Q,\tau^2}$ for all $t \in [0,T]$ by Theorem 2.1. The condition in (2.14) holds since $w \in \mathbf{L}_2^m$ and the state trajectories remain in $\Omega^{cc,0}_{V,R^2}$ by (2.15). □

Obviously, the procedure described in Theorem 2.3 can be used to relax the conditions on w such that $x(t)$ remains in Ω_{Q,τ^2}.

Theorem 2.6. *Assume the conditions of Theorems 2.3 and 2.4 hold, and $\Omega^{cc,0}_{Q,\tau^2} \subseteq \Omega^{cc,0}_{V,R^2}$. Then for all x_0 with $M(x_0) < \tau_e^2$, $T > 0$, and all $w \in \mathbf{L}_2^m$ with $w(t) \in \mathcal{W}$ for all t and $\|w\|^2_{2,T} \leq \tau_e^2 - M(x_0)$, the solution satisfies $x(t) \in \Omega^{cc,0}_{M,\tau_e^2}$ for all $t \in [0,T]$ and $\|z\|^2_{2,T} \leq \gamma^2 V(x_0) + \gamma^2 \|w\|^2_{2,T}$.*

Proof. The solution satisfies $x(t) \in \Omega^{cc,0}_{M,\tau_e^2}$ for all $t \in [0,T]$ by Theorem 2.3. The condition in (2.14) holds since $w \in \mathbf{L}_2^m$ and the state trajectories remain in $\Omega^{cc,0}_{V,R^2}$ by (2.15) and (2.9). □

Remark 2.5. *Theorems 2.5 and 2.6 can be applied to locally Lipschitz continuous f, g and h by enforcing that $\Omega^{cc,0}_{V,R^2}$ is bounded.*

2.7 Reachability and gain estimates for uncertain systems

This section extends the conditions in Theorems 2.1 and 2.4 to systems with dynamic uncertainty. The uncertainty is modeled in the standard linear fractional transformation framework, with the uncertain element obeying multiple, known, dissipativeness conditions.

Consider the dynamics of a multivariable, nonlinear system, G

$$\begin{array}{rcl} \dot{x}(t) & = & f(x(t)) + g_1(x(t))w_1(t) + g_2(x(t))w_2(t), \\ z_1(t) & = & h_1(x(t)), \\ z_2(t) & = & h_2(x(t)), \end{array} \tag{2.16}$$

where $x(t) \in \mathbb{R}^n$, $z_1(t) \in \mathbb{R}^{p_1}$, $z_2(t) \in \mathbb{R}^{p_2}$, $w_1(t) \in \mathbb{R}^{m_1}$, and $w_2(t) \in \mathbb{R}^{m_2}$. For notational simplicity, define $\tilde{f} : \mathbb{R}^{n \times m_1 \times m_2} \to \mathbb{R}^n$ as $\tilde{f}(x, w_1, w_2) := f(x) + g_1(x)w_1 + g_2(x)w_2$.

Likewise, let $\Delta : \mathbf{L}_{2e}^{p_2} \to \mathbf{L}_{2e}^{m_2}$ be a bounded, causal operator. Assume $\{r_i : \mathbb{R}^{p_2} \times \mathbb{R}^{m_2} \to \mathbb{R}\}_{i=1}^N$ is a collection of supply rates for which the operator Δ is dissipative with respect to. This means that the behavior of Δ guarantees that the constraints

$$\int_0^T r_i(q(t), (\Delta(q))(t))dt \geq 0 \tag{2.17}$$

are satisfied for all $q \in \mathbf{L}_{2,e}^{p_2}$ and all $T > 0$.

Assume that Δ and G form a well-posed interconnection through the constraint

$$w_2(t) = (\Delta(z_2))(t) \tag{2.18}$$

as shown in Figure 2.1, meaning that for any $w_1 \in \mathbf{L}_{2e}^{m_1}$, and any initial condition x_0, there exists unique $w_2 \in \mathbf{L}_{2e}^{m_2}$ and absolutely continuous functions x, z_1 and z_2 satisfying equations (2.16) and (2.18), and all causally dependent on w_1. This is an assumption about the interaction of G and Δ. It is true, for instance, if Δ is governed by nonlinear ODEs of the form

$$\begin{array}{rl} \dot{\xi}(t) & = a(\xi(t)) + b(\xi(t))z_2(t) \\ w_2(t) & = c(\xi(t)) + d(\xi(t))z_2(t) \end{array}$$

for Lipschitz continuous functions a, b, c and d.

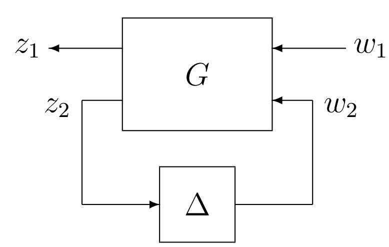

Figure 2.1: G-Δ interconnection.

The following proposition analyzes the interconnection, establishing $\mathbf{L}_2$ gain bounds from w_1 to z_1 valid for all operators Δ that are dissipative with respect to the supply rates $r_1, \ldots, r_N$.

Theorem 2.7. *Suppose $\mathcal{W}_1 \subseteq \mathbb{R}^{m_1}$. Assume that f, g_1, g_2, h_1, and h_2 in (2.16) are Lipschitz continuous. Let $r_1, \ldots, r_N : \mathbb{R}^{p_2 \times m_2} \to \mathbb{R}$. Suppose there exist constant $\tau > 0, R > 0$,* *nonnegative* $\lambda_i \in \mathbb{R}$*,* $\beta_i \in \mathbb{R}$*, differentiable functions* $Q : \mathbb{R}^n \to \mathbb{R}$ *and* $V : \mathbb{R}^n \to \mathbb{R}$ *that satisfy* $Q(0) = V(0) = 0$*,*

$$\Omega^{cc,0}_{Q,\tau^2} \subseteq \Omega^{cc,0}_{V,R^2} \tag{2.19}$$

$$\Omega^{cc,0}_{V,R^2} \setminus 0 \subseteq \{x \in \mathbb{R}^n : V(x) > 0\} \tag{2.20}$$

$$\Omega^{cc,0}_{Q,\tau^2} \subseteq \Big\{x \in \mathbb{R}^n : \nabla Q(x) \cdot \tilde{f}(x, w_1, w_2) \le w_1^T w_1 - \sum_{i=1}^{N} \lambda_i r_i(h_2(x), w_2),\ \forall (w_1, w_2) \in \mathcal{W}_1 \times \mathbb{R}^{m_2}\Big\} \tag{2.21}$$

$$\Omega^{cc,0}_{V,R^2} \subseteq \Big\{x \in \mathbb{R}^n : \nabla V(x) \cdot \tilde{f}(x, w_1, w_2) \le w_1^T w_1 - \frac{1}{\gamma^2} h_1^T(x) h_1(x) - \sum_{i=1}^{N} \beta_i r_i(h_2(x), w_2),\ \forall (w_1, w_2) \in \mathcal{W}_1 \times \mathbb{R}^{m_2}\Big\}. \tag{2.22}$$

Consider $x_0 \in \Omega^{cc,0}_{Q,\tau^2}$ *with* $Q(x_0) < \tau^2$*,* $T > 0$*, and* $w_1 \in \mathbf{L}_2^{m_1}$ *with* $w_1(t) \in \mathcal{W}_1$ *for all* t*. If* $\|w_1\|^2_{2,T} < \tau^2 - Q(x_0)$*, then for all* Δ *dissipative with respect to the supply rates* $r_1, \ldots, r_N$*, the solution to* (2.16)-(2.18) *with* $x(0) = x_0$ *satisfies* $Q(x(t)) < \tau^2$ *for all* $t \in [0, T]$*, and*

$$\|z_1\|^2_{2,T} \le \gamma^2 V(x_0) + \gamma^2 \|w_1\|^2_{2,T} \tag{2.23}$$

Proof. Suppose there is a finite $T > 0$ such that $Q(x(T)) = \tau^2$ and $Q(x(t)) < \tau^2$ for all $0 \le t < T$. Hence $x(t) \in \Omega^{cc,0}_{Q,\tau^2}$ for all $t \le T$. Integrate $\dot{Q}$ from 0 to T, using that $x(t) \in \Omega^{cc,0}_{Q,\tau^2}$, and $\Omega^{cc,0}_{Q,\tau^2}$ is contained in the region where the dissipation inequality in (2.21) holds. This yields

$$\begin{aligned} Q(x(T)) - Q(x_0) &\le \|w_1\|^2_{2,T} - \textstyle\sum_{i=1}^{N} \displaystyle\int_0^T \lambda_i r_i(h_2(x(t), w_2(t)))dt \\ &\le \|w_1\|^2_{2,T}. \end{aligned}$$

But $Q(x(T)) = \tau^2$, therefore $\|w_1\|^2_{2,T} \ge \tau^2 - Q(x_0)$, and the first claim is established. Recall $\Omega^{cc,0}_{Q,\tau^2} \subseteq \Omega^{cc,0}_{V,R^2}$, so the solutions remain in $\Omega^{cc,0}_{V,R^2}$ as well. Integrating $\dot{V}$ gives

$$\begin{aligned} V(x(T)) - V(x_0) &\le \|w_1\|^2_{2,T} - \tfrac{1}{\gamma^2} \|z_1\|^2_{2,T} - \textstyle\sum_{i=1}^{N} \displaystyle\int_0^T \beta_i r_i(h_2(x(t), w_2(t)))dt \\ &\le \|w_1\|^2_{2,T} - \tfrac{1}{\gamma^2} \|z_1\|^2_{2,T}. \end{aligned}$$

Since $V \ge 0$ on $\Omega^{cc,0}_{V,R^2}$, $V(x(T)) \ge 0$, and therefore $\|z_1\|^2_{2,T} \le \gamma^2 V(x_0) + \gamma^2 \|w_1\|^2_{2,T}$. □

Remark 2.6. *In a manner analogous to Theorem 2.2 and Corollary 2.1, results for locally Lipschitz* f*,* g_1*,* g_2*,* h_1 *and* h_2 *can be derived. Assume conditions* (2.19)-(2.22) *hold, and in addition,* $\Omega^{cc,0}_{Q,\tau^2}$ *is bounded. Use Lemma 7.2 to define globally Lipschitz functions* $\tilde{f}$*,* $\tilde{g}_i$ *and*

$\tilde{h}_i$ which equal, respectively, f, g_i and h_i on $\Omega^{cc,0}_{Q,\tau^2}$. Following (2.16), define a multivariable system $\tilde{G}$ using these functions. If Δ and $\tilde{G}$ form a well-posed interconnection, and Δ is dissipative with respect to $\{r_i\}_{i=1}^N$, then the conclusions of Theorem 2.7, still regarding the behavior of the interconnection of Δ and G, remain true.

2.8 Example: Application to a Locally Stable Scalar System

In this section, we calculate, analytically, the performance characterizations, focusing on a 1-state, not-uncertain system, in order to maintain simplicity in the example.

Reachability calculations

Since (2.1) restricts the vector fields $(f(x) + g(x)w)$ to be affine in w, the maximizing $w \in \mathbb{R}^n =: \mathcal{W}$ in (2.4) is $\frac{1}{2}g^T(x)\nabla Q^T(x)$. Plugging this in and setting the maximum to be zero (the limit for (2.4) to be satisfied) gives a quadratic equality in $\nabla Q(x)$. Considering scalar systems ($n = m = 1$), notate $Q'(x) := \nabla Q(x)$, and we obtain $\frac{1}{4}g^2(x)Q'^2(x) + Q'(x)f(x) = 0$. Hence, $Q'(x) = -\frac{4f(x)}{g^2(x)}$ and

$$Q(x) = \int_0^x -\frac{4f(\xi)}{g^2(\xi)}\, d\xi. \tag{2.24}$$

If $g(x) \neq 0$ for all x, then $Q(x)$ is well defined for all x and $Q'(x)\,[f(x) + g(x)w] \leq w^2$ holds for all $x \in \mathbb{R}$ and for all $w \in \mathbb{R}$. Thus, (2.3) holds for any $\tau > 0$. For such τ, if f and g are Lipschitz continuous, then the conditions of Theorem 2.1 are satisfied. If f and g are locally Lipschitz continuous and $\Omega^{cc,0}_{Q,\tau^2}$ is bounded, then the conditions of Theorem 2.2 are satisfied.

For example, consider the system $\dot{x} = -x + x^3 + w$, which has finite escape times for some inputs with $\|w\|_2 \geq 1$. Equation (2.24) yields $Q(x) = 2x^2 - x^4$, illustrated in Figure 2.2.

Since $f = -x + x^3$ is only locally Lipschitz continuous, we must choose τ such that $\Omega^{cc,0}_{Q,\tau^2}$ is bounded in order to apply Theorem 2.2. Clearly, for $0 < \tau < 1$, $\Omega^{cc,0}_{Q,\tau^2}$ is bounded and the conditions of Theorem 2.2 are satisfied. For example, if $\tau^2 = 0.95$, then $\Omega^{cc,0}_{Q,\tau^2} = [-0.88, 0.88]$, illustrated in Figure 2.2. Thus, for all $|x_0| < 0.88$ and $w \in \mathbf{L}_2$ with $\|w\|_2 < 0.95 - Q(x_0)$, solutions satisfy $|x(t)| \leq 0.88$ for all t.

$\mathbf{L}_2$ Gain calculations

Analogously to reachability, the maximizing w in the DIE (2.12) is $\frac{1}{2}g^T(x)\nabla V^T(x)$ and setting the maximum to zero yields a quadratic inequality in $\nabla V(x)$. Again, we solve this

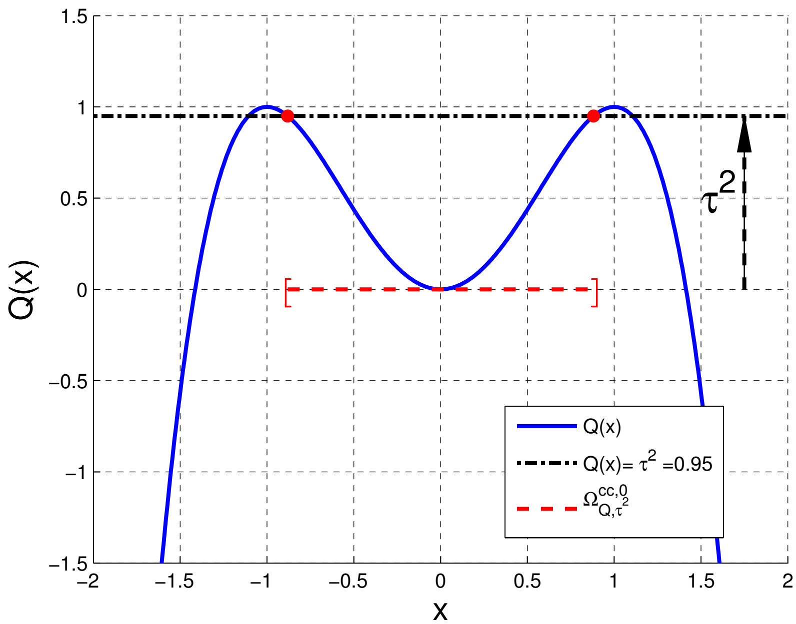

Figure 2.2: Illustration of Q

for a scalar system. At the maximizing w with zero as the maximum we obtain

$$\frac{1}{4}g^2(x)V'^2(x) + V'(x)f(x) + \frac{1}{\gamma^2}h^2(x) = 0. \tag{2.25}$$

Applying the quadratic formula to (2.25) yields

$$V'(x) = \begin{cases} \frac{2\left(-f(x)-\sqrt{f^2(x)-\frac{1}{\gamma^2}g^2(x)h^2(x)}\right)}{g^2(x)} & \text{for } x < 0 \\ \\ \frac{2\left(-f(x)+\sqrt{f^2(x)-\frac{1}{\gamma^2}g^2(x)h^2(x)}\right)}{g^2(x)} & \text{for } x \geq 0. \end{cases}$$

Setting $V(0) = 0$ gives $V(x) = \displaystyle\int_0^x V'(\xi)\, d\xi$. Assume $g(x) \neq 0$ for all $x \in \mathbb{R}$. Note that $V'(x)$ is real for all x such that $f^2(x) - \frac{1}{\gamma^2}g^2(x)h^2(x) \geq 0$. Let R be such that $\Omega^{cc,0}_{V,R^2} \setminus 0 \subseteq \{x : V(x) > 0\}$. The inequality

$$V'(x)\left[f(x) + g(x)w\right] \leq w^2 - \frac{1}{\gamma^2}h^2(x) \tag{2.26}$$

holds for all $x \in \Omega^{cc,0}_{V,R^2}$ and for all $w \in \mathbb{R}$. If f, g, and h are Lipschitz continuous and $V(0) = 0$, then the assumptions of Theorem 2.4 are satisfied. If f, g and h are locally

Lipschitz continuous, $V(0) = 0$, and $\Omega^{cc,0}_{V,R^2}$ is bounded, then the assumptions of Corollary 2.1 are satisfied.

Plugging in $f(x) = -x + x^3$, $g(x) = 1$ and $h(x) = x$ into (2.26) yields

$$V'(x) = \begin{cases} 2\left(x - x^3 - x^2\sqrt{x^4 - 2x^2 + 1 - \frac{1}{\gamma^2}}\right) & \text{for } x < 0, \\ \\ 2\left(x - x^3 + x^2\sqrt{x^4 - 2x^2 + 1 - \frac{1}{\gamma^2}}\right) & \text{for } x \geq 0 \end{cases}$$

and the resultant V is illustrated in Figure 2.3 with a choice of $\gamma = 2$.

Let $\alpha = \sqrt{\frac{\gamma - 1}{\gamma}}$ and note that $V(x)$ is real for all x such that $|x| \leq \alpha$. Thus, for any $R^2 < V(\alpha)$, $\Omega^{cc,0}_{V,R^2}$ is bounded and $\Omega^{cc,0}_{V,R^2} \setminus 0 \subseteq \{x : V(x) > 0\}$, satisfying Corollary 2.1. For example, let $\gamma = 2$ (V as in Figure 2.3) and $R^2 = 0.62$, then $\Omega^{cc,0}_{V,R^2} = [-0.68, 0.68]$. Thus, by Corollary 2.1, the solution satisfies $|x(t)| < 0.68$ and

$$\|y\|_{2,T} \leq 2\sqrt{V(x_0)} + 2\|w\|_{2,T}$$

for all $|x_0| < 0.68$, $T > 0$, and all $w \in \mathbf{L}_2$ with $\|w\|^2_{2,T} \leq 0.62 - V(x_0)$.

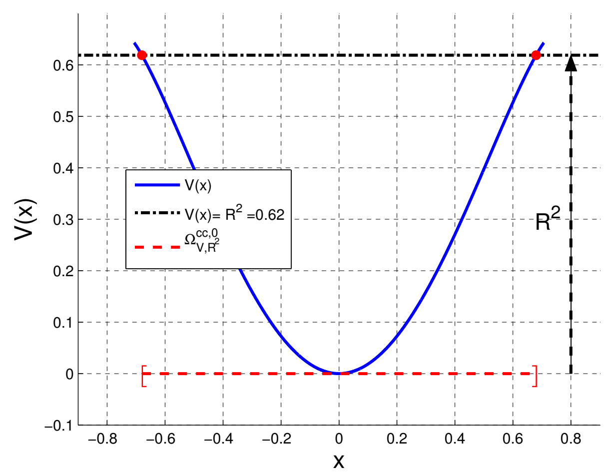

Figure 2.3: Illustration of V with $\gamma = 2$

We can further improve the bound on the gain by exploiting the reachability argument. From Theorem 2.5, given $\gamma > 0$, we restrict $\Omega^{cc,0}_{Q,\tau^2} \subseteq \Omega^{cc,0}_{V,R^2}$. In the case of $\gamma = 2$ and $R^2 = 0.62$, we simply equate $\Omega^{cc,0}_{Q,\tau^2} = \Omega^{cc,0}_{V,0.62} = [-0.68, 0.68]$, which results in $\tau^2 = 0.711$. Thus, the bound on $\|w\|_2$ is increased from $\|w\|^2_{2,T} \leq 0.62 - V(x_0)$ to $\|w\|^2_{2,T} \leq 0.711 - Q(x_0)$, while the bound on the gain remains $\gamma = 2$. The increase is shown in Figure 2.4. We repeat this procedure for a range of γ values to obtain a curve, shown in Figure 2.5, of the gain based on the size of the input assuming $x_0 = 0$. Note that the bound on the input approaches 1 as the gain increases, which is expected since the dynamical system under consideration has finite escape times for some inputs $\|w\|_{2,T} \geq 1$.

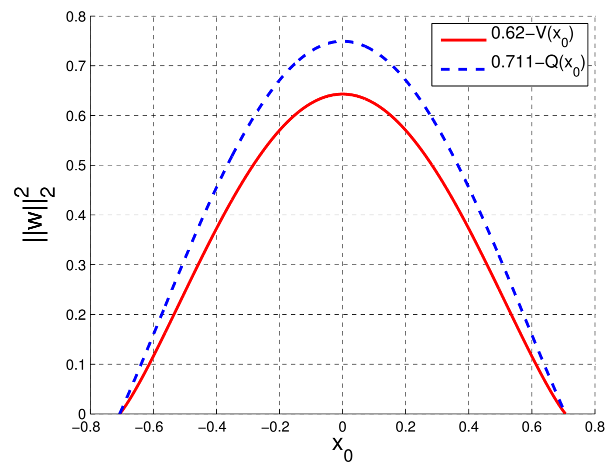

Figure 2.4: The bound on the input is shown as a function of the initial condition x_0. After applying Theorem 2.5, the input bound is increased for all $x_0 \in [-0.68, 0.68]$

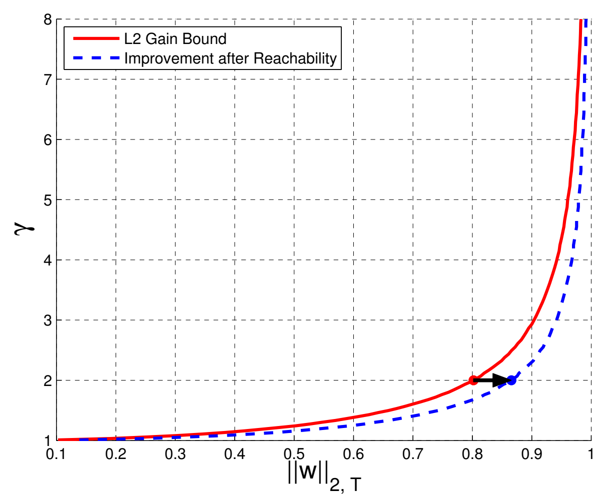

Figure 2.5: The $\mathbf{L}_2$ gain bound is improved after applying Theorem 2.5

2.9 Sum-of-Squares Conditions

In this section we outline the computational methods used to verify the $\mathbf{L}_2$ gain and reachability conditions using SOS programming [56, 67, 66], introduced in the Appendix. Assume f, g, and h in (2.1) are polynomials, and are therefore locally Lipschitz continuous. The Q and V in Theorem 2.2, Corollary 2.1 and Remark 2.5 will also be restricted to be polynomial. The S-procedure, in the Appendix, gives a sufficient condition to verify containments of sets described by inequalities.

Reachability, Refinement and $\mathbf{L}_2$ Gain Formulations

The results of Theorems 2.1, 2.2, and 2.3 show that a partial differential inequality yields an outer bound on states reachable from a given initial condition, driven by a ball of $\mathbf{L}_2$ disturbances. For various sets of disturbances, the exact reachable set can be described in terms of a sublevel set of a generalized storage function that satisfies a PDE [88]. The numerical methods outlined in this section search for storage functions from a limited class (eg., polynomial, of degree 4). Generally, this class will not include the "real" storage function, and as a consequence, the PDE has been relaxed, for example, into the inequality presented in theorems, in order to admit meaningful solutions from a prespecified function class. The relaxed partial differential inequalities, by their nature, have many solutions. Among all solutions, the techniques presented are geared towards solutions which improve the reachability bound relative to a particular shape the analyst proposes. The analyst can specify a set $\mathcal{P}$ for which the goal is to show that all states reachable from $x(0) = 0$ and $\|w\|_2^2 < \tau^2$ are contained in $\mathcal{P}$. Augmenting the conditions of Theorems 2.1 and 2.2 (and corollaries) with the requirement $\Omega_{Q,\tau^2} \subseteq \mathcal{P}$ ensures the containment. More flexible is to use an adjustable region derived from a given function $p : \mathbb{R}^n \to \mathbb{R}$, called the *shape-factor function*, defining $\mathcal{P} := \Omega_{p,\beta}$. The function p is usually simple (e.g., quadratic), so that even in high dimensions, its sub-level sets are easily interpreted (in contrast to Q, whose sub-level sets may be difficult to quantify). Much like a weight parameter as part of a cost function in optimal control, the positive-definite function p is chosen by the analyst to reflect the relative importance of the individual state elements. Hence, assume a shape-factor function p, with bounded sublevel sets, is given. The condition $\Omega^{cc,0}_{Q,\tau^2} \subseteq \Omega_{p,\beta}$ is enforced with the S-procedure, which actually certifies more, namely $\Omega_{Q,\tau^2} \subseteq \Omega_{p,\beta}$. The set $\mathcal{W}$ is defined in terms of a sublevel set of a polynomial function p_W, with $\mathcal{W} := \{w : p_W(w) \leq 0\}$.

Translating the results of Theorem 2.1 (and Theorem 2.2) into SOS conditions employs two applications of the S-procedure: one to for $\Omega_{Q,\tau^2} \subseteq \Omega_{p,\beta}$ in $\mathbb{R}^n$, and one for DIE containment of (2.4), in $\mathbb{R}^n \times \mathbb{R}^m$. For given $\tau > 0$, the conditions

$$s_1 \in \Sigma_{n+m}, \quad s_2 \in \Sigma_n, \quad s_p \in \Sigma_{n+m}, \quad Q \in \Sigma_n, \quad Q(0) = 0 \tag{2.27}$$

$$w^T w - \nabla Q \cdot (f + gw) - s_1 \cdot (\tau^2 - Q) - s_p p_W \in \Sigma_{n+m}, \tag{2.28}$$

$$\beta - p - s_2 \cdot (\tau^2 - Q) \in \Sigma_n \tag{2.29}$$

guarantee the hypothesis of Theorem 2.2 and $\Omega_{Q,\tau^2} \subseteq \Omega_{p,\beta}$.

Implementation of Theorem 2.3 is relatively simple, since a storage function Q and constant τ that satisfy Theorem 2.2 are given. The computational objective is to find a suitable function $k : [0\ \tau^2] \to \mathbb{R}$ satisfying (2.7). The simplest approach restricts k to be piecewise constant. For example, take $N > 0$, and let $\{k_i\}_{i=1}^N$ denote the function values, with k given by nearest neighbor interpolation, defining $\epsilon := \frac{\tau^2}{N}$ and $k(\xi) = k_i$ for all $\epsilon(i-1) \leq \xi < \epsilon i$ for $i = 1, \ldots, N$. Employing the S-procedure, obtaining optimal values for the $\{k_i\}_{i=1}^N$ only requires N uncoupled, linear SOS optimizations, namely for $i = 1, \ldots, N$

$$\begin{aligned}
&\underset{s_{1i}, s_{2i}, s_{pi}}{\text{minimize}} && k_i \\
&\text{subject to} && s_{1i} \in \Sigma_{n+m}, \quad s_{2i} \in \Sigma_{n+m}, \quad s_{pi} \in \Sigma_{n+m}, \\
& && -\left[(\epsilon i - Q)s_{1i} + (Q - \epsilon(i-1))s_{2i} + \nabla Q(f + gw) - k_i w^T w\right] - s_{pi} p_W \in \Sigma_{n+m}.
\end{aligned}$$

Using the resultant piecewise-constant k yields $\tau_e^2 = \displaystyle\int_0^{\tau^2} k^{-1}(\xi) d\xi = \epsilon \sum_{i=1}^{N} k_i^{-1}$.

For $\mathbf{L}_2$ gain, only one application of the S-procedure is used for the DIE containment of (2.12) in $\mathbb{R}^n \times \mathbb{R}^m$, which requires the SOS polynomial to be in Σ_{n+m}. If V is restricted to be a polynomial, for given $\gamma > 0$, $R > 0$, and polynomial $l(x) > 0$ for all $x \neq 0$, $l(0) = 0$, the conditions $V(0) = 0$,

$$s_3 \in \Sigma_{n+m}, \quad s_p \in \Sigma_{n+m}, \tag{2.30}$$

$$V - l \in \Sigma_n, \tag{2.31}$$

$$w^T w - \frac{1}{\gamma^2} h(x)^T h(x) - \nabla V \cdot (f(x) + g(x)w) - s_3(R^2 - V) - s_p p_W \in \Sigma_{n+m} \tag{2.32}$$

satisfy the conditions of Corollary 2.1. Note that (2.32) actually implies the DIE holds on Ω_{V,R^2}, not just the connected component, as in (2.12). Following Theorem 2.5, the reachability equations, (2.27)-(2.29) can be solved (maximizing τ) with $p := V$ and $\beta := R^2$. It is straightforward to show that any V and R in (2.30)-(2.32), $\tau := R$ and $Q := V$ are feasible for (2.27)-(2.29).

The corresponding robust versions, for use with Theorem 2.7, simply include the supply rates of the known dissipativeness conditions, and account for the signals w_1, w_2, z_1 and z_2. For example, condition (2.22) is ensured by a generalization to (2.32),

$$w_1^T w_1 - \frac{1}{\gamma^2} h_1^T h_1 - \nabla V \cdot (f + g_1 w_1 + g_2 w_2) - s_3(R^2 - V) - s_p p_W - \sum_{i=1}^{n} \beta_i r_i(h_2, w_2) \in \Sigma_{n+m_1+m_2} \tag{2.33}$$

constrained by $\beta_i \geq 0$, as well as the original constraints on the various SOS multipliers.

Feasibility Guarantee

Many standard results from nonlinear system theory show that properties of the linearized dynamical system carry over to local properties of the nonlinear system, for instance, exponential stability of an equilibrium point of an autonomous system, [52]. In [90], we explored how properties of the linearized system implied the corresponding feasibility of the SOS formulations (Section 2.9), using quadratic storage functions, for three types of problems: region-of-attraction, reachability and $\mathbf{L}_2$ gain. In [90], the vector field was limited to cubic polynomials and the proof techniques geared toward systems of that class. In this section, we extend these results to polynomial vector fields of any degree. For brevity, we only consider $\mathbf{L}_2$ gain formulation (section 2.5) although the other results follow as well, including the uncertain $\mathbf{L}_2$ gain formulation from Section 2.7.

The results here are similar in spirit, though different and of significantly weaker theoretical value, to other results in the literature. The work of [69] establishes the optimality of polynomial storage functions for certain stability questions, using Positivestellensatz-based proofs (generalization of the simple S-procedure). By contrast, the results in [2] are negative, showing the inadequacy of polynomial storage functions in answering stability questions for a special class of nonlinear autonomous systems.

A simple technical lemma (proof in Appendix) is used in the subsequent claim.

Lemma 2.1. *Let $d \geq 2$ be a positive integer. Let $V(x) := x^T Q x$ with $0 \prec Q = Q^T \in \boldsymbol{R}^{n\times n}$. Let $r(x)$ denote the vector of all monomials of degree 1 through degree $d-1$ and $s(x) = r(x)^T r(x)$. Similarly let $z(x)$ denote the vector of all monomials of degree 2 through degree d. The length of z is denoted n_z. There exists $H \in \boldsymbol{R}^{n_z \times n_z}$ with $H = H^T \succ 0$ and $s(x)V(x) = z(x)^T H z(x)$.*

Proof. Since $Q \succ 0$ there exists $\alpha > 0$ such that $\tilde{Q} := Q - \alpha I \succ 0$. Define the perturbed polynomial $\tilde{V}(x) := x^T \tilde{Q} x$. By assumption, $s(x) = \sum_i r_i(x)^2$ and hence

$$s(x)\tilde{V}(x) = \sum_i r_i(x)^2 x^T \tilde{Q} x = \sum_i (r_i(x)x)^T \tilde{Q} (r_i(x)x). \tag{2.34}$$

Each term in the sum is SOS with positive definite Gram matrix $\tilde{Q}$. Thus $s(x)\tilde{V}(x)$, being a sum of SOS terms, is itself an SOS polynomial. Since $s(x)\tilde{V}(x)$ contains all monomials of degree 4 through degree $2d$ it has a Gram matrix decomposition of the form $z(x)^T \tilde{H} z(x)$. The existence of a Gram matrix $\tilde{H} \succeq 0$ follows because $s(x)\tilde{V}(x)$ is SOS.

Finally, $s(x)V(x) = s(x)\tilde{V}(x) + \alpha \sum_i r_i(x)^2 x^T x$. $\sum_i r_i(x)^2 x^T x$ is a sum of monomials squared. The sum includes squares of all monomials in $z(x)$ possibly with repeats. Therefore this sum has a Gram matrix decomposition of the form $z(x)^T D z(x)$ where D is diagonal and positive-definite. Thus $s(x)V(x)$ has a Gram representation $z^T(x) H z(x)$ where $H = \tilde{H} + \alpha D \succ 0$. □

Now, write the affine-in-w system in (2.1) as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t) + f_2(x(t)) + g_1(x(t))w(t) \\ y(t) &= Cx(t) + h_2(x(t)) \end{aligned} \tag{2.35}$$

where f_2, g_1 and h_2 are polynomials, respectively consisting of terms of degree 2, 1, and 2 (and higher). Let $\partial(f_2)$, $\partial(h_2)$ and $\partial(g_1)$ denote the highest degree of monomials within each function. Define $d := max\{\partial(f_2), \partial(h_2), \partial(g_1)+1\}$. Suppose the linearization has A Hurwitz, and for some $\gamma > 0$, $\|C(sI-A)^{-1}B\|_\infty < \gamma$. By the bounded-real lemma, there exists $P = P^T \succ 0$ such that

$$\begin{bmatrix} A^T P + PA + \frac{1}{\gamma^2} C^T C & PB \\ B^T P & -I \end{bmatrix} \prec 0.$$

Defining $V(x) := x^T Px$ and $s := \alpha r^T(x)r(x)$ leads to the main SOS constraint as

$$2x^T P\left[Ax + Bw + f_2(x) + g_1(x)w\right] - w^T w + \frac{1}{\gamma^2}\left[x^T C^T + h_2^T(x)\right]\left[Cx + h_2(x)\right] + \alpha\left(R^2 - x^T Px\right) r^T(x)r(x).$$

This is a quadratic form in $[x; w; z(x)]$ as follows. There exists a matrices F and H such that $f_2(x) = Fz(x)$ and $h_2(x) = Hz(x)$. Likewise, there exists a matrix G such that $x^T Pg_1(x)w = w^T Gz(x)$. By Lemma 2.1, there exists a positive-definite matrix M_P such that $r^T(x)r(x)x^T Px = z^T(x)M_P z(x)$. Finally, there exists a matrix E such that $r^T(x)r(x) = x^T x + z(x)^T Ez(x)$. Combining, the expression is

$$\begin{bmatrix} x \\ w \\ z(x) \end{bmatrix}^T \underbrace{\begin{bmatrix} A^T P + PA + \frac{1}{\gamma^2}C^T C + \alpha R^2 I & PB & PF + \frac{1}{\gamma}C^T H \\ B^T P & -I & G \\ F^T P + \frac{1}{\gamma}HC & G^T & \alpha R^2 E + \frac{1}{\gamma^2}H^T H - \alpha M_P \end{bmatrix}}_{K(R,\alpha)} \begin{bmatrix} x \\ w \\ z(x) \end{bmatrix}$$

At $R := 0$, the top/left 2×2 block is negative definite, and $M_P \succ 0$. Hence for α sufficiently large, it follows that $K(0, \alpha) \prec 0$. With such an α chosen, by continuity there exist nonzero R such that $K(R, \alpha) \prec 0$. The above reasoning is summarized in a theorem.

Theorem 2.8. *Assume $x = 0$ is an equilibrium of* (2.1)*, and express the vector field with linear and nonlinear terms separated, as in* (2.35)*. If A is Hurwitz, and $\|C(sI - A)^{-1}B\|_\infty < \gamma$, then there exists $R > 0$, $\epsilon > 0$, quadratic V and polynomial s_3 (with $s_p = 0$) such that equations* (2.30)-(2.32) *are feasible using $l(x) := \epsilon x^T x$.*

Iteration Strategy

Equations (2.27)-(2.29) and (2.30)-(2.32) constitute a nonconvex optimization problem, namely a linear objective subject to bilinear matrix inequality constraints. Acknowledging the theoretical implications [37], [54], we nevertheless push forward with iterative schemes to generate feasible solutions, and further optimize the cost. We outline an iteration for (2.30)-(2.32). Analogous iterations are possible for (2.27)-(2.29) by replacing V with Q, R with τ, and s_3 with s_1, s_2.

1. Based on the polynomials f, g and h (and their degrees), choose basis functions for the unknowns s_p, s_3 and V. At present, computational restrictions (memory, numerical conditioning, algorithms etc.) place a practical restriction on the overall degree of the DIE polynomial in terms of the number of independent variables $(n + m)$, which in turn, limits the degrees of s_p, s_3 and V.

2. If the linearization is stable, use the LMI derived in Section 2.9 to obtain feasible values for s_3 and V (using $s_p = 0$). If the linearization is not stable, the DIE is relaxed and the constraint violation is minimized. If this minimum is less than 0, feasible values for V and s_3 for the original problem are obtained.

3. [**R Maximization:**] Hold V fixed, and maximize R, by choice of s_3 and s_p, such that (2.30) and (2.32) hold. This step requires a bisection in R, where for each fixed value of R, determining feasible s_3, s_p is a linear SOS problem.

4. [**V Recenter:**] Hold s_3 fixed, and "recenter" V by finding the analytic center (in R^2, parameters of s_p, parameters of V, and parameters associated with the kernel representation of the SOS problem, [66]) of system of LMI constraints defined by equations (2.31) and (2.32).

5. Return to the [**R Maximization**] step, and repeat.

2.10 Examples

In the following examples, we utilize the SOS optimization tool SOSOPT and associated nonlinear systems analysis software, available at `http://www.aem.umn.edu/~AerospaceControl/`. Supporting material, documentation, and additional examples can be found at [8].

Scalar Example from Section 2.8

We revisit the example in Section 2.8 and compare those results with the SOS-based iteration from Section 2.9. Using quadratic V, Q, the results are compared in Figure 2.6. The V obtained from the $\mathbf{L}_2$ analysis is used as the shape factor function in the reachability analysis, which improves the bound, and then improved further with refinement. A power-algorithm from [87] attempts to find input signals, of a given norm, which maximize the resultant output norm, yielding a lower bound on the system $\mathbf{L}_2$ gain (also shown).

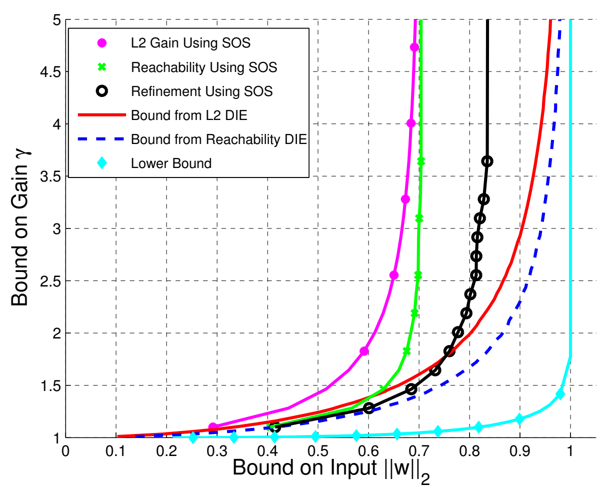

Figure 2.6: Comparison of the SOS approach with the algebraic approach in Section 2.8

Bilinear System with Nonzero Initial Condition

Consider the bilinear system $\dot{x} = (w-1)x$, $y = x$, which is unstable for $w(t) > 1$. Note that for $x(0) = 0$, the solution is $x(t) = 0$, regardless of w. We use degree 6 V in the SOS iteration. In this example, the refinement analysis does not yield any benefit. Under the assumption $x_0 = 0$, we obtain a narrow bound on the input $[1.41, 1.46]$, with little effect from the bound imposed on the gain. However, for nonzero initial conditions, the bound on the input varies widely, depending on the bound imposed on the gain, shown in Figure 2.7.

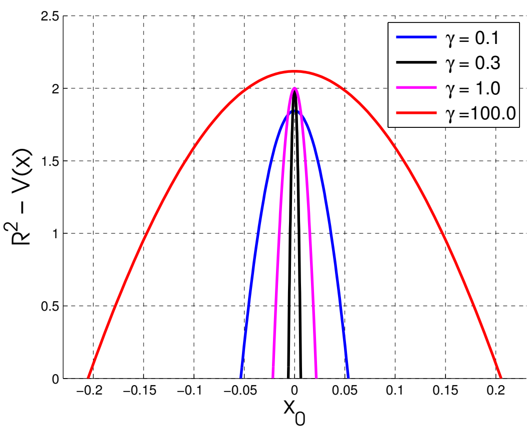

Figure 2.7: Bound on input as a function of the initial condition x_0

Three State Reachability Example

Consider the three state system, extending an example in [43]: $\dot{x}_1 = -x_3 + x_2 - x_3x_2^2$; $\dot{x}_2 = -x_2x_3^2 - x_2 + w$; $\dot{x}_3 = \frac{1}{2}(x_1 - x_3)$. For purposes of illustration, choose $p(x) := 8x_1^2 - 8x_1x_2 + 4x_2^2 + x_3^2$. Given $\beta > 0$, and a basis for Q, we maximize τ such that the conditions of Theorem 2.1 hold and $\Omega_{Q,\tau^2} \subseteq \Omega_{p,\beta}$, which will further imply that Theorem 2.2 holds since Ω_{Q,τ^2} is bounded. We perform the analysis for $\beta \in (0\ 50]$ using both quadratic and quartic Q. In both cases, the SOS multipliers s_1 and s_2 are chosen constant and quadratic, respectively. Conversely, a power algorithm from [87] attempts to find inputs on a finite-horizon, of a given norm, which maximize $p(x(T_{\text{final}}))$. The algorithm is globally convergent for linear systems, but can also be applied to nonlinear systems as an ad-hoc manner to find the worst-case input. In that context, the results it produces are lower bounds on the worst-case, since convergence to the global maximum may not occur. Finally, a single global analysis with a quartic degree Q is also performed (i.e, positive-definite Q such that $\Omega^{cc,0}_{Q,\tau^2}$ is bounded for all $\tau > 0$ and $\nabla Q(x) \cdot [f(x) + g(x)w] \leq w^T w$ on $\mathbb{R}^n \times \mathbb{R}^m$).

These results are shown in Figure 2.8. The various axes show subsets of the upper-bounds for ease of comparison. The (same) lower bound is shown in all three subplots. The upper-left axes shows the bounds obtained from the single global analysis with quartic Q

and the many local analysis with quadratic Q (and refinements, with $N = 20$, as in Theorem 2.3). The benefits of the refinement step are obvious. The upper-right axes compares results obtained with quadratic Q to those with quartic Q. The improvement in the upper bound is expected, but comes at an increased computational cost. Lastly, the lower-left axes shows the significant effect of imposing an additional $\mathbf{L}_\infty$ constraint on w, $|w(t)| \leq 2.5$ for all t. with bounds from the reachability analysis using a quartic Q (refinement provides a negligible improvement and is not shown).

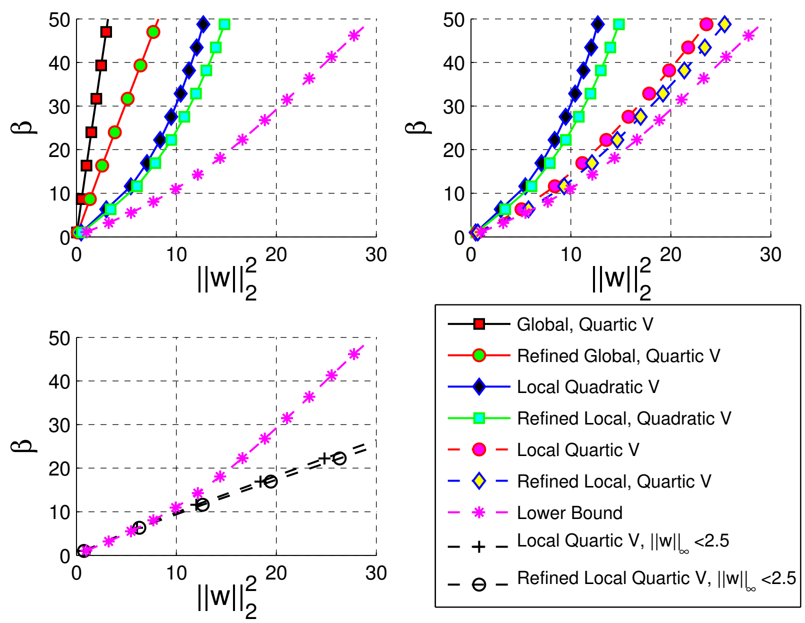

Figure 2.8: Estimations of reachability, using quadratic and quartic Q, and with $\|w\|_\infty \leq 2.5$

$\mathbf{L}_2$ Induced Analysis of Generic Transport Longitudinal Model

This section performs nonlinear analyses for NASA's Generic Transport Model (GTM). The GTM is a remote-controlled 5.5 percent scale commercial aircraft [25, 62]. The main GTM aircraft and environmental parameters are: wing area $S = 5.902$ ft^2, mean aerodynamic chord $\bar{c} = 0.9153$ ft, mass $m = 1.542$ slugs, pitch axis moment of inertia $I_{yy} = 4.254$ slugs-ft^2, air density $\rho = 0.002375$ slugs/ft^3, and gravitational acceleration $g = 32.17$ ft/s^2. The longitudinal dynamics of the GTM are described by a standard four-state model [81]:

$$\begin{aligned}
\dot{V} &= \left(-D - mg\sin(\theta - \alpha) + T_x\cos\alpha + T_z\sin\alpha\right)/m \\
\dot{\alpha} &= q + \left(-L + mg\cos(\theta - \alpha) - T_x\sin\alpha + T_z\cos\alpha\right)/(mV) \\
\dot{q} &= \left(M + T_m\right)/I_{yy} \\
\dot{\theta} &= q
\end{aligned}$$

where V is air speed (ft/s), α is angle of attack (rad), q is pitch rate (rad/s) and θ is pitch angle (rad). Control inputs are elevator deflection δ_{elev} (deg) and engine throttle δ_{th} (percent). The drag force D (lbs), lift force L (lbs), and aerodynamic pitching moment M (lb-ft) are given by $D = \bar{q}SC_D(\alpha, \delta_{elev}, \hat{q})$, $L = \bar{q}SC_L(\alpha, \delta_{elev}, \hat{q})$ and $M = \bar{q}S\bar{c}C_m(\alpha, \delta_{elev}, \hat{q})$, where $\bar{q} := \frac{1}{2}\rho V^2$ is the dynamic pressure (lbs/ft^2) and $\hat{q} := \frac{\bar{c}}{2V}q$ is the normalized pitch rate (unitless). C_D, C_L, and C_m are unitless aerodynamic coefficient functions provided as look-up tables by NASA. The GTM has one engine on the port side and one on the starboard side of the airframe. The thrust from a single engine T (lbs) is a function of the throttle setting δ_{th} (percent). $T(\delta_{th})$ is specified as a ninth-order polynomial in NASA's high fidelity GTM simulation model. T_x (lbs) and T_z (lbs) denote the projection of the total engine thrust along the body x-axis and body-z axis, respectively. T_m (lbs-ft) denotes the pitching moment due to both engines.

The following terms of the longitudinal model are approximated by low-order polynomials: Trigonometric functions; Engine model; Rational dependence on speed ($1/V$); and Aerodynamic coefficients (C_D, C_L, C_m). The trigonometric functions are approximated by Taylor series expansions. For the engine model, least squares is used to approximate the ninth order polynomial function $T(\delta_{th})$ by a third order polynomial. Least squares is also used to compute a linear fit to $1/V$ over the desired range of interest from 100 ft/s to 200 ft/s. Finally, polynomial least squares fits are computed for the aerodynamic coefficient look-up tables provided by NASA. A degree seven polynomial model, provided in [17], is obtained after replacing all non-polynomial terms with their polynomial approximations.

The polynomial model takes the form $\dot{x} = f_7(x, u)$ where $x := [V(\text{ft/s}),\ \alpha(\text{rad}),\ q(\text{rad/s}),\ \theta(\text{rad})]^T$, and $u := [\delta_{elev}(\text{deg}),\ \delta_{th}(\%)]^T$. The subscript in f_7 denotes that the vector field is a degree seven polynomial in x. The quality of the polynomial approximation was assessed by comparing the trim conditions and simulation responses of the polynomial and original non-polynomial models. The following straight and level flight condition was computed for this model: $V_t = 150$ ft/s, $\alpha_t = 0.047$ rad, $q_t = 0$ rad/s, $\theta_t = 0.047$ rad, with $\delta_{elev,t} = 0.051$ rad, $\delta_{th,t} = 14.78$ %. The subscript "t" denotes a trim (equilibrium) value. A cubic order polynomial longitudinal model is extracted from the 4-state, degree seven polynomial model by holding δ_{th} at its trim value and retaining terms up to cubic order. The cubic order model is $\dot{x} = f_3(x, u)$ with 4 states $[V,\ \alpha,\ q,\ \theta]^T$ and one input $u := \delta_{elev}$. This cubic model is used for all analyses described in the remainder of the section. Additional details on the polynomial modeling are provided in [17] and files containing the model can be found at [8].

An $\mathbf{L}_2 \rightarrow \mathbf{L}_2$ gain analysis was first performed on the open-loop model by injecting a disturbance w at the elevator input. Figure 2.9 indicates how the induced gain of this open-loop system varies as the size of the elevator disturbances $||w||_2$ increases. The horizontal axis indicates the size of the elevator disturbances, $||w||_2$, around the trim input value and the vertical axis shows the bounds of the induced gain from disturbance w to pitch rate q. The bounds are calculates for several fixed values of γ by maximizing R over the choice of V and s_3 such that (2.30)-(2.32) hold using the strategy from Section 2.9. Figure 2.9 shows the open-loop results for both a quadratic (black dashed-'$\diamondsuit$') and a quartic (black solid-'$\diamondsuit$') storage function V. The higher (quartic) degree storage function provides a less conservative bound on the gain, as expected. The induced gain for the linearized open-loop system is 23.9. Both nonlinear bounds converge to this linearized gain as $\|w\|_2 \to 0$.

The open loop dynamics of the GTM are slightly underdamped. Inner loop pitch rate feedback is typically used to improve the damping of the aircraft. A proportional pitch rate feedback is used to improve the damping of the GTM aircraft. Combining with the input disturbance gives $delta_{elev} = K_q q + w = 0.0698q + w$ where w is the input disturbance at the elevator channel. Figure 2.9 also shows bounds on the closed-loop $\mathbf{L}_2$ gain from elevator disturbances to pitch rate for a quadratic (red dashed-'x') and a quartic (blue solid-'x') storage function V. Again, the higher (quartic) degree storage function provides a less conservative bound on the gain. Moreover, the pitch-rate damping lowers the induced gain for the linearized closed-loop system to 16.6. Both nonlinear bounds for the closed-loop system converge to this linearized gain as $\|w\|_2 \to 0$. Finally, as expected, the gain bounds for the closed-loop system are both below the open-loop bounds.

The refinement procedures were used to improve the computed bounds for the closed-loop system. First,the reachability analysis is performed by setting V as the shape factor function and maximizing τ over choice of Q, s_1 and s_2 such that (2.27)-(2.29) hold. Figure 2.9 shows the results obtained when the degree of Q is restricted to be quadratic (red dashed-'o') and quartic (blue solid-'o'). Finally, the refinement is performed on the quadratic (red dashed-'+') and quartic (blue solid-'+') results from the reachability. These results show the bound on the allowable input is improved from the reachability analysis, and improved further by the refinement procedure. As expected, the upper bounds on the gain using quartic V and Q are improvements over their quadratic counterparts.

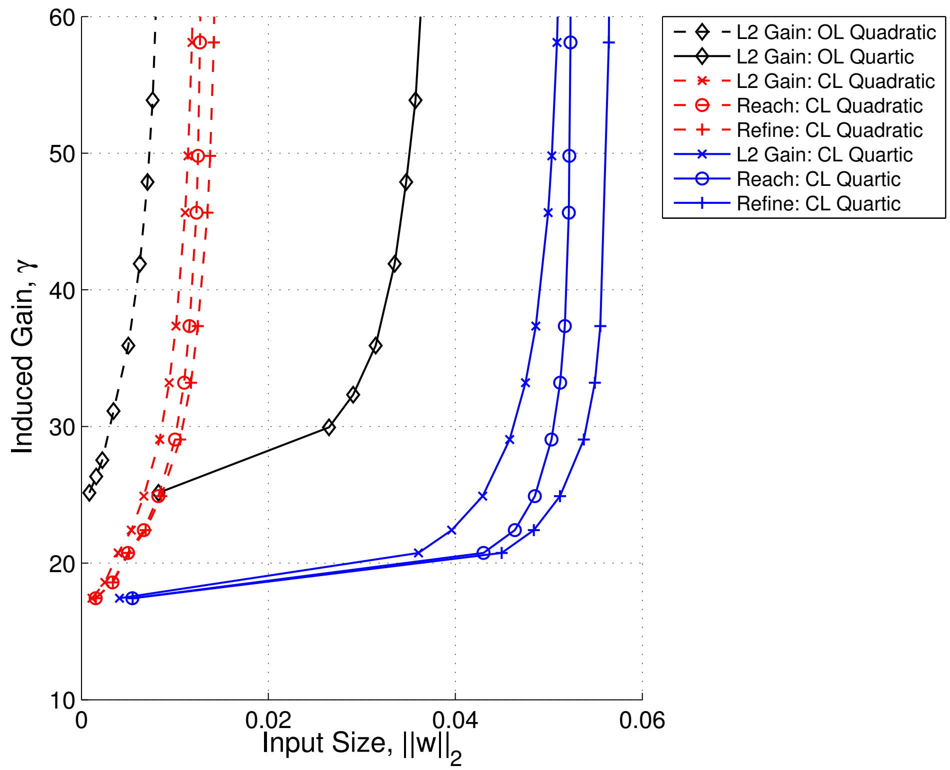

Figure 2.9: Upper bounds of $\mathbf{L}_2 \to \mathbf{L}_2$ gain from w to q for open and closed-loop GTM model.

Chapter 3

Integral Quadratic Constraints

Integral quadratic constraints (IQCs) [59, 46], encapsulate input-output properties of a system. The definitions and theorems in [59] only apply to systems who are defined on all of $\mathbf{L}_{2e}$ and are bounded on $\mathbf{L}_2$. However, in many cases systems do not meet this criteria. In this section, we develop the notion of a local IQC, which may be satisfied for systems who are only defined locally. We establish facts about local IQCs and local operators which will be used in Chapter 4 to extend the stability proof to include local operators.

3.1 Background

For continuity, we review the definitions and stability theory for IQCs [59], and performance analysis [4] using IQCs.

Definition of an Integral Quadratic Constraint

Let $\Pi : j\mathbb{R} \to \mathbb{C}^{(l+m)\times(l+m)}$ be a measurable, bounded Hermitian-valued function.

Definition 3.1. *The signals $u \in \mathbf{L}_2^l$, $y \in \mathbf{L}_2^l$ are said to* ***satisfy the IQC defined by*** Π *if*

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{y}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{y}(j\omega) \end{bmatrix} d\omega \geq 0$$

holds.

Definition 3.2. *A bounded, causal operator Δ mapping $\mathbf{L}_{2e}^l \to \mathbf{L}_{2e}^m$ is said to satisfy* ***the IQC defined by*** Π*, if for all $u \in \mathbf{L}_2^l$, with $y = \Delta(u)$, the inequality*

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{y}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{y}(j\omega) \end{bmatrix} d\omega \geq 0 \tag{3.1}$$

holds. This is notated as $\Delta \in \mathcal{I}(\Pi)$.

Time-Domain Representation

If Φ is real, rational and uniformly bounded on the imaginary axis and $\Pi(j\omega) = \Phi(s)|_{s=j\omega}$, then (3.1) can be expressed equivalently in the time-domain. We factorize Φ as $\Phi(s) = \Psi^{\sim}(s)M\Psi(s)$, where M is a constant matrix, Ψ is stable and proper. By Parseval's Theorem, the inequality in (3.1) is expressed as:

$$\int_0^\infty y_\Psi^T(t) M y_\Psi(t)\, dt \geq 0, \tag{3.2}$$

where $y_\Psi = \Psi \left(\begin{smallmatrix} u \\ y \end{smallmatrix} \right)$. This is notated as $\Delta \in \mathcal{I}(\Psi, M)$.

A stricter notion of satisfying and IQC is related to the time domain representation.

Definition 3.3. *If*

$$\int_0^T y_\Psi^T(t) M y_\Psi(t)\, dt \geq 0, \tag{3.3}$$

*for all T, then Δ is said to satisfy the **strict** IQC defined by Ψ and M and is notated as $\Delta \in \mathcal{I}_S(\Psi, M)$.*

Stability Test

We recall the following definitions and theorem from [59]:

Definition 3.4. *The interconnection of G and Δ in Figure 3.1 is **well-posed** if the map from $(z, w) \longmapsto (f_1, f_2)$ defined by*

$$\begin{aligned} w &= f_1 + \Delta(z) \\ z &= f_2 + Gw \end{aligned}$$

has a causal inverse on $\mathbf{L}_{2e}$.

Definition 3.5. *The interconnection is **stable** if, in addition to being well-posed, the inverse is bounded, i.e. there exists a constant C such that*

$$\|z\|_{2,T}^2 + \|w\|_{2,T}^2 \leq C \left(\|f_1\|_{2,T}^2 + \|f_2\|_{2,T}^2 \right)$$

for any $T \geq 0$ and for any solution of w and z.

Theorem 3.1. *Assume Δ is a bounded, causal operator and G is proper and has no poles in the right half plane. If*

1. *for every $\tau \in [0,\ 1]$, the interconnection of G and $\tau\Delta$ is well-posed;*

2. *for every $\tau \in [0,\ 1]$, $\tau\Delta \in \mathcal{I}(Pi)$;*

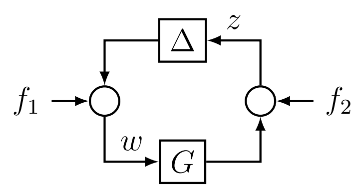

Figure 3.1: Exogenous Inputs in the interconnection of (G, Δ)

3. there exists an $\epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \preceq -\epsilon I \tag{3.4}$$

for all $\omega \in \mathbb{R}$, then the feedback interconnection of G and Δ is stable.

Remark 3.1. *If $\Delta \in \mathcal{I}(\Pi_i)$ for $i = 1, \ldots, n$, then the interconnection of G and Δ is stable if there exists $\alpha_i \geq 0$ such that Theorem 3.1 holds with $\Pi := \sum_{i=1}^n \alpha_i \Pi_i$.*

Performance Analysis with IQCs

Consider the interconnection of G and Δ in Figure 3.2. Suppose Δ satisfies a collection of IQCs. One might wonder, what is the bound on the $\mathbf{L}_2 \to \mathbf{L}_2$ gain from d to e, are the signals from d to e output strictly passive, or (more generally) which IQCs do the signals d and e satisfy? We quantify performance metrics using a "performance IQC", by applying techniques from [4]. By characterizing the desired performance metric as an IQC, and using the stability test in [59], we can test whether d and e satisfy performance criteria.

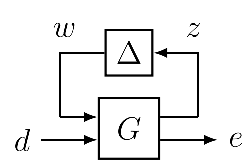

Figure 3.2: Feedback Interconnection of (G, Δ)

Let Π_p represent a performance IQC between the exogenous input d and output e. Our goal is to show that for all $d \in \mathbf{L}_2$

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{d}(j\omega) \\ \hat{e}(j\omega) \end{bmatrix}^* \Pi_p(j\omega) \begin{bmatrix} \hat{d}(j\omega) \\ \hat{e}(j\omega) \end{bmatrix} d\omega \geq 0 \tag{3.5}$$

holds.

Using information about the IQCs which Δ satisfies, IQCs can be certified for the signals d and e. We recall the following theorem from [4]:

Theorem 3.2. *Let $\Delta : \mathbf{L}_{2e} \to \mathbf{L}_{2e}$ be causal and bounded and $\Delta \in \mathcal{I}(\Pi)$. Let $G \in \mathbb{R}\mathbf{H}_\infty$ be causal, finite dimensional, and denote G_{11} as the channels in G from w to z. Assume that the interconnection in Figure 3.3 is well-posed and stable. If there exists and $\epsilon > 0$ such that*

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} \Pi_{11}(j\omega) & 0 & \Pi_{12}(j\omega) & 0 \\ 0 & -\Pi_{p,22}(j\omega) & 0 & -\Pi_{p,12}^*(j\omega) \\ \Pi_{12}^*(j\omega) & 0 & \Pi_{22}(j\omega) & 0 \\ 0 & -\Pi_{p,12}(j\omega) & 0 & -\Pi_{p,11}(j\omega) \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \preceq -\epsilon I \tag{3.6}$$

for all $\omega \in \mathbb{R}$, then for all $d \in \mathbf{L}_2$, $e \in \mathbf{L}_2$ and (3.5) is satisfied.

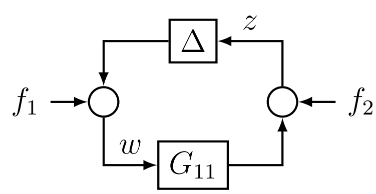

Figure 3.3: Exogenous Inputs in the interconnection of (G_{11}, Δ)

Proof. The well-posedness assumption of the interconnection in Figure 3.3 means that for all $f_1, f_2 \in \mathbf{L}_{2e}$ there exists unique w, z satisfying

$$\begin{aligned} w &= \quad f_1 + \Delta(z) \\ z &= \quad f_2 + G_{11} w. \end{aligned} \tag{3.7}$$

Furthermore, since the interconnection in Figure 3.3 is stable and there exists $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathbb{R}_+$ such that for all $f_1, f_2 \in \mathbf{L}_{2e}$, there exists unique $w, z \in \mathbf{L}_{2e}$ that satisfy (3.7) and

$$\begin{aligned} \|w\|_{2,T} &\le \gamma_1 \|f_1\|_{2,T} + \gamma_2 \|f_2\|_{2,T} \\ \|z\|_{2,T} &\le \gamma_3 \|f_1\|_{2,T} + \gamma_4 \|f_2\|_{2,T}. \end{aligned} \tag{3.8}$$

Suppose $d \in \mathbf{L}_{2e}$. Defining $f_1 := 0$ and $f_2 := G_{12} d$ recovers the relationship of w and z from Figure 3.2. Substituting into (3.8) yields

$$\begin{aligned} \|w\|_{2,T} \le \gamma_2 \|G_{12} d\|_{2,T} &\le \gamma_2 \|\hat{G}_{12}\|_\infty \|d\|_{2,T} \\ \|z\|_{2,T} \le \gamma_4 \|f_2\|_{2,T} &\le \gamma_4 \|\hat{G}_{12}\|_\infty \|d\|_{2,T}. \end{aligned} \tag{3.9}$$

Hence, $w, z \in \mathbf{L}_{2e}$ and

$$\|e\|_{2,T} \le \|\hat{G}_{21}\|_\infty \|w\|_{2,T} + \|\hat{G}_{22}\|_\infty \|d\|_{2,T}$$

and $e \in \mathbf{L}_{2e}$. Furthermore, for all $d \in \mathbf{L}_2$, the signals $w, z, e \in \mathbf{L}_2$.

Rearranging rows and columns in (3.6) (and dropping $j\omega$ for clarity) yields

$$\begin{bmatrix} G_{11} & G_{12} \\ I & 0 \\ 0 & I \\ G_{21} & G_{22} \end{bmatrix}^* \begin{bmatrix} \Pi & 0 \\ 0 & -\Pi_p \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ I & 0 \\ 0 & I \\ G_{21} & G_{22} \end{bmatrix} \preceq -\epsilon I \tag{3.10}$$

which implies

$$\begin{bmatrix} 0 & I \\ G_{21} & G_{22} \end{bmatrix}^* \Pi_p \begin{bmatrix} 0 & I \\ G_{21} & G_{22} \end{bmatrix} \succ \begin{bmatrix} G_{11} & G_{12} \\ I & 0 \end{bmatrix}^* \Pi \begin{bmatrix} G_{11} & G_{12} \\ I & 0 \end{bmatrix} \tag{3.11}$$

Note that from the interconnection in Figure 3.2, if $d \in \mathbf{L}_2$ then w, z, e are in $\mathbf{L}_2$ and the Fourier transforms satisfy

$$\begin{bmatrix} \hat{z}(\omega) \\ \hat{w}(\omega) \end{bmatrix} = \begin{bmatrix} G_{11}(j\omega) & G_{12}(j\omega) \\ I & 0 \end{bmatrix} \begin{bmatrix} \hat{w}(\omega) \\ \hat{d}(\omega) \end{bmatrix}, \qquad \begin{bmatrix} \hat{d}(\omega) \\ \hat{e}(\omega) \end{bmatrix} = \begin{bmatrix} 0 & I \\ G_{21}(j\omega) & G_{22}(j\omega) \end{bmatrix} \begin{bmatrix} \hat{w}(\omega) \\ \hat{d}(\omega) \end{bmatrix}. \tag{3.12}$$

Hence, substituting (3.12) into (3.11), multiplying the right and left sides by $\begin{bmatrix} w \\ d \end{bmatrix}$, and integrating yields

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{d}(j\omega) \\ \hat{e}(j\omega) \end{bmatrix}^* \Pi_p(j\omega) \begin{bmatrix} \hat{d}(j\omega) \\ \hat{e}(j\omega) \end{bmatrix} d\omega \succeq \int_{-\infty}^{\infty} \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega. \tag{3.13}$$

Since $\Delta \in \mathcal{I}(\Pi)$, the right hand side of (3.13) is non-negative. Hence, for all $d \in \mathbf{L}_2$ (3.5) holds. $\square$

Remark 3.2. *If $\Pi_{p,22} \preceq 0$, for all $\tau \in [0,1]$ the interconnection of $(G_{11}, \tau\Delta)$ is well-posed, and $\tau\Delta \in \mathcal{I}(\Pi)$, then the assumption that the interconnection in Figure 3.3 is stable is implicit since the $(1,1)$ entry of the frequency domain inequality in (3.10) provides the remaining requirement for stability by Theorem 3.1.*

As an example, we consider the simple feedback network in Figure 3.4. Suppose that Δ_1 satisfies a gain roll-off IQC at corner frequency ω_c and gain γ_1, defined by

$$\Pi(j\omega) = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1+\left(\frac{\omega}{\omega_c}\right)^2}{\gamma_1^2} \end{bmatrix}$$

(for more on roll-off IQCs, see Section 5.4). Additionally, suppose that Δ_2 satisfies a gain property such that $\|w\|_2 \leq \gamma_2 \|e\|_2$. Using IQCTOOLs, we can verify that if $\gamma_1\gamma_2 < 1$, then the gain from d to e also satisfies a roll-off IQC with corner frequency ω_c and any gain γ satisfying $\gamma \geq \frac{\gamma_1}{1-\gamma_1\gamma_2}$. The following MATLAB script demonstrates how quick and simple it is to verify this gain property using IQCTOOLs using the `iqcperf` method. The performance IQC will be satisfied for any choice of $\gamma_1\gamma_2 < 1$. An alternate software suite IQC-beta [58] can be used to verify IQCs as well.

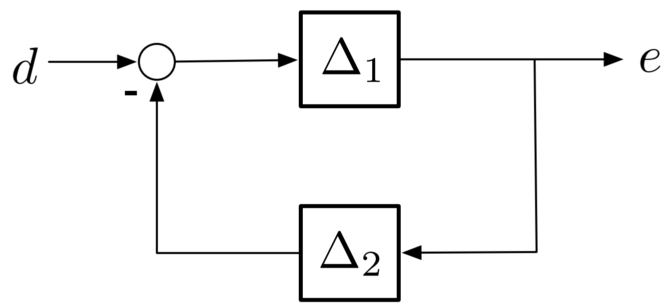

Figure 3.4: We verify performance criteria for the signals d and e

```
% Roll Off Performance IQC example
wc = 0.5;
gamma1 = 0.4;
gamma2 = 2.4;

%rolloff IQC for Delta1
iqcDelta1 = rolloffIQC(gamma1, wc);

% gain IQC for Delta2
iqcDelta2 = gainIQC(gamma2);

%performance IQC
perfIQC = rolloffIQC(abs(gamma1/(1-gamma1*gamma2)), wc);

%frequency grid
Omega = logspace(log10(wc/100), log10(wc*100), 40);

%Define udyns for Delta1 and Delta2
delta1 = udyn('Delta1', [1 1], 'UserData', {iqcDelta1});
delta2 = udyn('Delta2', [1 1], 'UserData', {iqcDelta2});

%build interconnection
H = feedback(delta1, delta2, -1);

%test performance IQC
[perfparm,Sopt,xopt] = iqcperf(H,Omega,{perfIQC});

if(perfparm==1)
```

```
    fprintf('The performance IQC is satisfied\n')
end
```

3.2 Local IQCs Definition

Megretski and Rantzer [59] define IQCs for causal operators which are defined on $\mathbf{L}_{2e}$ and bounded on $\mathbf{L}_2$. If the operator is not (or is not known to be) bounded (or even defined) on all of $\mathbf{L}_2$, then the notion of *locally* satisfying the IQC is relevant. Recall, the notation $\mathbb{B}_R\mathbf{L}_2 := \{u \in \mathbf{L}_2 : \|u\|_2 \leq R\}$.

Definition 3.6. *Suppose $\Delta : \mathbb{B}_R\mathbf{L}_2 \to \mathbf{L}_2$ is a bounded, causal operator. Then Δ is said to locally satisfy the IQC by Π on R if*

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{\Delta(u)}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{\Delta(u)}(j\omega) \end{bmatrix} d\omega \geq 0$$

holds for all $u \in \mathbb{B}_R\mathbf{L}_2$. This is notated as $\Delta \in \mathcal{I}(\Pi, [\ ], R)$.

If $\Delta : \mathbb{B}_R\mathbf{L}_2 \to \mathbf{L}_2$ locally satisfies the IQC defined by Π for all $u \in \mathbb{B}_R\mathbf{L}_2$, $\Pi(j\omega) = \Phi(s)|_{s=j\omega}$, and $\Phi(s) = \Psi(s)^\sim M\Psi(s)$, we notate this as $\Delta \in \mathcal{I}(\Psi, M, R)$ and is illustrated in Figure 3.5.

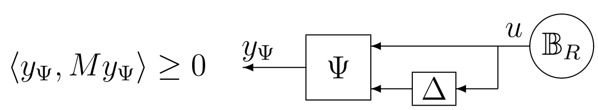

Figure 3.5: Illustration of local, factorized IQC, $\Delta \in \mathcal{I}(\Psi, M, R)$

Definition 3.7. *If*

$$\int_0^T y_\Psi^T(t) M y_\Psi(t)\, dt \geq 0, \tag{3.14}$$

for all T and for all $u \in \mathbb{B}_R\mathbf{L}_2$, then Δ is said to locally satisfy the ***strict*** *IQC defined by Ψ, M, and R and is notated as $\Delta \in \mathcal{I}_S(\Psi, M, R)$.*

3.3 Establishing Local, Strict IQCs

Given a nonlinear system, we want to generate many IQC which that system satisfies. In generating these IQCs, we will encapsulate input-output properties of the system and be able to perform various analyses without using the system dynamics. A procedure is outlined to generate locally satisfied IQCs for a nonlinear dynamical system, using linear offsets, linear weighting functions and estimates of local $\mathbf{L}_2$ gains.

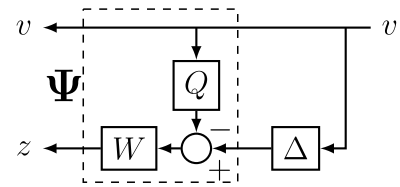

Figure 3.6: IQC Interconnection

Theorem 3.3. *If Δ is a bounded, causal operator mapping $\mathbb{B}_R\mathbf{L}_2 \to \mathbf{L}_2$, and Q and W are linear, time-invariant, stable, then $W(\Delta - Q)$ is bounded and causal on $\mathbb{B}_R\mathbf{L}_2$. Furthermore, if $\|W(\Delta - Q)\|_{\mathbf{L}_2,\mathbf{L}_2} \leq 1$ on $\mathbb{B}_R\mathbf{L}_2$, then $\Delta \in \mathcal{I}(\Pi, [\,], R)$ where*

$$\Pi(j\omega) = \begin{bmatrix} I - Q^*(j\omega)W^*(j\omega)W(j\omega)Q(j\omega) & Q^*(j\omega)W^*(j\omega)W(j\omega) \\ W^*(j\omega)W(j\omega)Q(j\omega) & -W^*(j\omega)W(j\omega) \end{bmatrix}. \tag{3.15}$$

Proof. For $v \in \mathbb{B}_R\mathbf{L}_2$, define $z := W(\Delta(v) - Qv)$, as shown in Figure 3.6. Clearly $z \in \mathbf{L}_2$ and $\|z\| \leq \|v\|$ by assumption on the local $\mathbf{L}_2$ gain of $W(\Delta - Q)$. In terms of Fourier transforms,

$$\begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{z}(j\omega) \end{bmatrix} = \begin{bmatrix} I & 0 \\ -W(j\omega)Q(j\omega) & W(j\omega) \end{bmatrix} \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{\Delta(v)}(j\omega) \end{bmatrix}. \tag{3.16}$$

By Parseval's theorem, $\|z\| \leq \|v\|$ is equivalent to

$$\int_R \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{z}(j\omega) \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{z}(j\omega) \end{bmatrix} d\omega \geq 0. \tag{3.17}$$

Direct substitution of (3.16) into (3.17) yields

$$\int_R \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{\Delta(v)}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{\Delta(v)}(j\omega) \end{bmatrix} d\omega \geq 0 \tag{3.18}$$

as desired. □

Remark 3.3. *A factorization of this IQC is*

$$\Psi := \begin{bmatrix} I & 0 \\ -WQ & W \end{bmatrix}, M := \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

Remark 3.4. *By causality $\|z\|_{2,T} \leq \|v\|_{2,T}$ for all T. Hence $\Delta \in \mathcal{I}_S(\Psi, M, R)$.*

As an example, we consider Δ governed by:

$$\begin{aligned}\dot{x} &= -x + x^3 + u \\ y &= x.\end{aligned}$$

This is a locally stable system, but can exhibit finite-escape time solutions when $\|u\|_2 \geq 1$. We establish simple local IQCs for this system by selecting the linear offset $Q(s) = \frac{1}{s+1}$, which is the linearization of the system, and a stable, minimum phase weight W. Next, using the procedure in Section 2.9, we estimate the induced $\mathbf{L}_2 \to \mathbf{L}_2$ gain of the locally stable operator $W(\Delta - Q)$. The gain, which depends on the norm-bound of the input is 0 for arbitrarily small inputs, and goes to ∞ as the norm of the input is allowed to approach 1. Two bounds on the gain, as a function of input-norm level, are shown below in Figure 3.7 using $W = 1$ (red) and the bandpass filter $W = \frac{0.05s^2+s+0.05}{s^2+s+1}$ (dashed, blue). The curve was obtained with the SOS analysis suite [8] using storage functions V and Q of degree 6.

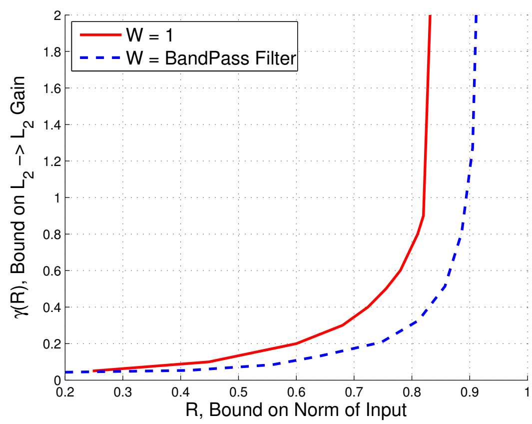

Figure 3.7: Bound of $\mathbf{L}_2 \to \mathbf{L}_2$ Gain for system $W(\Delta - Q)$

The horizontal axis is labeled R, which is the bound on the input u. Each point on the upper curve gives rise to a local IQC, with $Q(s)$ as defined, R as the horizontal coordinate, and $\bar{W}(s) = \frac{1}{\gamma(R)} W(s)$, which is the reciprocal of the norm bound (ie., the reciprocal of the vertical component of the point). Hence, many local IQCs characterized by $\left(\bar{W} = \frac{1}{\gamma(R)} W(s), Q(s) = \frac{1}{s+1}, R\right)$ can be generated for a single Δ by sampling different points $(R, \gamma(R))$ on a gain curve. Further, many of these gain curves can be established by using different weights W and linear offsets Q.

3.4 Extending Local Operators to Global Operators

One approach to verify stability of an interconnection given local IQCs using frequency domain methods is to attempt to use the results of Megretski and Rantzer in [59]. However,

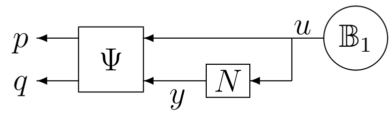

Figure 3.8: Local IQC

the theorems and proofs [59] rely on the operators being defined on $\mathbf{L}_{2e}$, bounded on $\mathbf{L}_2$, and satisfying the IQC on $\mathbf{L}_2$. This raises the idea of extending an operator that is defined on $\mathbb{B}_R\mathbf{L}_2$ to all of $\mathbf{L}_2$ such that the extended operator is bounded, causal and satisfies the IQC globally.

In particular, we focus on the operators $N : \mathbb{B}_1\mathbf{L}_2 \to \mathbf{L}_2$ which satisfy the property $\|p\|_2 \geq \|q\|_2$ for $u \in \mathbb{B}_1\mathbf{L}_2$, where $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] = \Psi\left[\begin{smallmatrix} u \\ N(u) \end{smallmatrix}\right]$, illustrated in Figure 3.8. Our goal is to define an extension of N, called N_e, such that $N_e : \mathbf{L}_{2e} \to \mathbf{L}_{2e}$ is bounded, causal and $\|p\|_2 \geq \|q\|_2$ for all in $u \in \mathbf{L}_2$, where $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] = \Psi\left[\begin{smallmatrix} u \\ N_e(u) \end{smallmatrix}\right]$. The dimensions satisfy $n_q = n_y$ and $n_p = n_u$.

We first recall a simple lemma about linear system inverses.

Lemma 3.1. *Consider the linear system H*

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C & D_1 & D_2 \end{bmatrix} \begin{bmatrix} x(t) \\ u_1(t) \\ u_2(t) \end{bmatrix},$$

with D_2 square and invertible. Define another system H_z, driven by u_1 as

$$\begin{bmatrix} \dot{\xi}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A - B_2D_2^{-1}C & B_1 - B_2D_2^{-1}D_1 \\ -D_2^{-1}C & -D_2^{-1}D_1 \end{bmatrix} \begin{bmatrix} \xi(t) \\ u_1(t) \end{bmatrix}.$$

If $x(0) = \xi(0)$ and $u_2(t) := z(t)$, then $y(t) = 0$ for all $t \geq 0$. Furthermore, if $A - B_2D_2^{-1}C$ is Hurwitz, then there exists a γ such that for all $u_1 \in \mathbf{L}_2$, $\|z\|_2 \leq \gamma\|u_1\|_2$.

Theorem 3.4. *If Ψ_{22}^{-1} is stable, then there exists an extension $N_e : \mathbf{L}_{2e} \to \mathbf{L}_{2e}$ of N that is bounded, casual, and globally satisfies the strict IQC factorization $\mathcal{I}_S\left(\Psi, M = \left[\begin{smallmatrix} I_{n_u} & 0 \\ 0 & -I_{n_y} \end{smallmatrix}\right]\right)$.*

Proof. Let

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1u(t) + B_2N(u)(t), \\ p(t) &= C_1x(t) + D_{11}u(t) + D_{12}N(u)(t) \\ q(t) &= C_2x(t) + D_{21}u(t) + D_{22}N(u)(t) \end{aligned}$$

be the response of Ψ as depicted in Figure 3.8. The matrix A is Hurwitz, D_{22} is invertible, and it follows from the assumption that Ψ_{22}^{-1} is stable that $A - B_2D_{22}^{-1}C_2$ is Hurwitz.

Define the bounded operator $N_e : \mathbf{L}_{2e} \to \mathbf{L}_{2e}$ by first describing it's action on any $u \in \mathbf{L}_{2e}$ as follows:

$$T_c := \sup_{T>0} \int_0^T \|u(t)\| dt \leq 1.$$

For $t \leq T_c$, let

$$(N_e(u))(t) := (N(u))(t).$$

If $T_c < \infty$,

$$\xi_0 := \int_0^{T_c} e^{A(T_c-\tau)}[B_1 u(\tau) + B_2 N(u)(\tau)]d\tau$$

and note that ξ_0 is a copy of the value of the state $x(t)$ of Ψ at $t = T_c$. For $t > T_c$, $(N_e(u))(t) := z(t)$ where z is governed by the system H_z

$$\begin{bmatrix} \dot{\xi}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A - B_2 D_{22}^{-1} C_2 & B_1 - B_2 D_{22}^{-1} D_{21} \\ -D_{22}^{-1} C_2 & -D_{22}^{-1} D_{21} \end{bmatrix} \begin{bmatrix} \xi(t) \\ u(t) \end{bmatrix}, \quad \xi(T_c) := \xi_0.$$

For brevity, we denote A_z, B_z, C_z, D_z as the state space matrices for H_z.

By Lemma 3.1, defining $(N_e(u))(t) := z(t)$ renders $q(t) = 0$ for all $t \geq T_c$ and hence

$$\|p\|_{2,T} \geq \|q\|_{2,T}$$

for all $T \geq 0$. Hence, N_e globally satisfies the strict IQC factorization, so $N_e \in \mathcal{I}_S(\Psi, M)$.

Clearly N_e is a causal operator, since it's output at any time depends only on input values up to that time. We will now prove the boundedness of N_e. There exists $\beta_1 = \sum_i \left(\sum_j \| \left(e^{At} B_1\right)_{ij} \|_{2,[0,\infty]} \right)^2$, $\beta_2 = \sum_i \left(\sum_j \| \left(e^{At} B_2\right)_{ij} \|_{2,[0,\infty]} \right)^2$, $\beta_N = \sup_{\|u\|_2 \leq 1} \|N(u)\|_2$, $\beta_3 = \|\bar{\sigma}(C_2 e^{At})\|^2_{2,[0,\infty]}$, $\beta_4 = \|C_z(sI - A_z)^{-1} B_z + D_z\|_\infty$ such that

$$\begin{aligned} \|\xi_0\|_2 &\leq (\beta_1 + \beta_2 \beta_N)\|u\|_{2,[0\ T_c]}, \\ \|y\|_{2,[0\ T_c]} &\leq \beta_N \|u\|_{2,[0\ T_c]}, \\ \|y\|_{2,[T_c,\ \infty]} &\leq \beta_3 \|\xi_0\|_2 + \beta_4 \|u\|_{2,[T_c\ \infty]}. \end{aligned}$$

(See facts 7.3-7.4 in the Appendix for derivations of the bounds on $\beta_1, \beta_2, \beta_3$.) Note that

$$\begin{aligned} \|y\|^2_{2,[0\ \infty]} &= \|y\|^2_{2,[0\ T_c]} + \|y\|^2_{2,[T_c\ \infty]} \\ &\leq \beta_N^2 \|u\|^2_{2,[0\ T_c]} + \left(\beta_3(\beta_1 + \beta_2\beta_N)\|u\|_{2,[0\ T_c]} + \beta_4 \|u\|_{2,[T_c\ \infty]}\right)^2 \\ &\leq \beta_N^2 \|u\|^2_{2,[0\ \infty]} + \left(\beta_3(\beta_1 + \beta_2\beta_N) + \beta_4\right)^2 \|u\|^2_{2,[0\ \infty]} \\ &\leq 2\max\left(\beta_N^2, ((\beta_3(\beta_1 + \beta_2\beta_N) + \beta_4)^2\right) \|u\|^2_{2,[0\ \infty]}. \end{aligned}$$

Hence, N_e is a bounded operator on $\mathbf{L}_2$. Furthermore, since N_e maps $\mathbf{L}_2$ to $\mathbf{L}_2$ and it is causal, N_e also maps $\mathbf{L}_{2e}$ to $\mathbf{L}_{2e}$. This completes the extension. $\square$

Remark 3.5. *Although N_e is an extension of the operator N it is clear that the extension also depends on (Ψ, M). Hence, if an operator satisfies multiple local IQCs, it is not clear that there is one extension of the operator for all of the IQCs. This issue will arise in subsequent sections.*

Remark 3.6. *In Section 3.3, the IQCs developed by weighted local $\mathbf{L}_2$ gains have the property that $\Psi_{22} = W$. Note that if W^{-1} is stable, then conditions of Theorem 3.4 are satisfied, and hence operators which satisfy them locally can be extended as described above.*

3.5 Consistent IQC Multipliers

As mentioned in Remark 3.5, if an operator satisfies many IQCs locally, the extended operator that is suitable for one IQC is different than the extended operator for another IQC. In a particular robustness analysis, the convex optimization in Remark 3.1 transforms the many IQCs that the operator satisfies into a single IQC that is ultimately useful for this analysis. Even if extensions of the operator for the individual IQCs existed, does an extension exist for this specific element from the positive cone of the multipliers? In this section, we address this for the specific form of the IQC from Section 3.3.

Lemma 3.2. *Let $\{W_k\}_{k=1}^n \in \mathbb{R}\mathbf{H}_\infty^{n_y \times n_y}$ with $\{W_k^{-1}\}_{k=1}^n \in \mathbb{R}\mathbf{H}_\infty^{n_y \times n_y}$, $L \in \mathbb{R}\mathbf{H}_\infty^{n_y \times n_u}$, $\{\alpha_k > 0\}_{k=1}^n$. Suppose $N : \mathbb{B}_R\mathbf{L}_2 \to \mathbf{L}_2$ is bounded, causal and for all k*

$$N \in \mathcal{I}_S \left(\Psi_k = \begin{bmatrix} I & 0 \\ -W_k L & W_k \end{bmatrix}, M = \begin{bmatrix} I_{n_u} & 0 \\ 0 & -I_{n_y} \end{bmatrix}, R \right).$$

Let $W \in \mathcal{U}_{\mathbb{R}\mathbf{H}_\infty}^{n_y \times n_y}$ satisfy $W^*W = \sum_{k=1}^n \alpha_k W_k^* W_k$ (spectral factorization). Then

$$N \in \mathcal{I}_S \left(\Psi = \begin{bmatrix} \sqrt{\sum_{k=1}^n \alpha_k} I & 0 \\ -WL & W \end{bmatrix}, M = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, R \right).$$

Proof. Let $u \in \mathbb{B}_R\mathbf{L}_2$ and define $q := N(u) - Lu$. Since $N : \mathbb{B}_R\mathbf{L}_2 \to \mathbf{L}_2$ is bounded, causal and satisfies $\mathcal{I}_S(\Psi_k, M, R)$ for $k = 1, \ldots, n$, it follows that $q \in \mathbf{L}_2$ and $\|W_k q\|_{2,T} \le \|u\|_{2,T}$

for all $T \geq 0$. Hence, for all $T \geq 0$

$$\begin{aligned}
\|Wq\|_{2,T}^2 &\leq \|Wq\|_2^2 \\
&= \int_{-\infty}^{\infty} \hat{q}^*(\omega) W^*(j\omega) W(j\omega) \hat{q}(\omega) d\omega \\
&= \int_{-\infty}^{\infty} \hat{q}^*(\omega) \left(\sum_{k=1}^{n} \alpha_k W_k^*(j\omega) W_k(j\omega) \right) \hat{q}(\omega) d\omega \\
&= \sum_{k=1}^{n} \alpha_k \int_R \hat{q}^*(\omega) W_k^*(j\omega) W_k(j\omega) \hat{q}(\omega) d\omega \\
&= \sum_{k=1}^{n} \alpha_k \|W_k q\|_2^2 \\
&\leq \sum_{k=1}^{n} \alpha_k \|u\|_2^2
\end{aligned}$$

By causality, $\|Wq\|_{2,T} \leq \sqrt{\sum_{k=1}^{n} \alpha_k} \|u\|_{2,T}$. By Theorem 3.3,

$$N \in \mathcal{I}_S \left(\Psi = \begin{bmatrix} \sqrt{\sum_{k=1}^{n} \alpha_k} I & 0 \\ -WL & W \end{bmatrix}, M = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, R \right).$$

□

Lemma 3.3. *The IQC multiplier Π_k associated with (Ψ_k, M) is*

$$\Pi_k = \begin{bmatrix} I - L^* W_k^* W_k L & L^* W_k^* W_k \\ W_k^* W_k L & -W_k^* W_k \end{bmatrix}.$$

The multiplier Π associated with (Ψ, M), $\Pi = \Psi^{\sim} M \Psi$ satisfies

$$\Psi^{\sim} M \Psi = \sum_{k=1}^{n} \alpha_k \Pi_k.$$

Proof. This simple verification is left to the reader. □

Remark 3.7. *Since $\Psi_{22}^{-1} = W^{-1} \in \mathbb{R}\mathbf{H}_\infty$, by Theorem 3.4 there exists a bounded, casual extension of N, $N_e : \mathbf{L}_{2e} \to \mathbf{L}_{2e}$ such that $N_e \in \mathcal{I}(\Psi, M)$.*

Remark 3.8. *This lemma will be used in Chapter 4, where we examine interconnections of operators which satisfy many IQCs of this form.*

3.6 Invariants of Integral Quadratic Constraints

The methods outlined for establishing for a particular system in Section 3.3 require computation of gain bounds using SOS techniques from Section 2.9. This raises the question, once an IQC is established for a particular system, is it possible to reduce or eliminate the computation time needed to establish IQCs for similar systems?

We establish invariants of factorized, local integral quadratic constraints (IQCs) for parameterized systems through and input-output scaling and a scaling of the units of the original system. Given a list of IQCs for a parameterized system with a particular choice of parameters, we generate lists of IQCs for any other system in the parameter class.

Dimensional analysis [9] relies on the fact that a physical law must be independent of the units used to measure the physical variables. By scaling the units of an equation, we obtain a new equation with different quantities for the physical variables that describes the same relationship in terms of the new units. While dimensional analysis plays a huge roll in many branches of engineering and the physical sciences, its role in control is much less visible. Recent work to reduce the dimension of a relevant parameter space in a family of control designs, [15], [16], [70], [14], show great promise.

For a parameterized system with an IQC, we show that through scaling the units, we obtain a scaled system which satisfies a scaled IQC. We establish invariants among local IQCs through the use of an input-output scaling, time scaling, and coordinate change. If an IQC is established for one particular system in a parameterized class, then (through the invariants) a manifold of systems within the class satisfy IQCs obtained through transformation. Thus, through this reduction in dimensionality, obtaining a rich collection of IQCs for the entire parameterized class is easier. This advantage significantly reduces computation time for establishing lists of IQCs. Utilizing these invariants is crucial to the success of the ultimate goal of building a large library of IQCs that these (small) subsystems satisfy.

Notation and Definitions

We recall the following definitions from [9]:

Definition 3.8. ***Fundamental units*** *are a set of units of measurement, which are arbitrarily defined. (ex: length, time, mass).*

Definition 3.9. *A set of fundamental units that is sufficient for measuring the properties of the class of phenomena under consideration is called a* ***system of units*** *(ex: meter-second).*

Definition 3.10. *A set of systems of units that differ only in the magnitudes (but not in the physical nature) of the fundamental units is called a* ***class of systems of units*** *(ex: meter and inches).*

For example, if we refer to the meter system as the original system, then the units for an arbitrary system in the class are

$$\text{unit of length} \;=\; \frac{\text{meter}}{L}$$

where $L > 0$, and indicate the factor by which the fundamental units of length *decreases*. The numerical value for length is *increased* by a factors of L. If we choose inches to be the arbitrary system, then $L = 39.37$,

$$\text{units of inches} \;=\; \frac{1}{39.37}\text{units of meters},$$

and if a is the length of some object in units of meters, then $b = 39.37a$ is the length of the object in units of inches.

An instance of a model of a dynamical system is written with a specific system of units. If S is a model of a dynamical system, $\mathcal{U}_S$ will denote the system of units being used in this system model. For clarity, $\mathbf{L}_{2,\mathcal{U}}$ indicates that the time units are drawn from the system of units $\mathcal{U}$.

Let N be an operator described by the finite dimensional ODE of the form

$$\begin{aligned} x'(\cdot) &= f(x(\cdot), u(\cdot)), \\ y(\cdot) &= h(x(\cdot)). \end{aligned}$$

A few words about time units are in order. Local satisfaction of an IQC $\mathcal{I}(\Psi, M, R)$ by a system (f, h) where Ψ is defined by (A, B, C, D) means that all solutions of

$$\begin{aligned} x'(\cdot) &= f(x(\cdot), u(\cdot)), \\ y(\cdot) &= h(x(\cdot)), \\ x'_\Psi(\cdot) &= Ax_\Psi(\cdot) + B\begin{bmatrix} u(\cdot) \\ y(\cdot) \end{bmatrix}, \\ y_\Psi(\cdot) &= Cx_\Psi(\cdot) + D\begin{bmatrix} u(\cdot) \\ y(\cdot) \end{bmatrix}, \end{aligned}$$

with $\|u\|_2 \leq R$ satisfy $\langle y_\Psi, My_\Psi \rangle \geq 0$. This is a property of the tuple (f, h, A, B, C, D, M, R), and not a property of the time unit. The specific time unit is denoted by the $(\cdot)$ placeholder; the symbol $'$ means differentiation with respect to that chosen time unit; and the integration in the $\mathbf{L}_2$ norm of u is also in this time unit.

Input-Output Scaling

We obtain the first invariant for a local IQC from an input-output scaling. Suppose a $N \in \mathcal{I}(\Psi, M, R)$. For any $\alpha_1 \neq 0, \alpha_2 \neq 0$, let the

$$\Psi_{\alpha_1,\alpha_2} := \begin{bmatrix} \alpha_1\Psi_{11} & \alpha_2\Psi_{12} \\ \alpha_1\Psi_{21} & \alpha_2\Psi_{22} \end{bmatrix}$$

indicate that the first input of Ψ is scaled by α_1 and the second input of Ψ is scaled by α_2.

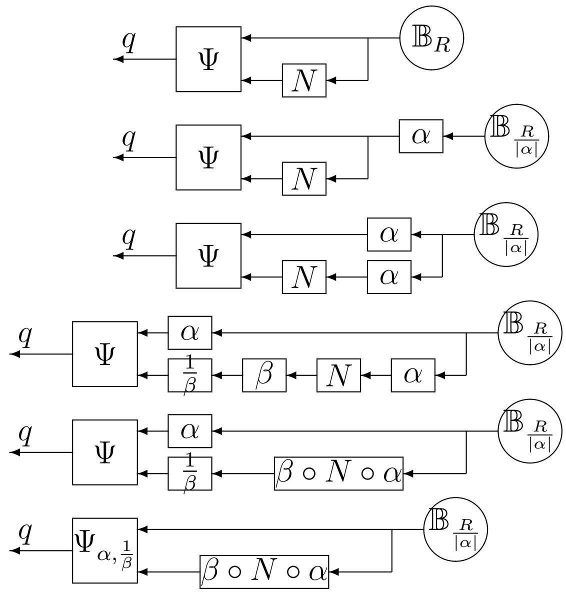

Figure 3.9: Illustrated proof of Theorem 3.5.

Theorem 3.5. *For any $\alpha, \beta \neq 0$,*

$$N \in \mathcal{I}(\Psi, M, R) \iff \beta \circ N \circ \alpha \in \mathcal{I}\left(\Psi_{\alpha,\frac{1}{\beta}}, M, \frac{R}{|\alpha|}\right).$$

Proof. As illustrated in Figure 3.9, the original system can be equivalently represented by scaling the ball of inputs $\mathbb{B}_R$ by $\frac{1}{|\alpha|}$, so that $v \in \mathbb{B}_{\frac{R}{|\alpha|}}$ while simultaneously scaling the first input of Ψ by α. Likewise, the original system is equivalent under an output scaling of the second input of Ψ by $\frac{1}{\beta}$ and an output scaling of N by β. This results in an IQC factorization of $(\Psi_{\alpha,\frac{1}{\beta}}, M, \frac{R}{|\alpha|})$, which $\beta \circ N \circ \alpha$ satisfies. $\square$

Time Scale and Variable Transformation

For the time scale and variable transformation, we employ dimensional analysis. Let $S : \mathbb{B}_{R,\mathcal{U}_S} \to \mathbf{L}_{2,\mathcal{U}_S}$ be a bounded, causal operator defined by

$$\frac{d\eta}{d\tau} = f_S(\eta(\tau), u(\tau)), \tag{3.19}$$
$$y(\tau) = h_S(\eta(\tau), u(\tau)), \tag{3.20}$$

and $P : \mathbb{B}_{R,\mathcal{U}_P} \to \mathbf{L}_{2,\mathcal{U}_P}$ is a bounded, causal operator defined by

$$\frac{dx}{dt} = f_P(x(t), u(t)), \tag{3.21}$$
$$z(t) = h_P(x(t), u(t)). \tag{3.22}$$

Let Ψ^H and Ψ^L be stable, proper linear systems defined by their respective (A, B, C, D) matrices with the appropriate subscripts to denote H or L.

Theorem 3.6. *If there exist invertible matrix T and constant $\lambda > 0$ such that*

$$f_S(\eta, u) \equiv \frac{T}{\lambda} f_P(T^{-1}\eta, u), \tag{3.23}$$

$$h_S(\eta, u) \equiv h_P(T^{-1}\eta, u), \tag{3.24}$$

for all $(\eta, u) \in \mathbb{R}^n \times \mathbb{R}^m$ and

$$\begin{aligned} A_L = & \frac{A_H}{\lambda}, \\ B_L = & \frac{B_H}{\lambda}, \\ C_L = & C_H, \\ D_L = & D_H, \end{aligned} \tag{3.25}$$

then $P \in \mathcal{I}\left(\Psi^H, M, R\right)$ if and only if $S \in \mathcal{I}\left(\Psi^L, M, \sqrt{\lambda}R\right)$.

Proof. The proof follows by representing the P system in terms of the units of the S system through a time scaling and a coordinate change. Let (x, u) be solution trajectories of (3.21). Define

$$\eta(\tau) := Tx(t)\big|_{t=\lambda^{-1}\tau}, \tag{3.26}$$

$$v(\tau) := u(t)\big|_{t=\lambda^{-1}\tau}. \tag{3.27}$$

Then

$$\begin{aligned} \left.\frac{d\eta}{d\tau}\right|_{\tau} &= \left.\frac{T}{\lambda}\frac{dx(t)}{dt}\right|_{t=\lambda^{-1}\tau} \\ &= \left.\frac{T}{\lambda} f_P\left(x(t), u(t)\right)\right|_{t=\lambda^{-1}\tau} \\ &= \frac{T}{\lambda} f_P\left(T^{-1}\eta(\tau), v(\tau)\right) \\ &= f_S(\eta(\tau), v(\tau)), \end{aligned}$$

showing that (η, v) in (3.26) and (3.27) solve (3.19). The outputs h_S and h_P are equivalent under the coordinate change (3.26) and the time scaling $\tau = \lambda t$ used above. Hence, P and S are systems that describe the same physical law with different units. Using the same approach, input/state/output trajectories of Ψ^H and Ψ^L are related by the same time scaling by virtue of (3.25).

Finally, note that u and v from (3.27) satisfy:

$$\|v\|^2_{2,\mathcal{U}_S} = \int_0^\infty v^2(\tau)\, d\tau = \int_0^\infty u^2(\tau\lambda^{-1})\, d\tau = \int_o^\infty u^2(t)\, (\lambda dt) = \lambda \|u\|^2_{2,\mathcal{U}_P}$$

Hence, $\|u\|_{2,\mathcal{U}_P} \leq R$ if and only if $\|v\|_{2,\mathcal{U}_S} \leq \sqrt{\lambda}R$. $\square$

Example

This example combines the input-output and time-scaling invariants and exhibits a one-to-one correspondence between local IQCs for one system and local IQCs for another, related system.

Consider the first-order parametrized system, denoted as the operator N, as

$$\begin{aligned} f_N(x,u) &= ax + bx^m + cu, \\ h_N(x,u) &= dx, \end{aligned} \tag{3.28}$$

where $m \in \mathbb{N}$, $m \geq 2$, and $a > 0, b \geq 0, c \geq 0, d \geq 0$ are arbitrary parameters. Let the operator P describe the dynamics of a particular parameter choice for N, where $a = b = c = d = 1$:

$$\begin{aligned} f_P(x,u) &= -x + x^m + u, \\ h_P(x,u) &= x. \end{aligned} \tag{3.29}$$

Let

$$\lambda := \frac{1}{a}, \tag{3.30}$$

$$T := \left(\frac{1}{b\lambda}\right)^{\frac{1}{m-1}}, \tag{3.31}$$

$$\alpha := \frac{c\lambda}{T}, \tag{3.32}$$

$$\beta := dT. \tag{3.33}$$

Let Ψ^H and Ψ^L be stable, proper linear systems with (A,B,C,D) matrices related by (3.25).

Lemma 3.4. *$P \in \mathcal{I}\left(\Psi^H, M, R\right)$ if and only if $N \in \mathcal{I}\left(\Psi^L_{\alpha,\frac{1}{\beta}}, M, \frac{R\sqrt{\lambda}}{|\alpha|}\right)$*

Proof. Define a system S such that (3.23)-(3.24) hold. Substituting (3.30)-(3.31) into (3.23)-(3.24) yields

$$\begin{aligned} f_S(\eta,u) &= -a\eta + b\eta^m + \frac{T}{\lambda}u, \\ h_S(x,u) &= \frac{\eta}{T}. \end{aligned} \tag{3.34}$$

Note that (3.28), (3.32), (3.33), (3.34), and imply that $N = \beta \circ S \circ \alpha$. By Theorem 3.6, $P \in \mathcal{I}\left(\Psi^H, M, R\right)$ if and only if $S \in \mathcal{I}\left(\Psi^L, M, \sqrt{\lambda}R\right)$. Moreover, by Theorem 3.5, $S \in \mathcal{I}\left(\Psi^L, M, \sqrt{\lambda}R\right)$ if and only if $N \in \mathcal{I}\left(\Psi^L_{\alpha,\frac{1}{\beta}}, M, \frac{\sqrt{\lambda}R}{|\alpha|}\right)$

□

As a concrete example, let $m = 3, a = 2, b = 7, c = 3, d = 5$ so that

$$\begin{aligned} f_P(x,u) &= -x + x^3 + u, \\ h_P(x,u) &= x \end{aligned}$$

and

$$\begin{aligned} f_N(x,u) &= -2x + 7x^3 + 3u, \\ h_N(x,u) &= 5x. \end{aligned}$$

Using the procedure in Section 2.9 with degree 6 V and Q, the induced $\mathbf{L}_2 \to \mathbf{L}_2$ gain of $\|W(P-Q)\|$ is estimated, where $Q(s) = \frac{1}{s+1}$ and $W(s) = \frac{0.05s^2+s+0.05}{s^2+s+1}$, which gives rise to a gain curve. Sampling a point from this curve, for inputs less than $R = 0.7473$, the gain bound is less than $\gamma_R = 0.2078$. Hence, by Theorem 3.3, $P \in \mathcal{I}\left(\Psi^P, M, R = 0.7473\right)$, where $M = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$ and the state space realization of Ψ^P is

$$\Psi^P = \left[\begin{array}{ccc|c:c} -1 & -1 & -2 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 2.2854 & 0 & -0.2406 & 0 & 0.2406 \end{array}\right]$$

By Lemma 3.4, $N \in \mathcal{I}\left(\Psi^N, M, R = 0.1883\right)$, where the state space realization of Ψ^N is

$$\Psi^N = \left[\begin{array}{ccc|c:c} -2 & -2 & -4 & 0 & 1.4967 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 5.612 & 0 \\ \hline 0 & 0 & 0 & 2.806 & 0 \\ 2.2854 & 0 & -0.2406 & 0 & 0.09 \end{array}\right].$$

Remark 3.9. *By Lemma 3.4, a library of IQCs which N satisfies, with any choice of a, b, c, d, can be trivially generated using a pre-established library of IQCs which P satisfies.*

In order to find IQCs for (3.28) it is only necessary (and sufficient) to find IQCs for (3.29). Moreover, such IQCs would apply to (3.28) for any choice of (a, b, c, d) via (3.30), (3.31), (3.32), (3.33). This shows that if one wants to establish a large collection of parameterized IQCs (by (a, b, c, d)) for (3.28), one needs only to establish a large collection of IQCs for the single system given in (3.29).

Chapter 4

Performance Analysis of Interconnected Locally Stable Systems

We are interested in model-based certification of large-scale dynamical systems in the presence of input and model uncertainty. The complexity of such problems often dictates a decomposition approach, [3], [92], breaking the system into a complex interconnection of smaller subsystems. Individual analysis on the isolated subsystems reveals coarse properties of the subsystems (eg., passive, small-gain or generalizations, such as dissipativeness with respect to various supply rates, etc.). In some cases, the coarse properties, coupled with the interconnection topology is enough to verify the overall behavior, [99], [47], [92]. In special cases, optimization can select which coarse properties are most important [92]. Moreover, this verification step, which involves the coarse properties and the interconnection topology is scalable, with semidefinite programming as the foundational computational engine.

We present an IQC-based analysis technique to analyze the input/output gain properties of interconnections of locally stable subsystems. We assume that each subsystem satisfies a collection of IQCs on inputs with $\mathbf{L}_2$ norm less than or equal to 1. The subsystems may be unstable (or even undefined) on larger norm input signals. The goal of the analysis is to exploit these local IQCs to obtain a local bound on the gain of the interconnection. We extend these ideas to the situation where the decomposition leads to component models which are not globally stable, and may not even be defined on all inputs (eg., certain inputs may lead to finite-escape time solutions).

4.1 Introduction

Consider the system in Figure 4.1, specified by the equations

$$\begin{bmatrix} z \\ e \end{bmatrix} = G \begin{bmatrix} w \\ d \end{bmatrix},$$

where $G \in \mathbb{R}\mathbf{H}_\infty$, the operator $\Delta : \mathbb{B}_1\mathbf{L}_2 \to \mathbf{L}_2$ is bounded and causal, and $\Delta \in \mathcal{I}(\Pi, [\,], 1)$.

We wish to verify if a performance IQC Π_p is satisfied for the exogenous input d to the output e, in Figure 4.1, which is valid under some (unknown at this point) bound on $\|d\|$. The analysis will be accomplished by using the information about Δ contained in the IQC defined by Π. We will first investigate the gain from d to z, by temporarily setting $e := z$, which yields the structure shown in Figure 4.2. The system G_z embodies this error redefinition.

Both a frequency domain and state-space approach to verifying the performance IQC are presented. For each approach, we will first analyze properties of the interconnection of G with an extension $\Delta_e : \mathbf{L}_{2e} \to \mathbf{L}_{2e}$ of Δ, as in Figures 4.3-4.4. Then, connections to the original problem, represented in Figures 4.1-4.2, will be established. It is important to note (as will be clear in the derivations) that the requirements on the extended operator Δ_e are different in the two approaches.

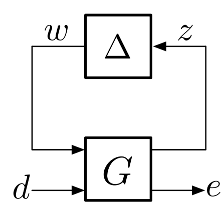

Figure 4.1: (G, Δ)

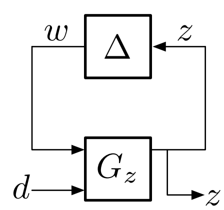

Figure 4.2: (G_z, Δ)

4.2 Frequency Domain Approach

We apply performance analysis using IQCs in Section 3.1 and the extension for local operators in Section 3.4 to verify a performance IQC Π_p for the signals d and e.

Theorem 4.1. *Assume* Π *and* Π_p*, describing* Δ *and the performance objective, as outlined in Section 4.1 are defined. Let* G *be a causal, finite dimensional, linear time-invariant operator and let* Δ *be a causal, bounded operator mapping* $\mathbb{B}_1\mathbf{L}_2 \to \mathbf{L}_2$*.* $\Delta \in \mathcal{I}(\Pi, [\,], 1)$*. Assume there exists a bounded, causal extension* $\Delta_e : \mathbf{L}_{2e} \to \mathbf{L}_{2e}$ *of* Δ *with* $\Delta_e(z) = \Delta(z)$ *for* $z \in \mathbb{B}_1\mathbf{L}_2$ *and* $\Delta_e \in \mathcal{I}(\Pi)$*. Further assume*

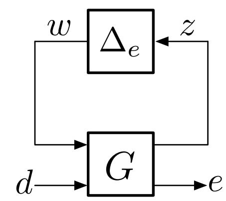

Figure 4.3: (G, Δ_e)

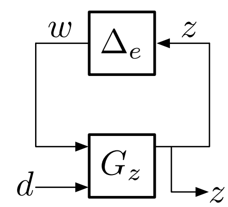

Figure 4.4: (G_z, Δ_e)

1. *for every* $\tau \in [0,1]$ *the interconnection of* G *and* $\tau\Delta_e$ *is well-posed,*

2. *for every* $\tau \in [0,1]$, $\tau\Delta_e \in \mathcal{I}(\Pi)$,

3. *the interconnection of* G_{11} *and* Δ_e *is well-posed and stable.*

4. *there exists* $\rho > 0$ *and* $\epsilon_1 > 0$ *such that*

$$\begin{bmatrix} G_z(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} \Pi_{11}(j\omega) & 0 & \Pi_{12}(j\omega) & 0 \\ 0 & I & 0 & 0 \\ \Pi_{12}^*(j\omega) & 0 & \Pi_{22}(j\omega) & 0 \\ 0 & 0 & 0 & -\rho^2 I \end{bmatrix} \begin{bmatrix} G_z(j\omega) \\ I \end{bmatrix} \preceq -\epsilon_1 I \tag{4.1}$$

   for all $\omega \in \mathbb{R}$

5. *there exists* $\epsilon_2 > 0$ *such that*

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} \Pi_{11}(j\omega) & 0 & \Pi_{12}(j\omega) & 0 \\ 0 & \Pi_{p,22} & 0 & \Pi_{p,12}^* \\ \Pi_{12}^*(j\omega) & 0 & \Pi_{22}(j\omega) & 0 \\ 0 & \Pi_{p,12} & 0 & \Pi_{p,11} \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \preceq -\epsilon_2 I \tag{4.2}$$

   for all $\omega \in \mathbb{R}$,

then for all $\|d\|_2 < \frac{1}{\rho}$ *there exist unique solutions to the equations describing the interconnection of* (G, Δ). *Moreover,* z *satisfies* $\|z\|_2 \leq 1$ *and* (d, e) *satisfy the IQC defined by* Π_p *(see Definition 3.1).*

Proof. We begin by establishing facts about the signals in the (G, Δ_e) interconnection. Since $\Delta_e \in \mathcal{I}(\Pi)$ and the assumptions 3-5 hold, by Theorem 3.2, for all $d \in \mathbf{L}_2$, there exist unique z and e are in $\mathbf{L}_2$ solving the equations. Moreover, $\|z\|_2 \leq \rho\|d\|_2$, and (d, e) satisfy the IQC defined by Π_p.

Note that if $\|d\|_2 = \alpha < \frac{1}{\rho}$, then $\|z\|_2 \leq \rho\alpha < 1$. Since Δ_e equals Δ on $\mathbb{B}_1\mathbf{L}_2$, $\Delta_e(z) = \Delta(z)$. Hence, the unique solution of the loop equations involving (G, Δ_e) is also a solution for the (G, Δ) interconnection. Suppose that other solutions for (G, Δ) existed. If so, then either $\|z\|_2 \leq \rho\alpha$ or there are finite times when $\|z\|_{2,T} > \rho\alpha$. The case where $\|z\|_2 \leq \rho\alpha$ is not possible, since it would constitute another solution with (G, Δ_e). If at some finite time, $\rho\alpha < \|z\|_{2,T} < 1$, then $\Delta_e(z) = \Delta(z)$ on $[0\ T]$. So, on $[0\ T]$, this is also a solution to the interconnection of (G, Δ_e), but violates the already established gain $\|z\|_{2,T} \leq \rho\alpha$ (on a finite horizon) in the (G, Δ_e) loop.

Hence, we conclude that for all $\|d\|_2 < \frac{1}{\rho}$ there exist unique solutions to the equations describing the interconnection of (G, Δ). Moreover, z satisfies $\|z\|_2 \leq 1$ and (d, e) satisfy the IQC defined by Π_p. □

Remark 4.1. *In the theorem, we considered $z \in \mathbb{B}_1\mathbf{L}_2$. In a more general case, if $z \in \mathbb{B}_R\mathbf{L}_2$, then it is easy to extend the theorem and proof by replacing $\|d\|_2 < \frac{R}{\rho}$ and $\|z\|_2 \leq R$.*

4.3 Interconnections of One System with Many IQCs

In Sections 4.1-4.2, we considered the interconnection with G, of a single Δ which satisfies one IQC, $\Delta \in \mathcal{I}(\Pi, [\,], R)$. Often Δ is known to satisfy a collection of IQCs. We extend previous results in Section 4.2 to cover the case with a single Δ, which satisfies many IQCs.

Theorem 4.2. *Assume $\{\Pi_j\}_{j=1}^K$ and Π_p are defined. Let $G \in \mathbb{R}\mathbf{H}_\infty$ and let Δ is a causal, bounded operator mapping $\mathbb{B}_1\mathbf{L}_2 \to \mathbf{L}_2$ with $\Delta \in \mathcal{I}(\Pi_j, [\,], 1)$ for $j = 1, \ldots, K$. Assume*

1. *there exists $\{\alpha_j > 0\}_{j=1}^K$, $\rho > 0$ and $\epsilon_1 > 0$ such that*

$$\begin{bmatrix} G_z(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} \sum_{j=1}^K \alpha_j \Pi_{j,11}(j\omega) & 0 & \sum_{j=1}^K \alpha_j \Pi_{j,12}(j\omega) & 0 \\ 0 & I & 0 & 0 \\ \sum_{j=1}^K \alpha_j \Pi_{j,12}^*(j\omega) & 0 & \sum_{j=1}^K \alpha_j \Pi_{j,22}(j\omega) & 0 \\ 0 & 0 & 0 & -\rho^2 I \end{bmatrix} \begin{bmatrix} G_z(j\omega) \\ I \end{bmatrix} \preceq -\epsilon_1 I \quad (4.3)$$

   for all $\omega \in \mathbb{R}$

2. *there exists an extension $\Delta_e^\alpha : \mathbf{L}_{2e} \to \mathbf{L}_{2e}$ of Δ with $\Delta_e^\alpha(z) = \Delta(z)$ for $z \in \mathbb{B}_1\mathbf{L}_2$ and $\Delta_e^\alpha \in \mathcal{I}\left(\sum_{j=1}^K \alpha_j \Pi_j\right)$*

3. *for every $\tau \in [0, 1]$ the interconnection of G and $\tau\Delta_e^\alpha$ is well-posed,*

4. *for every $\tau \in [0, 1]$, $\tau\Delta_e^\alpha \in \mathcal{I}\left(\sum_{j=1}^K \alpha_j \Pi_j\right)$,*

Then for all $\|d\| < \frac{1}{\rho}$ *there exists unique solutions* z, e, w *to the equations describing the interconnection of* (G, Δ). *Moreover,* $z, e, w \in \mathbf{L}_2$.

Further assume

6. *there exists* $\{\beta_j > 0\}_{j=1}^K$, $\epsilon_2 > 0$ *such that*

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} \sum_{j=1}^K \beta_j \Pi_{j,11}(j\omega) & 0 & \sum_{j=1}^K \beta_j \Pi_{j,12}(j\omega) & 0 \\ 0 & \Pi_{p,22} & 0 & \Pi_{p,12}^* \\ \sum_{j=1}^K \beta_j \Pi_{j,12}^*(j\omega) & 0 & \sum_{j=1}^K \beta_j \Pi_{j,22}(j\omega) & 0 \\ 0 & \Pi_{p,12} & 0 & \Pi_{p,11} \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \preceq -\epsilon_2 I \tag{4.4}$$

for all $\omega \in \mathbb{R}$,

Then this (d, e) *pair satisfy the IQC defined by* Π_p.

Proof. We begin by establishing facts about the signals in the (G, Δ_e^α) interconnection. Since $\Delta_e^\alpha \in \mathcal{I}\left(\sum_{j=1}^K \alpha_j \Pi_j\right)$ and the assumptions 1-5 hold, by Theorem 3.2, for all $d \in \mathbf{L}_2$, there exist unique z and e in $\mathbf{L}_2$ solving the equations. Moreover, $\|z\|_2 \leq \rho \|d\|_2$.

The argument for the proof of Theorem 4.1 also applies, showing that if $\|d\| < \frac{1}{\rho}$, then the resulting $z \in \mathbb{B}_1 \mathbf{L}_2$ and, hence, (z, e, w) are also the unique solution of (G, Δ). Condition 6 is applied to these $\mathbf{L}_2$ signals (in the same manner as in 3.2) to infer (d, e) satisfy the IQC defined by Π_p. □

Remark 4.2. *In Chapter 3, we defined local IQCs arising from weighted, local* $\mathbf{L}_2$ *norm bounds. We showed operators which satisfy these local IQCs can always be extended to all of* $\mathbf{L}_{2e}$*, and all input/output pairs of the extended operator satisfy the IQC. Furthermore, we showed that positive combinations of multipliers associated with a group of such IQCs can be viewed as another IQC of the same form. This means that for any operator which satisfies all of the IQCs individually, and any positive combination of multipliers, there is an extension which satisfies the combined multiplier globally. Hence, for these types of IQCs, the seemingly stringent condition 2 in Theorem 4.2 is* ***automatically satisfied****, and in that sense, Theorem 4.2 generalizes the results of [59] and [4] to conditional IQCs of this form.* ***This combination of the results from Chapter 3 along with Theorem 4.2 is the main original contribution of these two chapters.*** *Continued collaborative work on analysis with conditional IQCs is ongoing with my colleagues in the UC Berkeley research group (Packard, Meissen and Lessard).*

4.4 Interconnections of Many Systems with Many IQCs

In Sections 4.1-4.3, we have considered interconnections with a single Δ. For general interconnected systems, Δ in Figure 4.2 describes a collection of many subsystems interconnected through G and each satisfying a collection of IQCs.

Consider the system in Figure 4.5, which is specified by the equations

$$\begin{bmatrix} z \\ e \end{bmatrix} = G \begin{bmatrix} w \\ d \end{bmatrix}, \tag{4.5}$$

$$w_i = \Delta_i(z_i), i = 1, \ldots, N \tag{4.6}$$

where $G \in \mathbb{R}\mathbf{H}_\infty$ and the subscript i denotes the i'th component of a partitioned vector or diagonal system, the operators $\Delta_i : \mathbb{B}_1\mathbf{L}_2 \to \mathbf{L}_2$ are bounded and causal, each $\Delta_i \in \mathcal{I}(\Pi_{ij}, [\,], 1)$ for $j = 1, \ldots, K_i$, $z_i \in \mathbb{R}^{n_{zi}}$, $w_i \in \mathbb{R}^{n_{wi}}$, $n_z = \sum_{i=1}^N n_{zi}$, $n_w = \sum_{i=1}^N n_{wi}$, and $e \in \mathbb{R}^{n_e}$, $d \in \mathbb{R}^{n_d}$.

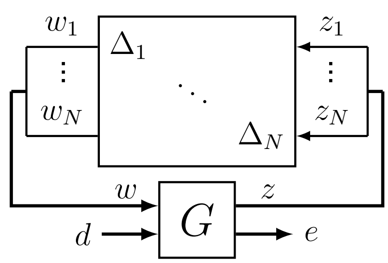

Figure 4.5: Feedback Interconnection of (G, Δ)

Again, we wish to verify if a performance IQC Π_p is satisfied for the exogenous input d to the output e, which is valid under some bound on $\|d\|$. We take a similar approach as before, by investigating the gain from d to any one of the components of z, z_i, by temporarily setting $e := z_i$. The system G_i embodies this error redefinition.

Frequency Domain Approach

Remark 4.3. *Block diagonal transformation of the system in Figure 4.5 yields an interconnection with a single Δ, which satisfies many IQCs. Thus, Theorem 4.2 can be applied to this interconnection after a basic manipulation.*

State-Space Approach

In previous sections, we focused on satisfying a performance IQC Π_p for the signals (d, e). In this section, we examine the satisfaction of a specific performance IQC, namely

$$\Pi_p = \begin{bmatrix} \gamma^2 I_{n_d} & 0 \\ 0 & -I_{n_e} \end{bmatrix}, \tag{4.7}$$

which encapsulates the gain bound property $\|d\|_2 \leq \gamma \|e\|_2$.

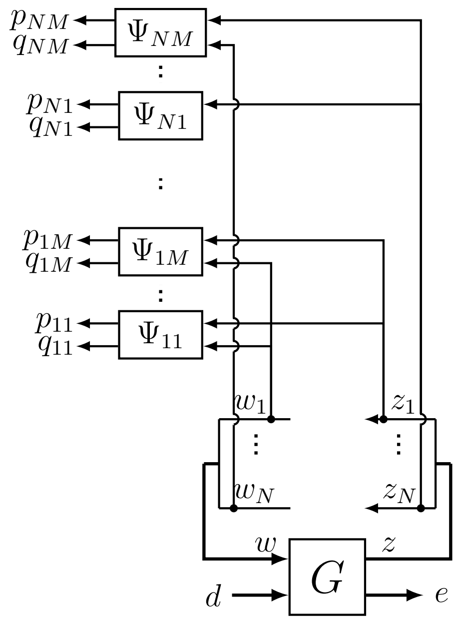

Figure 4.6: Interconnection with Δ_i replaced by the IQCs

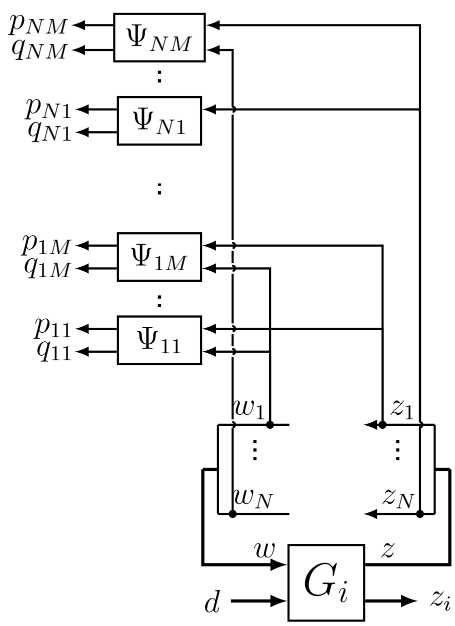

Figure 4.7: Interconnection with Δ_i replaced by the IQCs and $e := z_i$

We assume that the Π_{ij} can be factorized as

$$\Pi_{ij} = \Psi_{ij}^{\sim} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \Psi_{ij} \qquad (4.8)$$

$$\forall i = 1, \ldots, N, \forall j = 1, \ldots, M,$$

where $\Psi_{ij} \in \mathbb{R}\mathbf{H}_\infty$ with

$$\dot{x}_{\Psi_{ij}} = A_{ij}x_{\Psi_{ij}} + B_{1,ij}z_i + B_{2,ij}w_i \tag{4.9}$$

$$\begin{bmatrix} p_{ij} \\ q_{ij} \end{bmatrix} = C_{ij}x_{\Psi_{ij}} + D_{1,ij}z_i + D_{2,ij}w_i. \tag{4.10}$$

Let Δ denote the block diagonal concatenation of the Δ_i operators. Each Δ_i can be trivially extended to all of $\mathbf{L}_{2e}$ by defining

$$(\Delta_{e,i}(v))(t) = \begin{cases} (\Delta_i(u))(t) & \text{if } \|u\|_t \leq 1, \\ 0 & \text{if } \|u\|_t > 1. \end{cases} \tag{4.11}$$

Theorem 4.3. *Let $G(s)$ be a causal, finite dimensional, linear time-invariant operator and let $\{\Delta_i\}_{i=1}^N$ be causal, bounded operators mapping $\mathbb{B}_1\mathbf{L}_2 \to \mathbf{L}_2$. For each i, Δ_i locally (on $\mathbb{B}_1\mathbf{L}_2$) satisfies the IQCs defined by $\{\Pi_{ij}\}_{j=1}^M$. Each Ψ_{ij} is represented by a linear system, as in* (4.8)-(4.10)*. Let x be the state of G, and x_Ψ be the concatenated state of all Ψ_{ij}. Finally, let $\Delta_{e,i}$ denote the extension introduced in* (4.11)*. For notational simplicity, define $\Delta := diag\{\Delta_i\}$ as the block diagonal concatenation. Similarly, $\Delta_e := diag\ \{\Delta_{e,i}\}$. Assume $\alpha > 0, \beta > 0$ and*

1. *the interconnection of G and Δ_e, shown in Figure 4.8, is well-posed;*
2. *for each $1 \leq i \leq N$ there exist positive semidefinite quadratic function $V_i(x, x_\Psi) = \left[\begin{smallmatrix} x \\ x_\Psi \end{smallmatrix}\right]^* P_i \left[\begin{smallmatrix} x \\ x_\Psi \end{smallmatrix}\right]$ and $\{\lambda_{ijk}\} \geq 0$ such that the linear system shown in Figure 4.7 satisfies*

$$\dot{V}_i \leq \frac{1}{\beta^2}d^T d - z_i^T z_i + \sum_{j=1}^{N}\sum_{k=1}^{M} \lambda_{ijk}(q_{jk}^T q_{jk} - p_{jk}^T p_{jk}). \tag{4.12}$$

3. *there exist positive semidefinite quadratic function $V_0(x, x_\Psi) = \left[\begin{smallmatrix} x \\ x_\Psi \end{smallmatrix}\right]^* P_0 \left[\begin{smallmatrix} x \\ x_\Psi \end{smallmatrix}\right]$ and $\lambda_{0ij} \geq 0$ such that the linear system shown in Figure 4.6 satisfies*

$$\dot{V}_0 \leq \alpha^2 d^T d - e^T e + \sum_{j=1}^{N}\sum_{k=1}^{M} \lambda_{0jk}(q_{jk}^T q_{jk} - p_{jk}^T p_{jk}). \tag{4.13}$$

Then, the feedback interconnection of (G, Δ) is well-posed for all $d \in \mathbf{L}_2$ with $\|d\| < \beta$. Moreover, each z_i satisfies $\|z_i\| \leq 1$ and $\|e\| \leq \alpha\|d\|$.

Remark 4.4. *The inequalities in* (4.12) *and* (4.13) *are quadratic constraints on the variables (x, x_Ψ, d, w), parameterized by P_i. P_0, λ_{ijk}, and λ_{0ij}. Hence* (4.12) *and* (4.13) *are LMIs, [101], [12], in P and λ.*

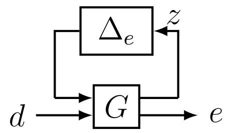

Figure 4.8: Feedback Interconnection of (G, Δ_e)

Proof. The proof is given for $\alpha = \beta = 1$. The interconnection of interest is (G, Δ). However, we initially quantify the behavior of the well-posed interconnection (G, Δ_e), shown in Figure 4.8 and, at the end of the proof, relate the solutions of (G, Δ_e) to (G, Δ). Let $d \in \mathbf{L}_{2e}$, $\|d\| < 1$. Since $d \in \mathbf{L}_{2e}$, unique solutions in $\mathbf{L}_{2e}$ exist. Suppose there exists a i such that $\|z_i\|_{\bar{T}} > 1$ at some $\bar{T} > 0$. Since $z_i \in \mathbf{L}_{2e}$, $\|z_i\|_T$ is a continuous, nondecreasing, function of T and is equal to 0 at $T = 0$. Therefore, at some $T_1 < \bar{T}$, there exists an index m (possibly equal to i) such that $\|z_m\|_{T_1} = 1$ and $\|z_k\|_{T_1} \leq 1$ for all $k \neq m$. Note that for all $T_2 \leq T_1$ and all k, $\|z_k\|_{T_2} \leq 1$. Therefore on the time interval $[0, T_2]$, the hard IQCs for each $\Delta_{e,i}$ are satisfied. Hence for all j, k

$$\|q_{jk}\|_{T_2} \leq \|p_{jk}\|_{T_2}. \tag{4.14}$$

From well-posedness of the interconnection of (G, Δ_e), we can integrate (4.12) with the initial condition $x(0) = 0$ and $x_\Psi(0) = 0$, yielding for all i

$$\|z_i\|_{T_2}^2 + V_i(x(T_2), x_\Psi(T_2)) \leq \|d\|_{T_2}^2 + \sum_{j=1}^{N} \sum_{k=1}^{M} \lambda_{ijk}(\|q_{jk}\|_{T_2}^2 - \|p_{jk}\|_{T_2}^2). \tag{4.15}$$

From the positive semidefiniteness of V_i and the hard IQC conditions in (4.14)

$$\|z_i\|_{T_2}^2 \leq \|d\|_{T_2}^2 \tag{4.16}$$

holds for all i. However, with $i = m$, we know $\|z_i\|_{T_2}^2 = 1$, which contradicts $\|d\| < 1$. The proof of $\|e\| \leq \|d\|$ follows similarly by integrating (4.13).

Summarizing, for (G, Δ_e), we have shown that $\|d\| < 1$ implies all $\|z_i\| \leq 1$ and $\|e\| \leq \|d\|$. However, we are ultimately interested in the interconnection of (G, Δ), in Figure 4.5. Since $\Delta_{e,i}|_{\mathbb{B}_1 \mathbf{L}_2} = \Delta_i|_{\mathbb{B}_1 \mathbf{L}_2}$, any fact about the solutions of (G, Δ_e), which satisfies $\|z_i\| \leq 1$ for all i, is also true for (G, Δ). $\square$

4.5 Example

We conclude with an example to illustrate the ideas of this Chapter. A 1 state, nonlinear system Δ_i is governed by

$$\dot{x}_i = -x_i + x_i^3 + z_i \tag{4.17}$$

$$w_i = \frac{1}{3} x_i, \tag{4.18}$$

which is locally stable, but can exhibit finite escape time solutions if $\|u\|_2 > 1$. We establish simple local IQCs of the form in Section 3.3. We choose a linear offset $L = \frac{1}{s+1}$, which is the linearization of the system, and variety of 11 stable, minimum phase weights W_i. Specifically, we choose 10 different band pass filters and the identity as the weights. Next, using the procedure in Section 2.9, we estimate the induced $\mathbf{L_2} \to \mathbf{L_2}$ gain of the locally stable operator $W_i(\Delta - L)$ for $i = 1, \ldots, 11$. The gain, which depends on the norm-bound of the input is 0 for arbitrarily small inputs, and goes to ∞ as the norm of the input is allowed to approach 1. Each analysis produces a gain curve. We fix ten different sample points, and find the corresponding gain from each curve. Each point sampled on the gain curves gives rise to a local IQC. The local IQCs generated can be downloaded from `http://jagger.me.berkeley.edu/~erin/IQC.mat`.

We consider a simple interconnection of three Δ_i systems along with a disturbance d and error e. Let $w_i = \Delta_i(z_i)$. The inputs and outputs of the systems are described by

$$\begin{aligned}
z_1 &= -\frac{1}{3}w_1 + \frac{1}{3}w_2 \\
z_2 &= \frac{1}{6}w_1 - \frac{1}{3}w_2 + \frac{1}{3}w_3 + d \\
z_3 &= \frac{1}{3}w_2 - \frac{1}{3}w_3 \\
e &= w_2.
\end{aligned}$$

We compare three approaches for estimating the $\mathbf{L_2} \to \mathbf{L_2}$ bound from d to e, shown in Figure 4.9

1. applying the frequency domain methods in Section 4.3
2. direct approach using methods in 2.9, using quadratic V and Q,
3. lower bound, worst-case simulation analysis using methods from [87].

In this example, the bound achieved using the methods in Section 4.3 outperform the direct approach. Our hope is that these methods will become applicable to large-scale systems. We are optimistic since we have shown promising results for smaller interconnections of locally stable systems.

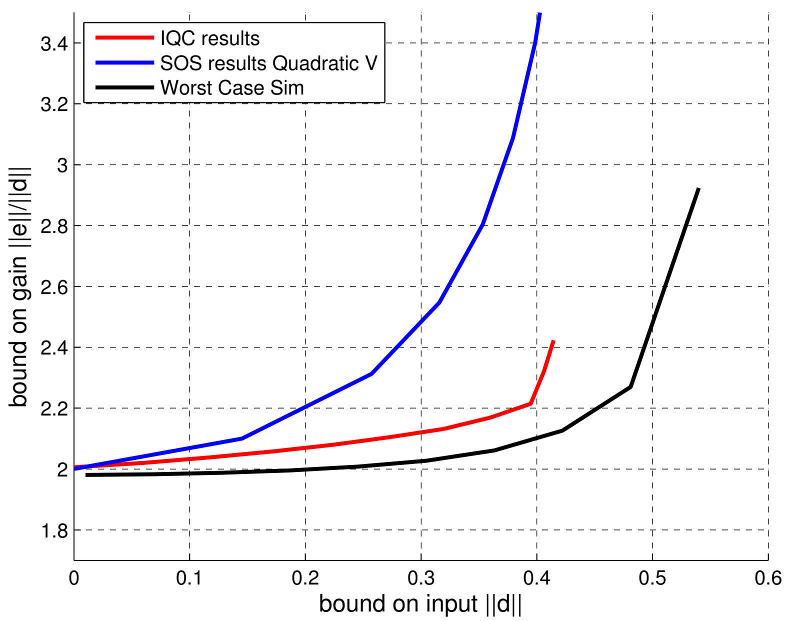

Figure 4.9: Analysis Results

Chapter 5

Delay Robustness of Interconnected Systems

5.1 Context and Acknowledgements

The ideas presented in this chapter embody an application of interconnections of systems which satisfy passivity and roll-off properties. This material is to appear in the IEEE Transactions on Automatic Control, co-authored by Murat Arcak and Andrew Packard, entitled "Delay Robustness of Interconnected Passive Systems: An Integral Quadratic Constraint Approach".

5.2 Introduction

Input-output notions in control theory, such as $\mathbf{L}_2$-stability and *passivity*, emerged from the far-reaching work of Zames [103] and Sandberg [75] who used these notions to formulate fundamental stability theorems for feedback systems. Extensions of these stability theorems to large-scale interconnected systems were developed by Willems [100], Moylan and Hill [61, 40], and Vidyasagar [95]. These stability theorems have permeated control theory and served as the starting point for numerous feedback design and analysis techniques [27, 77, 76]. Passivity, in particular, has been instrumental in nonlinear and adaptive control [55] and, more recently, in networked dynamical systems, such as multi-agent systems [5], biological networks [7], and communication networks [99].

Passivity is an abstraction of energy dissipation and is, thus, a practically relevant property that is inherent in numerous physical systems. However, analysis and design techniques that rely on passivity are often criticized for lack of stability guarantees in the presence of time delay. In contrast, a *small-gain* condition, which stipulates that the loop gain be smaller than one, guarantees stability for arbitrarily large delays in the feedback loop. This condition, however, fails to exploit the phase properties in the feedback loop, and may be overly conservative when the delay is small.

We derive stability conditions that converge to passivity estimates as the duration of delay approaches zero, and to small-gain estimates as the duration of delay approaches infinity. To accomplish this, we follow the IQC framework [59], [44] for stability of interconnections, and employ two IQCs simultaneously: The first one is an *output strict passivity* IQC, and the second one is a *roll-off* IQC that is frequency-dependent, and carries information about the time scales of the system it represents.

The roll-off IQC is indeed critical, because neither passivity nor an $\mathbf{L}_2$-gain description of a system encapsulates time-scale information. As an illustration, the first order dynamical system $\rho\dot{y} = -y + \gamma u$ is output strictly passive and has an $\mathbf{L}_2$-gain γ from input u to y regardless of the time constant $\rho > 0$. However, when this system is part of a feedback loop with delay T, a reasonable stability condition should restrict T/ρ, which is the magnitude of the delay relative to the time constant. The roll-off IQC introduced provides this essential time-scale information, and combines it with the gain and phase information encompassed by the output strict passivity IQC. The combination of these two IQCs then allows for sharp stability estimates that are sensitive to the duration of delay.

For a concrete demonstration of the advantage of the roll-off IQC, we study a *cyclic* interconnection structure for which a stability condition was derived in [7, 78] for the delay-free case, using output strict passivity properties of the subsystems. This bound is referred to as the "secant criterion," and has the form $\gamma \cos^n(\pi/n) < 1$ where n is the number of blocks in the feedback loop, and γ is the product of their gains. As an illustration, for $n = 3$ blocks, the secant criterion restricts the gain by $\gamma < \sec^3(\pi/3) = 8$. We first show that, in the presence of delay T, an application of the IQC stability theorem using only the output strict passivity IQC yields the small-gain condition $\gamma < 1$ regardless of the value of T. By including the roll-off IQC, we derive a new stability test in which the admissible gain γ is now a function of the delay T and the number of blocks, n. This function converges to the small-gain condition as $T \to \infty$ and to the secant criterion as $T \to 0$.

Next, we focus on the subsystems of the interconnection and discuss how to verify the output strict passivity and roll-off IQCs. In particular, we focus on an equilibrium-independent verification of these IQCs, since the equilibrium of the interconnection is sensitive to small perturbations in the subsystems and may not be accurately known. This is indeed a critical problem for biological networks where the parameters often exhibit wide variations, and for resource allocation algorithms in communication networks where the goal is to stabilize an optimal network equilibrium that is unknown to the users.

It is common in robust control, and specifically IQC literature, [48, 59], to treat known (and unknown) delays as uncertainties, usually "centered" at some finite-dimensional approximation (e.g., Pade), and employ a rational frequency-dependent IQC description, Π, to conservatively cover the difference. This is often done so that both the known (linear) interconnection system G, as well as the IQC multipliers remain rational. Rational G and Π are attractive so that the KYP lemma can be used to reformulate the frequency-domain inequality into a state-space linear matrix inequality (and avoid frequency gridding). We keep the delay as part of the interconnection system G, to avoid covering the delay with an IQC. Although this calls for frequency gridding in general, for special types of interconnections we derive exact solutions to the frequency-domain inequality.

The papers [49] and [31] implicitly use the idea of capturing magnitude roll-off at high frequencies. Unlike these references, which employ an IQC description of the delay element, we incorporate the delay within the known part of the plant, thereby avoiding undue conservatism. Work in [45] uses the IQC framework for network stability, but does not address time delays and relies on a critical symmetry assumption on the interconnections, which is avoided here.

5.3 Stability of Interconnected Output Strictly Passive Systems

Consider Figure 5.1, where G has the form:

$$G(s) = \sum_{k=1}^{M} G_k(s) e^{-sT_k} + G_0(s), \tag{5.1}$$

where G_k are proper, rational functions without poles in the closed right-half plane[1], and each Δ_i is a bounded, causal operator. The signals e and f are exogenous inputs that represent disturbances.

Our main interest is in the situation where Δ_i are SISO dynamical blocks representing the subsystems of a network, and G is $n \times n$ and represents their interconnection structure.

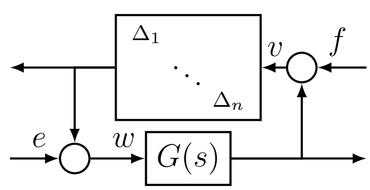

Figure 5.1: Feedback Interconnection of Δ and G. The Δ_i represent subsystems of a network and G represents the interconnection.

Consider the following IQC:

$$\Pi_{1,\gamma_1} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & -\frac{1}{\gamma_1} \end{bmatrix}, \tag{5.2}$$

$\gamma_1 > 0$, which encapsulates an *output strict passivity* (OSP) property:

$$\langle y, v \rangle - \frac{1}{\gamma_1} \|y\|_2^2 \geq 0. \tag{5.3}$$

[1] The authors in [59] consider a rational G, but the general form in (5.1) is admissible, as alluded to in [44].

In addition to passivity, this inequality implies an $\mathbf{L}_2$ gain of γ_1, as can be shown with a completion of squares argument.

Next we define

$$\Pi(j\omega) := \sum_{i=1}^{n} \alpha_i \Pi_{1,\gamma_1} \otimes e_i e_i^T = \Pi_{1,\gamma_1} \otimes P, \ \alpha_i \in \mathbb{R}, \tag{5.4}$$

where $P = \mathrm{diag}(\alpha_i)$. If each Δ_i satisfies the IQC defined by Π_{1,γ_1}, then the block diagonal concatenation $\Delta := \mathrm{diag}(\Delta_i)$ satisfies the IQC defined by (5.4) for any choice of $\alpha_i \geq 0$, $i = 1, \dots, n$.

From this point on, we assume that, for every $\kappa \in [0, 1]$, the interconnection of G and $\kappa\Delta$ is well-posed, as stipulated in [59]. The second condition in [59] is that, for every $\kappa \in [0, 1]$, the IQC defined by Π is satisfied by $\kappa\Delta$. The IQCs employed in this paper are structured such that if Δ satisfies the IQC, then so does $\kappa\Delta$, $\kappa \in [0, 1]$ (*cf.* [59, Remark 2]).

Proposition 5.1. *If there exists a $\epsilon > 0$ and a diagonal $P \in \mathbb{R}^{n\times n}$, $P \succ 0$, such that*

$$P(G(j\omega) - I) + (G(j\omega) - I)^* P \preceq -2\epsilon I \quad \forall \omega \in \mathbb{R}, \tag{5.5}$$

then the feedback interconnection of G and Δ is stable. □

Proof. We show that the proposition is equivalent to the third (and final) condition of the IQC stability theorem in [59]: If there exist $\epsilon > 0$ and $\alpha_i > 0$ with Π as in (5.4) such that

$$H_\Pi := \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \preceq -\epsilon I \ \forall \omega \in \mathbb{R} \tag{5.6}$$

holds, then the interconnection of G and Δ is stable.

To show this, we let $P := \mathrm{diag}(\alpha_i) \succ 0$ and note that

$$\begin{aligned} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} &= \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* (\Pi_{1,\gamma_1} \otimes P) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \\ &= \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} 0 & 0.5P \\ 0.5P & -\frac{1}{\gamma_1}P \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}. \end{aligned} \tag{5.7}$$

It then follows that condition (5.6) is equivalent to (5.5). □

5.4 Gain Roll-off Integral Quadratic Constraint

In order to evaluate the delay robustness of OSP systems interconnected as in Figure 5.1, we now introduce a roll-off IQC, which describes a reduction in the gain with increasing frequency. This IQC will be particularly useful when G contains delay elements, since the roll-off characterizes the time-constants within the Δ_i subsystems. The roll-off IQC has the form:

$$\Pi_{2,\tau,\gamma_2}(j\omega) := \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1+(\frac{\omega}{\omega_c})^2}{\gamma_2^2(1+\tau(\frac{\omega}{\omega_c})^2)} \end{bmatrix},$$

where $\gamma_2 > 0$, ω_c is the "corner frequency", and $0 < \tau \ll 1$ is introduced to render $\Pi_{2,\tau,\gamma_2}(j\omega)$ proper, as assumed in [59]. In the subsequent analysis, we will also refer to Π_{2,τ,γ_2} at the limit $\tau \to 0$, which we define as

$$\Pi_{2,\gamma_2}(j\omega) := \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1+(\frac{\omega}{\omega_c})^2}{\gamma_2^2} \end{bmatrix}.$$

If (3.1) holds for Π_{2,γ_2}, then (3.1) also holds for Π_{2,τ,γ_2} for any $\tau > 0$. Thus, if Δ_i satisfies Π_{2,τ,γ_2} for all $\tau > 0$, then we abbreviate the notation and say Δ_i satisfies Π_{2,γ_2}.

For systems that are both OSP and have roll-off, it is natural to use an IQC which combines Π_{1,γ_1} and Π_{2,γ_2}. Since Π_{1,γ_1} is a frequency-independent IQC, the time-scales of operators which satisfy Π_{1,γ_1} are not constrained. Likewise, the IQC Π_{2,γ_2} does not constrain the phase properties of operators. By combining both IQCs, we will be able to obtain less restrictive stability tests, particularly when $G(s)$ contains delay elements. Thus, we select the combined IQC:

$$\Pi(j\omega) := \sum_{i=1}^{n} (\alpha_{i1}\Pi_{1,\gamma_1} + \alpha_{i2}\Pi_{2,\gamma_2}(j\omega)) \otimes e_i e_i^T = \Pi_{1,\gamma_1} \otimes P_1 + \Pi_{2,\gamma_2} \otimes P_2, \tag{5.8}$$

where $P_1 = \mathrm{diag}(\alpha_{i1})$, $P_2 = \mathrm{diag}(\alpha_{i2})$, and search for $\alpha_{i1}, \alpha_{i2} \geq 0$ such that (5.6) holds with the form in (5.8). We refer to Π^τ as the combined IQC (5.8) when Π_{2,γ_2} is replaced with $\Pi_{2\tau,\gamma_2}$.

The condition in (5.6) can be evaluated for any Π by constructing a frequency grid for a finite set of ω and individually evaluating (5.6) for each ω in the grid. However, it may be numerically involved to employ a sufficiently dense grid for $\omega \in \mathbb{R}$. The following lemma proves that, in the special case where the delays T_k in (5.1) are commensurate[2], it is sufficient to check $H_\Pi \preceq -\epsilon I$ for a particular compact set of frequencies.

Lemma 5.1. *Suppose G_k in* (5.1) *are constant for $k = 0, \ldots, M$ and all of the delays T_k are commensurate so that there exists $\hat{T}$ such that $T_k = N_k \hat{T}$ for all $k = 1, \ldots, m$ and some $N_k \in$. Let H_Π be defined as in* (5.6) *with Π in* (5.8)*. Then, there exists an $\epsilon > 0$ such that $H_\Pi \preceq -\epsilon I$ holds for all $\omega \in \mathbb{R}$ if and only if $H_\Pi \preceq -\epsilon I$ holds for $\omega \in \Theta := \left[-\frac{\pi}{\hat{T}}, \frac{\pi}{\hat{T}}\right]$.*

[2]If all of the ratios between delays $\frac{T_k}{T_j}$ for $k, j = 1, \ldots, M$ are rational numbers, then the delays are said to be *commensurate* [35].

Proof. Let $P_1 = \operatorname{diag}(\alpha_{i1})$ and $P_2 = \operatorname{diag}(\alpha_{i2})$ and note that the combined IQC in (5.8) can be represented as

$$\Pi(j\omega) = \begin{bmatrix} P_2 & 0.5P_1 \\ 0.5P_1 & -\frac{1}{\gamma_1}P_1 - \frac{1+\left(\frac{\omega}{\omega_c}\right)^2}{\gamma_2^2}P_2 \end{bmatrix}.$$

Hence,

$$H_\Pi = G(j\omega)^* P_2 G(j\omega) + 0.5(G(j\omega)^* P_1 + P_1 G(j\omega)) - \frac{1}{\gamma_1}P_1 - \frac{1+\left(\frac{\omega}{\omega_c}\right)^2}{\gamma_2^2}P_2.$$

The two frequency dependent terms in H_Π are $G(j\omega)$, which is periodic, and the lower $n \times n$ block of Π, which is even and decreasing with increasing $\omega \in [0, \infty)$. Thus, $\lambda_{\max}(H_\Pi)$ will be achieved within the first period $\omega \in \Theta$. $\square$

We further show that, under the conditions of Lemma 5.1, verifying (5.6) with Π as in (5.8) is enough to ensure that (5.6) also holds for Π^τ when τ is sufficiently small. The use of Π yields a cleaner analysis that does not depend on the constant τ.

Lemma 5.2. *Suppose the conditions in Lemma 5.1 hold. If there exists $\epsilon > 0$, $\alpha_{i1}, \alpha_{i2} \geq 0$, such that $H_\Pi \preceq -\epsilon I$ for all $\omega \in \Theta$, then there exists a $\tilde{\tau} > 0, \tilde{\epsilon} > 0$, $\tilde{\alpha}_{i1}, \tilde{\alpha}_{i2} \geq 0$ such that*

$$H_{\Pi^\tau} \preceq -\tilde{\epsilon} I \tag{5.9}$$

holds for all $\omega \in \Theta$ and for all τ such that $0 < \tau \leq \tilde{\tau}$.

Proof. Assume there exists $\epsilon > 0, \alpha_{i1} \geq 0, \alpha_{i2} \geq 0$ such that $H_\Pi \preceq -\epsilon I$ for all $\omega \in \Theta$. Denote $\Pi^\tau(2,2)$ and $\Pi(2,2)$ as the lower right $n \times n$ blocks of Π^τ and Π. Let $\tilde{\alpha}_{i1} := \alpha_{i1}, \tilde{\alpha}_{i2} := \alpha_{i2}$ for $i = 1, \ldots, n$, and let

$$\hat{\alpha} = \max_i \ \tilde{\alpha}_{i2}.$$

Note that for any τ,

$$\begin{aligned} H_{\Pi^\tau} &= H_\Pi + \Pi^\tau(2,2) - \Pi(2,2) \quad \forall \omega, \text{ and} \\ H_{\Pi^\tau} &= H_\Pi \quad \text{for } \omega = 0. \end{aligned}$$

Moreover, for $\omega \in \{\Theta \setminus 0\}$, if

$$\tilde{\tau} < \frac{\gamma_2^2 \lambda_{\max}(-H_\Pi)}{\hat{\alpha}\left(\frac{\pi}{\hat{T}\omega_c}\right)^2\left(1 + \left(\frac{\pi}{\hat{T}\omega_c}\right)^2\right)}, \tag{5.10}$$

then

$$\overline{\sigma}\left(\Pi^\tau(2,2) - \Pi(2,2)\right) < \lambda_{\max}(-H_\Pi). \tag{5.11}$$

for any τ, such that $0 < \tau \leq \tilde{\tau}$. Note that the bound on $\tilde{\tau}$ in (5.10) is greater than 0. Hence, choosing $\tilde{\tau}$ such that (5.10) holds will preserve the negativity of $H_{\Pi^{\tilde{\tau}}}$ on $\omega \in \{\Theta \setminus 0\}$. Thus, for $\tilde{\alpha}_{i1} := \alpha_{i1}, \tilde{\alpha}_{i2} := \alpha_{i2}$ for $i = 1, \ldots, n$, $\tilde{\tau}$ in (5.10), there exists $\tilde{\epsilon} > 0$ and $\tau > 0$ such that (5.9) holds for all $\omega \in \Theta$ and for all τ such that $0 < \tau \leq \tilde{\tau}$. $\square$

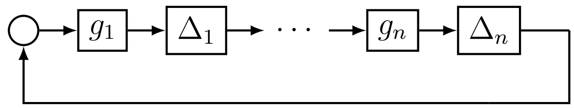

Figure 5.2: Interconnection of Δ and cyclic G. The gain, delay, and negative sign are distributed throughout the g_i blocks.

5.5 Example: Cyclic Interconnections with Delay

In this section we make the advantage of the combined IQC (5.8) explicit by studying a special interconnection structure whose stability properties in the absence of delay are characterized in [7, 78].

Let $G(j\omega)$ be of the form

$$G(j\omega) = \begin{bmatrix} 0 & 0 & \cdots & 0 & g_1(j\omega) \\ g_2(j\omega) & 0 & 0 & & 0 \\ 0 & g_3(j\omega) & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & g_n(j\omega) & 0 \end{bmatrix}, \tag{5.12}$$

where

$$g_i(j\omega) = \rho_i e^{j\beta_i(\omega)}, \ \rho_i > 0, \quad i = 1, \ldots, n \tag{5.13}$$

and $\beta_i(\omega)$ are real-valued functions such that

$$\prod_{i=1}^{n} \rho_i = \gamma \ \text{ and } \ \sum_{i=1}^{n} \beta_i(\omega) = \pi - \omega T. \tag{5.14}$$

Figure 5.2 represents the interconnection of Figure 5.1 with G defined as in (5.12). This interconnection structure plays an important role in biological oscillators (see [7] and references therein).

In this example, when applying Π_{1,γ_1} or Π_{2,γ_2} we will assume that γ_1 and γ_2 are equal. Under this assumption, we choose $\gamma_1 = \gamma_2 = 1$ without loss of generality, since we can modify $G(s)$ to absorb a different value of the gain. Let

$$\begin{aligned} \Pi_1 &:= \Pi_{1,1}, \\ \Pi_2 &:= \Pi_{2,1}. \end{aligned}$$

The phase condition in (5.14) means that this is a negative feedback loop with delay T. Now, we prove that the result of the IQC-based analysis depends only on the total gain γ and total delay T, and not the particular choice of each ρ_i and $\beta_i(\omega)$.

We present a lemma that will be used in the proof of Theorem 5.1.

Lemma 5.3. *Let G and $\tilde{G}$ represent two different cyclic matrices as in* (5.12) *such that*

$$\prod_{i=1}^{n} g_i(j\omega) = \prod_{i=1}^{n} \tilde{g}_i(j\omega), \quad \tilde{g}_i(j\omega), g_i(j\omega) \neq 0. \tag{5.15}$$

Then, there exists a diagonal nonsingular $D(j\omega) \in \mathbb{C}^{n\times n}$ such that

$$D(j\omega)^{-1}G(j\omega)D(j\omega) = \tilde{G}(j\omega). \tag{5.16}$$

Proof. Since (5.15) holds, choosing

$$d_1(j\omega) = 1, \tag{5.17}$$

$$d_i(j\omega) = d_{i-1}(j\omega)\frac{g_i(j\omega)}{\tilde{g}_i(j\omega)} \quad i = 2,\dots,n \tag{5.18}$$

and $D(j\omega) = \operatorname{diag}(d_1(j\omega),\dots,d_n(j\omega))$ provides a nonsingular, diagonal $D(j\omega)$ such that (5.16) holds. □

Theorem 5.1. *Let $G(j\omega)$ and $\tilde{G}(j\omega)$ represent two different cyclic matrices as in* (5.12) *and* (5.13)*, each with different choice of values for $g_i(j\omega)$ such that both matrices satisfy* (5.14) *with a common γ and T. Let $g_i(j\omega)$ and $\tilde{g}_i(j\omega)$ indicate the particular choice for $G(j\omega)$ and $\tilde{G}(j\omega)$. Let $\{\Pi_k(j\omega)\}_{k=1}^{p}$ represent an arbitrary set of IQCs.*

Define

$$\Pi(j\omega) = \sum_{k=1}^{p} \Pi_k(j\omega) \otimes P_k$$

and

$$\tilde{\Pi}(j\omega) = \sum_{k=1}^{p} \Pi_k(j\omega) \otimes \tilde{P}_k,$$

where $P_k = diag(\alpha_{ik}), \tilde{P}_k = diag(\tilde{\alpha}_{ik})$ for $i = 1,\dots,n$. There exist constants $\alpha_{ik} \geq 0$ and $\epsilon > 0$ such that for all $\omega \in \mathbb{R}$

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \preceq -\epsilon I, \tag{5.19}$$

if and only there if exist constants $\tilde{\alpha}_{ik} \geq 0$ and $\tilde{\epsilon} > 0$ such that for all $\omega \in \mathbb{R}$

$$\begin{bmatrix} \tilde{G}(j\omega) \\ I \end{bmatrix}^* \tilde{\Pi}(j\omega) \begin{bmatrix} \tilde{G}(j\omega) \\ I \end{bmatrix} \preceq -\tilde{\epsilon} I. \tag{5.20}$$

Proof. ($\Rightarrow$) From Lemma 5.3, we know that there exists a nonsingular, diagonal $D(j\omega)$ such that (5.16) holds. Moreover, note from (5.13), (5.17), (5.18), that $|d_i(j\omega)|$ are constant scalars. Thus, we use the notation $|d_i|$. Let $\overline{D}(j\omega) = I_2 \otimes D(j\omega)$. Since $D(j\omega)$ is nonsingular, multiplying (5.19) by $D(j\omega)$ will not effect the inequality constraint. Hence, the following holds for all ω:

$$D^*(j\omega)\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* (\overline{D}^*(j\omega))^{-1}\overline{D}^*(j\omega)\Pi(j\omega)$$
$$\overline{D}(j\omega)\overline{D}(j\omega)^{-1}\begin{bmatrix} G(j\omega) \\ I \end{bmatrix} D(j\omega) \preceq -\epsilon D^*(j\omega)D(j\omega). \quad (5.21)$$

The right hand term $-\epsilon D^*(j\omega)D(j\omega) = -\epsilon|d_i(j\omega)|^2 I$ is a constant, negative definite, diagonal matrix. Let $\tilde{\epsilon} = \epsilon \max(|d_i|^2)$. Since $G(j\omega)$ and $\tilde{G}(j\omega)$ are similar for $D(j\omega)$, (5.21) implies

$$\begin{bmatrix} \tilde{G}(j\omega) \\ I \end{bmatrix}^* \overline{D}^*(j\omega)\Pi(j\omega)\overline{D}(j\omega)\begin{bmatrix} \tilde{G}(j\omega) \\ I \end{bmatrix} \preceq -\tilde{\epsilon} I \quad (5.22)$$

holds for all ω. Expanding $\Pi(j\omega)$, $\overline{D}(j\omega)$ and rearranging yields

$$\begin{bmatrix} \tilde{G}(j\omega) \\ I \end{bmatrix}^* \sum_{k=1}^{p} \Pi_k(j\omega) \otimes \left(D^*(j\omega)P_k D(j\omega)\right)\begin{bmatrix} \tilde{G}(j\omega) \\ I \end{bmatrix} \preceq -\tilde{\epsilon} I \quad (5.23)$$

holds for all ω. Let $D^*(j\omega)P_kD(j\omega) = \tilde{P}_k$. Since $D(j\omega)$ and P_k are diagonal, $\tilde{P}_k = \operatorname{diag}(|d_i(j\omega)|^2\alpha_{ik})$ for $i = 1, \dots n$. Since, the $|d_i|$ terms are constant scalars, we remove the dependency on ω and note that $\tilde{P}_k = \operatorname{diag}(|d_i|^2\alpha_{ik})$. Hence, (5.23) holds if and only if

$$\begin{bmatrix} \tilde{G}(j\omega) \\ I \end{bmatrix}^* \sum_{k=1}^{p} \Pi_k(j\omega) \otimes \tilde{P}_k \begin{bmatrix} \tilde{G}(j\omega) \\ I \end{bmatrix} \preceq -\tilde{\epsilon} I$$

holds for all ω. Therefore, $\tilde{\epsilon} = \epsilon \max(|d_i|^2)$ and $\tilde{\alpha}_{ik} := |d_i|^2\alpha_{ik}$ are the appropriate, constant positive and non-negative multiples for the condition in (5.20) to hold for all ω.
($\Leftarrow$) From the symmetry, given the multipliers for (5.20), the multipliers of (5.19) can be recovered by the same argument.

□

Now we will study how delay affects the stability of the cyclic interconnection in (5.12). We first consider the case $T = 0$ and recall the following stability test from [7]:

Theorem 5.2. *The cyclic feedback interconnection of Δ and G is stable for all Δ satisfying the IQC defined by Π_1 if and only if*

$$\gamma \cos\left(\frac{\pi}{n}\right)^n < 1. \quad (5.24)$$

Although [7] did not use the IQC formalism, the stability criterion was identical to (5.5) with G as in (5.12)-(5.14) and $T = 0$. The existence of a diagonal $P \succ 0$ satisfying (5.5) was shown in [7, Theorem 1] to be equivalent to (5.24).

Now we consider the case $T \neq 0$, and show that employing the IQC Π_1 alone yields a conservative result that is independent of the duration of the delay.

Theorem 5.3. *Suppose $T > 0$ and each Δ_i satisfies the IQC defined by Π_1. There exists a diagonal $P \succ 0$ and $\epsilon > 0$ such that $H_\Pi \preceq -\epsilon I$ holds for all $\omega \in \mathbb{R}$ with $\Pi = \Pi_1$ if and only if $\gamma < 1$.*

Proof. Using Theorem 5.1, we choose $G(j\omega)$ such that

$$\begin{aligned} \rho_1 = \gamma, \quad \beta_1(\omega) = \pi - \omega T, \\ \rho_i = 1, \quad \beta_i(\omega) = 0, \quad i \geq 2, \end{aligned}$$

that is, $g_1(j\omega) = -\gamma e^{-j\omega T}$ and $g_i(j\omega) = 1, i \geq 2$.

($\Rightarrow$ Contradiction) Suppose $P \succ 0$ exists and $\gamma \geq 1$. At $\bar{\omega} = \frac{\pi}{T}$, $G(j\bar{\omega}) - I$ is a Metzler matrix of the form:

$$G(j\bar{\omega}) - I = \begin{bmatrix} -1 & 0 & \cdots & \gamma \\ 1 & -1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & 0 & 1 & -1 \end{bmatrix}. \tag{5.25}$$

From [10, Theorem 2.3], there exists an $\epsilon > 0$ and $P \succ 0$ such that $P[G(j\bar{\omega}) - I] + [G(j\bar{\omega}) - I]^* P \preceq -\epsilon I$ if and only if the principal minors of $-(G(j\bar{\omega} - I))$ are all positive. All principal minors of $-(G(j\bar{\omega} - I))$, except the minor which is the determinant of the matrix itself, are 1. The remaining principal minor, which is

$$\det(-(G(j\bar{\omega}) - I)) = 1 - \gamma(-1)^{n+1}(-1)^{n-1} = 1 - \gamma(-1)^{2n} = 1 - \gamma,$$

is positive only if $\gamma < 1$. Hence, when $\gamma \geq 1$ at $\omega = \bar{\omega}$ there does not exist a P such that (5.7) holds.

($\Leftarrow$) Assume $\gamma < 1$. We will first show that a P exists at $\bar{\omega} = \frac{\pi}{T}$, and then show that this P can be used for any value of ω. At $\omega = \bar{\omega}$, $G(j\bar{\omega}) - I$ is a Metzler matrix of the form in (5.25). Hence, since $\gamma < 1$, there exists a positive definite $P = \mathrm{diag}(p_1, \ldots, p_n)$ and $\epsilon > 0$ such that

$$x^*[P(G(j\bar{\omega}) - I) + (G(j\bar{\omega}) - I)^* P]x \leq -\epsilon |x|^2 \tag{5.26}$$

holds for all $x \in \mathbb{C}^n$. Expanding (5.26) yields the condition

$$\begin{aligned} &- p_1|x_1|^2 - \ldots - p_n|x_n|^2 + 0.5p_2(x_1^* x_2 + x_2^* x_1) + \ldots + 0.5p_n(x_{n-1}^* x_n + x_n^* x_{n-1}) \\ &\qquad + 0.5\gamma p_1(x_1^* x_n + x_n^* x_1) \leq -\epsilon |x|^2. \end{aligned} \tag{5.27}$$

Since (5.27) holds for all $x \in \mathbb{C}^n$, it also holds for all $x_i = |y_i|$ for $y \in \mathbb{C}^n$. Hence, it is clear that (5.27) implies that

$$- p_1|y_1|^2 - \ldots - p_n|y_n|^2 + \ldots + p_2|y_1||y_2| + \ldots + p_n|y_{n-1}||y_n| + \gamma p_1|y_1||y_n| \leq -\epsilon |y|^2 \tag{5.28}$$

for all $y \in \mathbb{C}^n$. Now we consider $y^*[P(G(j\omega)-I)+(G(j\omega)-I)^*P]y$ for all $\omega \in \mathbb{R}$. Expanding $y^*[P(G(j\omega)-I)+(G(j\omega)-I)^*P]y$ yields

$$\begin{aligned}&- p_1|y_1|^2 - \ldots - p_n|y_n|^2 + \ldots + 0.5p_2(y_1^*y_2 + y_2^*y_1) + \ldots \\ &\qquad + 0.5p_n(y_{n-1}^*y_n + y_n^*y_{n-1}) - 0.5\gamma p_1(e^{-j\omega T}y_1^*y_n + e^{j\omega T}y_n^*y_1),\end{aligned}$$

which is upper bounded by (5.28). Hence, there exists a positive definite $P = \mathrm{diag}(p_1, \ldots, p_n)$ and $\epsilon > 0$ such that (5.6) holds for all $\omega \in \mathbb{R}$. □

Note that $\gamma < 1$ in Theorem 5.3 is a small-gain condition. Since this condition does not depend on the duration of the delay, one would anticipate that using Π_1 alone may lead to a conservative result. We will present a theorem which shows that combining Π_1 with the roll-off IQC Π_2 gives a relaxed condition that depends on the delay T. To simplify the discussion, we assume that each of the Δ_i subsystems in the cyclic interconnection satisfies the IQC Π_2 with the same corner frequency ω_c.

We present Lemmas 5.4-5.6, which are used in the final proof of Theorem 5.4. We define

$$f(\omega) := 1 - \gamma^{1/n}\cos\left(\frac{\pi}{n} - \frac{\omega T}{n}\right) - \alpha(\gamma^{2/n} - (1+\omega^2)), \tag{5.29}$$

$$\overline{\omega} := \sqrt{\gamma^{2/n} - 1}, \tag{5.30}$$

$$\overline{T} := \frac{\pi - n\arctan(\overline{\omega})}{\overline{\omega}}, \tag{5.31}$$

$$\overline{\alpha} := \frac{\gamma^{1/n}\overline{T}}{2n\overline{\omega}}\sin\left(\frac{\pi - \overline{T}\overline{\omega}}{n}\right), \tag{5.32}$$

which are used in the following Lemmas.

Lemma 5.4. *If $\gamma > 1$, $\alpha = \overline{\alpha}$ and $T = \overline{T}$, then for f in (5.29)*

$$\operatorname*{argmin}_{\omega}\ f(\omega) = \overline{\omega}.$$

Proof. The lemma is proven true by inspecting the first, second and third derivatives of f at $\overline{\omega}$. □

Lemma 5.5. *If $\gamma > 1$, $\alpha = \overline{\alpha}$ and $T = \overline{T}$, then $\forall \omega \in \left[0, \frac{\pi}{T}\right]$*

$$f(\omega) \geq 0.$$

Proof. Since $f(\overline{\omega}) = 0$ and $\overline{\omega}$ is the global minimum by Lemma 5.4, $f(\omega) \geq 0 \quad \forall \omega \in \left[0, \frac{\pi}{T}\right]$. □

Lemma 5.6. *If $\gamma > 1$, for any $\hat{T}$ such that $0 \leq \hat{T} < \overline{T}$,*

$$\overline{\alpha}\omega^2 - \gamma^{1/n}\cos\left(\frac{\omega\hat{T} - \pi}{n}\right) + 1 + \overline{\alpha} - \overline{\alpha}\gamma^{2/n} > 0 \tag{5.33}$$

for all $\omega \in \left[0, \frac{\pi}{T}\right]$.

Proof. The proof follows by applying Lemma 5.5 and inspecting (5.33) after substituting in for $y = \overline{T}\omega$. □

Theorem 5.4. *For any $\gamma > 1$, the cyclic interconnection of G and Δ is stable for all Δ satisfying the IQCs defined by Π_1 and Π_2 with corner frequency ω_c if and only if*

$$\omega_c T < \frac{\pi - n \arctan\left(\sqrt{\gamma^{\frac{2}{n}} - 1}\right)}{\sqrt{\gamma^{\frac{2}{n}} - 1}}. \tag{5.34}$$

Proof. Without loss of generality, we prove the theorem for $\omega_c = 1$. This is because one can define the dimensionless time variable $t' = \omega_c t$ and, thus, take $\omega_c = 1$.
($\Rightarrow$) The condition in (5.34) is the time delay margin for a cascade of identical linear systems $\Delta_i = \frac{1}{(s+1)}$ for $i = 1, \ldots, n$, in feedback with gain γ. This is a particular system that satisfies the IQCs defined by Π_1 and Π_2 at $\omega_c = 1$, which proves the necessity of (5.34).
($\Leftarrow$) Using Theorem 5.1, we choose $G(j\omega)$ such that

$$g_i(j\omega) := g(j\omega) \triangleq \gamma^{\frac{1}{n}} e^{j\left(-\frac{\omega T}{n} + \frac{\pi}{n}\right)}, \quad i = 1, \ldots, n. \tag{5.35}$$

With $\alpha_{i1} = 1$ and $\alpha_{i2} = \alpha$, $i = 1, \cdots, n$, the IQC stability condition $H_\Pi \preceq -\epsilon I$ with the combined IQC (5.8) can be written as:

$$\frac{1}{2}(G(j\omega) - I) + \frac{1}{2}(G(j\omega) - I)^* + \alpha(|g(j\omega)|^2 - (1 + \omega^2))I \preceq -\epsilon I \quad \forall\omega. \tag{5.36}$$

Let $\Theta := \left[-\frac{\pi}{T}, \frac{\pi}{T}\right]$. From Lemma 5.1, the inequality (5.36) need only hold for $\forall\omega \in \Theta$.

Since

$$(G(j\omega) - I) + (G(j\omega) - I)^* \tag{5.37}$$

is a circulant matrix, its eigenvectors are [34]:

$$v_k = [1 \; e^{-j(k-1)\frac{2\pi}{n}} \; e^{-j(k-1)2\frac{2\pi}{n}} \cdots \; e^{-j(k-1)(n-1)\frac{2\pi}{n}}]^T \tag{5.38}$$

for $k = 1, \ldots n$ and, thus, the eigenvalues are the discrete Fourier transform coefficients of the first row which, for (5.37), are:

$$\lambda_k(j\omega) = -2 + g(j\omega)^* e^{-j(k-1)\frac{2\pi}{n}} + g(j\omega) e^{-j(k-1)(n-1)\frac{2\pi}{n}} \tag{5.39}$$

for $k = 1, \ldots n$.

Defining the matrix $V = [v_1 \cdots v_n]$ and noting that $V^{-1} = \frac{1}{n}V^*$, we conclude:

$$\frac{1}{n}V^*[(G(j\omega) - I) + (G(j\omega) - I)^*]V = \mathrm{diag}(\lambda_1(j\omega), \ldots, \lambda_n(j\omega)).$$

Multiplying (5.36) from the right by V and from the left by $\frac{1}{n}V^*$ yields a diagonal matrix on the left side of the inequality and does not effect the right side since $\frac{1}{n}V^*V = I$. Thus, the condition in (5.36) becomes

$$\frac{1}{2}\lambda_k(j\omega) + \alpha(\gamma^{2/n} - (1+\omega^2)) \leq -\epsilon, k = 1, \cdots, n \quad \forall \omega \in \Theta. \tag{5.40}$$

Substituting (5.35) into (5.39) and simplifying yields

$$\lambda_k(j\omega) = -2 + 2\gamma^{1/n}\cos\left(\frac{\pi}{n} + (k-1)\frac{2\pi}{n} - \frac{\omega T}{n}\right) \tag{5.41}$$

for $k = 1, \cdots, n$. We rewrite (5.40) as:

$$h(\omega,k) := -1 + \gamma^{1/n}\cos\left(\frac{\pi}{n} + (k-1)\frac{2\pi}{n} - \frac{\omega T}{n}\right) + \\ \alpha(\gamma^{2/n} - (1+\omega^2)) \leq -\epsilon k = 1, \cdots, n, \forall \omega \in \Theta.$$

Note that for $\omega \in \Theta$, $\frac{\omega T}{n} \in \left[-\frac{\pi}{n}, \frac{\pi}{n}\right]$. For $\omega \in \left[0, \frac{\pi}{T}\right]$, $h(\omega, k)$ has the largest value when $k = 1$. For $\omega \in \left[\frac{-\pi}{T}, 0\right]$, $h(\omega,k)$ has the largest value when $k = n$. However, note that $h(\omega, 1) = h(-\omega, n)$. Thus, we can restrict the range of interest to $\omega \in \left[0, \frac{\pi}{T}\right]$. Let

$$f(\omega) := -h(\omega, 1) = 1 - \gamma^{1/n}\cos\left(\frac{\pi}{n} - \frac{\omega T}{n}\right) - \alpha(\gamma^{2/n} - (1+\omega^2)). \tag{5.42}$$

Thus, if there exists an $\tilde{\epsilon} > 0$ such that $f(\omega) \geq \tilde{\epsilon}$ for all $\omega \in \left[0, \frac{\pi}{T}\right]$, then there exists an $\epsilon > 0$ such that $H_\Pi \preceq -\epsilon I$ for all $\omega \in \mathbb{R}$.

Note that (5.42) and (5.29) are equivalent. If $T < \overline{T}$, $\alpha = \overline{\alpha}$ and

$$\epsilon := \min_{\omega} f(\omega), \tag{5.43}$$

then by Lemma 5.6, $f(\omega) \to \infty$ as $\omega \to \infty$, $f(\omega) > 0$ $\forall \omega \in \mathbb{R}$, by our choice of T and α. Hence, there exists an $\epsilon > 0$ and $\alpha_{i1} := 1, \alpha_{i2} := \alpha$ for $i = 1, \ldots, n$, such that $H_\Pi \preceq -\epsilon I$ holds for all $\omega \in \left[0, \frac{\pi}{T}\right]$. Furthermore, by the symmetry of λ_k and Lemma 5.1, $H_\Pi \preceq -\epsilon I$ holds for all $\omega \in \mathbb{R}$.

□

Note that, when $T = 0$, (5.34) recovers the secant criterion in Theorem 5.2. Likewise, as $T \to \infty$, the small gain condition in Theorem 5.3 is recovered.

Moreover, we emphasize that condition in (5.34) is the exact stability bound for a cascade of identical linear systems $\Delta_i = \frac{\omega_c}{s+\omega_c}$ for $i = 1, \ldots, n$, in negative feedback with gain γ, as one can verify using classical Nyquist analysis criterion. Since this choice of Δ_i satisfies the IQCs defined by Π_1 and Π_2, the stability bound (5.34) is tight and cannot be relaxed without further assumptions on the Δ_i subsystems.

The special case $n = 2$ is of particular interest, as it relates to the classical Passivity Theorem [75, 103]. This theorem states that, in the absence of delay, the negative feedback interconnection of two passive systems is stable without any restriction on the loop gain γ. Theorem 5.4 shows how this gain must be restricted to accommodate the delay for a duration T in the feedback loop.

As a further example, in Figure 5.3 we show the bound on the gain γ as a function of the delay for a cyclic interconnection of $n = 3$ OSP systems with roll-off at $\omega_c = 1$. As $T \to 0$, the bound converges to 8, which is the secant criterion for $n = 3$ and as $T \to \infty$, the bound converges to 1, which is the small-gain condition.

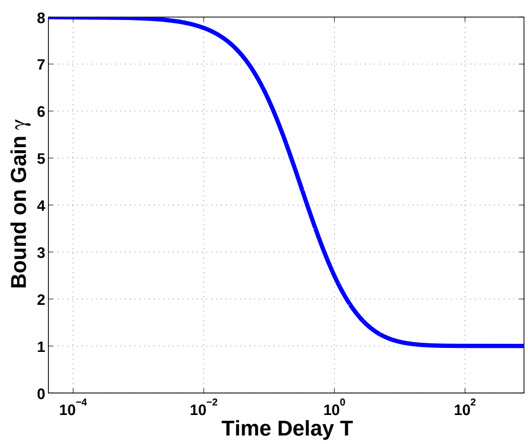

Figure 5.3: A stability bound on γ as a function of delay T for $n = 3$ from (5.34). The secant criterion is recovered as $T \to 0$ and the small-gain condition is recovered as $T \to \infty$.

5.6 Application to State Space Models

A series of recent publications presented a passivity approach for overcoming the complexity of high-order differential equation models arising in communication networks [99, 32], cooperative robotic vehicles [5, 42], and biochemical reaction networks [7, 6]. This approach decomposes the network into passive components and applies a stability test that is equivalent to the IQC test of this paper with Π_{1,γ_1} only. We now generalize this method to systems with time delays and incorporate the roll-off IQC Π_{2,γ_2} in the stability analysis.

Let Δ_i refer to a dynamical system of the form:

$$\begin{aligned} \dot{x}_i &= f_i(x_i, u_i) \\ y_i &= h_i(x_i), \end{aligned}$$

for $i = 1, \ldots, n$, where $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}$, $y_i(t) \in \mathbb{R}$. Let $N = \sum_{i=1}^{n} n_i$ and define $u \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $x \in \mathbb{R}^N$, by $u^T = [u_1 \ \ldots \ u_n]$, $y^T = [y_1 \ \ldots \ y_n]$, $x^T = [x_1^T \ \ldots \ x_n^T]$. The interconnection of these subsystems is described by the feedback law

$$u(t) = G_0 y(t) + \sum_{k=1}^{M} G_k y(t - T_k),$$

where G_k are constant for $k = 0, \ldots, M$. Assume that the interconnected system possesses an equilibrium $x^\star$, and let $x_i^\star$ denote the part corresponding to the *ith* subsystem. Let:

$$\begin{aligned}
y_i^\star &:= h_i(x_i^\star),\\
(y^\star)^T &:= [y_1^\star \ \ldots \ y_n^\star],\\
u^\star &:= \left(G_0 + \sum_{k=1}^{M} G_k\right) y^\star,\\
\bar{u}_i &:= u_i - u_i^\star,\\
\bar{y}_i &:= y_i - y_i^\star,\\
\bar{x}_i &:= x_i - x_i^\star.
\end{aligned}$$

Let $\bar{\Delta}_i$ represent the system with inputs $\bar{u}_i$, states $\bar{x}_i$, and outputs $\bar{y}_i$

$$\begin{aligned}
\dot{\bar{x}}_i(t) &= f(x_i^\star + \bar{x}_i(t), u_i^\star + \bar{u}_i(t))\\
\bar{y}_i(t) &= h(x_i^\star + \bar{x}_i(t)).
\end{aligned}$$

In order to apply the IQC test discussed in this paper, we need to verify the OSP and roll-off IQCs for each of the $\bar{\Delta}_i$ blocks, whose definitions depend on the network equilibrium $x^\star$ as shown above. However, the equilibrium of a network depends on the parameters of the subsystems. In many of our motivating applications, such as biological reaction networks and the Internet congestion control problem, the equilibrium of the network is not known *a priori* and it is essential to verify these IQCs without relying on the knowledge of $x^\star$. A procedure for equilibrium-independent verification is presented next.

5.7 Equilibrium-Independent Verification of Integral Quadratic Constraints

We now study the equilibrium-independent verification of Π_{1,γ_1} and Π_{2,γ_2}. In the previous sections we assumed that the Δ_i blocks are single-input single-output blocks for notational simplicity. However, this assumption is not essential, and identical results hold for m-input m-output blocks if G_k, $k = 0, \ldots, M$ in (5.1) are replaced with $G_k \otimes I_m$ and the IQCs Π_i, $i = 1, 2$ are replaced with $\Pi_i \otimes I_m$. Thus, in this section we study m-input m-output blocks for further generality.

Let Δ refer to a dynamical system of the form:

$$\dot{x} = f(x,u) \tag{5.44}$$

$$y = h(x), \tag{5.45}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$. We assume that for all $u^\star \in \mathbb{R}^m$ there exists a unique $x^\star \in \mathbb{R}^n$ such that $f(x^\star, u^\star) = 0$. We recall the following definition from [41]:

Definition 5.1. *Δ is said to be output strictly equilibrium-independent passive (OSEIP) with gain $\gamma_1 > 0$ if for every $u^\star \in \mathbb{R}^m$, there exists a once-differentiable storage function $S_{u^\star} : \mathbb{R}^n \to \mathbb{R}$ such that $S_{u^\star}(x) > 0 \ \forall x \neq x^\star$, $S_{u^\star}(x^\star) = 0$, and*

$$\nabla_x S_{u^\star} \cdot f(x,u) \leq (u-u^\star)^T(y-y^\star) - \frac{1}{\gamma_1}(y-y^\star)^T(y-y^\star) \tag{5.46}$$

for all $u \in \mathbb{R}^m, x \in \mathbb{R}^n$, where $y^\star = h(x^\star)$. □

By integrating (5.46) with respect to time from 0 to ∞, and assuming $\bar{x}(0) = 0$, it is not difficult to verify that (5.3) holds for the input-output pair $\bar{u}, \bar{y}$, which means that $\bar{\Delta}$ satisfies the IQC defined by $\Pi_{1,\gamma_1} \otimes I_m$. Clearly, the assumption $\bar{x}(0) = 0$ must be eliminated for stability analysis of the interconnected system in Figure 5.1. This can be done by assuming an appropriate *reachability* property for the interconnected system, which is a standard approach in going from input-output to state space stability [96, Section 6.3], [79]. In this approach one treats the system as if it had zero initial conditions and uses reachability to prescribe bounded and finite-duration (therefore, $\mathbf{L}_2$) exogenous signals that bring the state to the actual, non-zero initial condition. One then uses the input-output stability of the interconnection to conclude that the internal signals are in $\mathbf{L}_2$, and employs additional reachability or detectability conditions for the subsystems to conclude that the states converge to zero.

In order to verify that $\bar{\Delta}$ satisfies Π_{2,γ_2}, we cascade $\bar{\Delta}$ with a linear system whose gain "rolls-up" with corner frequency ω_c, illustrated in Figure 5.4. Next, we estimate the gain from the input $\bar{u}$ to the output $\bar{z}$ of the roll-up system. A bounded $\mathbf{L}_2$ gain from $\bar{u}$ to $\bar{z}$ implies that $\bar{\Delta}$ satisfies Π_{2,γ_2} with corner frequency ω_c.

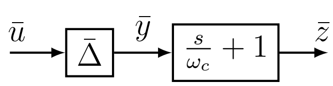

Figure 5.4: Cascade of Δ with roll-up. If the gain from $\bar{u}$ to $\bar{z}$ is bounded, then $\bar{\Delta}$ rolls off at ω_c.

The state equations for the new cascaded system are

$$\dot{\bar{x}} = f(x^\star + \bar{x}, u^\star + \bar{u}) \tag{5.47}$$

$$\bar{z} := \frac{\dot{\bar{y}}}{\omega_c} + \bar{y} \tag{5.48}$$

To verify the roll-off IQC, we bound the $\mathbf{L}_2$ gain of the cascaded system dynamics (5.47)-(5.48) from $\bar{u}$ to $\bar{z}$.

Definition 5.2. Δ *in* (5.44)-(5.45) *is said to have equilibrium-independent roll-off with corner frequency* $\omega_c > 0$ *and gain* $\gamma_2 > 0$ *if for every* $u^\star \in \mathbb{R}^m$*, there exists a once-differentiable storage function* $V_{u^\star}(x) : \mathbb{R}^n \to \mathbb{R}$ *such that* $V_{u^\star}(x) > 0\ \forall x \neq x^\star$, $V_{u^\star}(x^\star) = 0$*, and*

$$0 \leq -\nabla V_{u^\star} f(x,u) + \gamma_2^2 (u-u^\star)^T (u-u^\star) - \bar{z}^T \bar{z} \tag{5.49}$$

$\forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m$*, where*

$$\bar{z} = \frac{\partial h}{\partial x} f(x,u) \frac{1}{\omega_c} + y - y^\star.$$

We notate this as EIRO(γ_2, ω_c*).*

Thus, the EIRO(γ_2, ω_c) property may be used to show that $\bar{\Delta}$ satisfies $\Pi_{2,\gamma_2} \otimes I_m$ with corner frequency ω_c regardless of where the equilibrium is located.

As a special case of practical interest, we focus on the nonlinear system Δ with dynamics

$$\dot{x} = -\beta x - \phi(x) + u \tag{5.50}$$

$$y = x, \tag{5.51}$$

where $x \in \mathbb{R}$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, $\beta > 0$, $\phi(x) : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing so that for all $(x, x^\star) \in \mathbb{R} \times \mathbb{R}$,

$$(x - x^\star)(\phi(x) - \phi(x^\star)) \geq 0. \tag{5.52}$$

Claim 5.1. $\bar{\Delta}$ *satisfies* $\Pi_{1,\frac{1}{\beta}}$*.*

Proof. Consider the storage function

$$S_{u^\star}(x) := \frac{1}{2}(x - x^\star)^2. \tag{5.53}$$

Clearly $S_{u^\star}(x) > 0$ for all $x \neq x^\star$ and $S_{u^\star}(x^\star) = 0$. We show that the condition in (5.46) holds for all $x \in \mathbb{R}, u \in \mathbb{R}$ for the system Δ using the storage function (5.53).

We subtract $f(x^\star, u^\star)$, which is equal to zero, from (5.50) and multiply by $\nabla S_{u^\star}(x) = \bar{x}$, which yields

$$\begin{aligned}&\nabla S_{u^\star}(x)\left(f(x,u) - f(x^\star,u^\star)\right) \\ &\qquad = \bar{x}(-\beta\bar{x} - \phi(x) + \phi(x^\star) + \bar{u}) \leq -\beta\bar{x}^2 + \bar{x}\bar{u}\end{aligned}$$

from (5.52). Thus, Δ is OSEIP with $\gamma_1 = \frac{1}{\beta}$. □

Claim 5.2. $\bar{\Delta}$ *satisfies* $\Pi_{2,\frac{1}{\beta}}$ *with* $\omega_c > \beta$*.*

Proof. After cascading $\bar{\Delta}$ with the roll-up with ω_c, the new dynamics are

$$\dot{\bar{x}} = -\beta\bar{x} - (\phi(x) - \phi(x^\star)) + \bar{u} \tag{5.54}$$

$$\bar{z} = \frac{1}{\omega_c}\dot{\bar{x}} + \bar{x} \tag{5.55}$$

$$= \left(1 - \frac{\beta}{\omega_c}\right)\bar{x} - \frac{\phi(x) - \phi(x^\star)}{\omega_c} + \frac{\bar{u}}{\omega_c}. \tag{5.56}$$

With the choice of storage function

$$V_{u^\star}(x) = 2\gamma_2^2 \int_{x^\star}^{x} [\phi(y) - \phi(x^\star)]dy + \left(\beta\gamma_2^2 - \frac{1}{\omega_c}\right)\bar{x}^2, \tag{5.57}$$

we show that $V_{u^\star}(x^\star) = 0$, $V_{u^\star}(x) > 0$ for all $x \neq x^\star$, and (5.49) hold.

Trivial inspection of $V_{u^\star}(x)$ reveals that $V_{u^\star}(x^\star) = 0$ and $V_{u^\star}(x) > 0$ for all $x \neq x^\star$.

Let

$$\bar{\phi}_{x^\star}(x) := \phi(x) - \phi(x^\star).$$

Note that right hand side of (5.49) is a quadratic function in $\bar{u}$ and the inequality holds for all u. Thus, we can apply the Bounded Real Lemma and eliminate the dependence on $\bar{u}$. Assume $\gamma_2 = \frac{1}{\beta}$ and $\gamma_2\omega_c > 1$. Then (5.49) holds for all $x \in \mathbb{R}$ and $u \in \mathbb{R}$ if and only if both

$$\gamma_2^2\omega_c^2 > 1, \text{ and} \tag{5.58}$$

$$\left[\frac{2}{\omega_c}\left(\bar{x} - \frac{\beta\bar{x} + \bar{\phi}_{x^\star}(x)}{\omega_c}\right) + \nabla V_{u^\star}(x)\right]^2 + 4\left[\gamma_2^2 - \frac{1}{\omega_c^2}\right]\left(\bar{x} - \frac{\beta\bar{x} + \bar{\phi}_{x^\star}(x)}{\omega_c}\right)^2$$
$$+ 4\left[\gamma_2^2 - \frac{1}{\omega_c^2}\right]\nabla V_{u^\star}(x)(-\beta\bar{x} - \bar{\phi}_{x^\star}(x)) \leq 0 \tag{5.59}$$

hold. The conditions (5.58) and (5.59) ensure that right hand side of (5.49) is convex and has roots that are imaginary or zero. Thus, the right hand side of (5.49) is greater than or equal to zero for all $x \in \mathbb{R}, u \in \mathbb{R}$. By assumption, (5.58) holds. Note that

$$\nabla V_{u^\star}(x) = 2\gamma_2^2\bar{\phi}_{x^\star}(x) + 2\left(\beta\gamma_2^2 - \frac{1}{\omega_c}\right)\bar{x}.$$

Substituting $\nabla V_{u^\star}(x)$ into (5.59) and expanding yields an inequality of the form

$$c_1\bar{x}^2 + c_2\bar{x}\bar{\phi}_{x^\star}(x) + c_3\bar{\phi}_{x^\star}^2(x) \leq 0, \tag{5.60}$$

where

$$c_1 = \frac{4}{\omega_c^2}\left(1 - \gamma_2^2\omega_c^2\right)(\beta^2\gamma_2^2 - 1),$$

$$c_2 = \frac{8\beta\gamma_2^2}{\omega_c^2}\left(1 - \gamma_2^2\omega_c^2\right),$$

$$c_3 = \frac{4\gamma_2^2}{\omega_c^2}\left(1 - \gamma_2^2\omega_c^2\right).$$

Since $\gamma_2\omega_c > 1$ and $\gamma_2^2\beta^2 = 1$, the terms c_1, c_2, and c_3 are less than or equal to zero. From the condition in (5.52), $\bar{x}\bar{\phi}_{x^\star}(x) \geq 0$. Hence, (5.60) holds for all $\bar{x}$ and (5.59) holds for all x.

Since both (5.58) and (5.59) hold, (5.49) holds for all $x \in \mathbb{R}$ and $u \in \mathbb{R}$. Hence, and $\bar{\Delta}$ satisfies $\Pi_{2,\frac{1}{\beta}}$ with $\omega_c > \beta$. □

Note that the special case $\phi(x) \equiv 0$, in (5.50)-(5.51) gives a linear system with the transfer function $\frac{1}{s+\beta}$. In this case, the conditions $\gamma_2\omega_c > 1$, and $\gamma_2\beta > 1$ imply

$$\left|\frac{1}{s+\beta}\right|_{s=j\omega} < \left|\frac{\gamma_2}{\frac{s}{\omega_c}+1}\right|_{s=j\omega} \qquad \forall\omega,$$

which describes a roll-off with gain γ_2 and corner frequency ω_c.

5.8 Application to Internet Congestion Control

Establishing the stability of Internet congestion control algorithms has been a major research topic over the past decade [51], [57], [80], [31], [68], [63], [98]. A broadly applicable passivity approach was presented in [99], but this study did not take into account the forward and backward delays from the users to the routers. Stability estimates which bound the time delay are essential when achieving robustness and satisfactory performance for the Internet congestion control system, since time delay is an inherent property. As a motivating example, we apply both the IQC verification techniques and the IQC stability analysis to this problem.

Consider a set of N_U users and N_L links. The Internet congestion control problem is to design update algorithms for the sending rates x_i for $i = 1, \ldots, N_U$, and prices p_l for $l = 1, \ldots, N_L$, that are decentralized and have a stable network equilibrium that maximizes the aggregate utility

$$\sum_{i=1}^{N_U} U_i(x_i),$$

subject to the capacity constraints of the links, where $U_i(\cdot)$ is a concave utility function for user i.

The routing matrices R_f and R_b indicate connections with forward delay T_{li}^f and backward delay T_{li}^b as follows:

$$[R_f(s)]_{li} = \begin{cases} e^{-T_{li}^f s} & \text{if user } i \text{ uses link } l \\ 0 & \text{else} \end{cases} \tag{5.61}$$

$$[R_b(s)]_{li} = \begin{cases} e^{-T_{li}^b s} & \text{if user } i \text{ uses link } l \\ 0 & \text{else.} \end{cases} \tag{5.62}$$

Hence, the interconnection of the sources and links is described be the feedback law

$$\begin{bmatrix} q \\ z \end{bmatrix} = \begin{bmatrix} 0 & R_b^T \\ R_f & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix},$$

where $q := [q_1^T, \ldots, q_{N_U}^T]^T$, $x := [x_1^T, \ldots, x_{N_U}^T]^T$, $z := [z_1^T, \ldots, z_{N_L}^T]^T$, and $p := [p_1^T, \ldots, p_{N_L}^T]^T$. Here q_i denotes the price feedback received by user i, and z_l is the aggregate rate for link l.

We now verify Π_{1,γ_1} and Π_{2,γ_2} for a class of user algorithms:

$$\dot{x}_i = U_i'(x_i) - q_i. \tag{5.63}$$

Let Ω_i refer to the dynamics in (5.63) with input $-q_i$ and output x_i. We assume $U_i''(x_i) \leq -\beta_i$ for all $x_i \in \mathbb{R}$ and for some $\beta_i > 0$, which implies that $U_i(\cdot)$ is concave. We further assume that $U_i'(x_i) \to \infty$ as $x_i \to 0^+$, which renders $\mathbb{R}_+^{N_U}$ invariant and allows us to restrict our analysis to this domain.

Claim 5.3. *$\bar{\Omega}_i$ satisfies $\Pi_{1,\frac{1}{\beta}}$ and $\Pi_{2,\frac{1}{\beta}}$ with $\omega_c > \beta$.*

Proof. Since $U_i(\cdot)$ is concave, $U_i'(\cdot)$ can be written as:

$$U_i'(x_i) = -\beta_i x_i - \phi_i(x_i),$$

where ϕ_i is a nondecreasing function because the derivative of ϕ_i is

$$\phi_i'(x_i) = -U_i''(x_i) - \beta_i \geq 0.$$

Hence, $\phi_i(x_i)$ is strictly increasing and (5.52) holds. Clearly, Ω_i is a special case of the system in (5.50)-(5.52). Thus, by Claim 5.1 and Claim 5.2, $\bar{\Omega}_i$ satisfies $\Pi_{1,\frac{1}{\beta}}$ and $\Pi_{2,\frac{1}{\beta}}$ with $\omega_c > \beta$. □

Several classes of congestion control algorithms treat the links as static operators, interpreted as penalty functions that keep the link rate below capacity, and the users as dynamic operators [57]. For the link price we select the algorithm:

$$p_l = h_l(z_l), \tag{5.64}$$

where $h_l(\cdot)$ is a monotone penalty function. Let Δ_l describe the dynamics of the link price with input z_l and output p_l. Clearly, a static system will not satisfy the roll-off IQC Π_{2,γ_2}. However, since $h_l(\cdot)$ is strictly increasing, if the slope of h_l is less than or equal to γ_1, then $\bar{\Delta}_l$ satisfies Π_{1,γ_1}.

Let

$$\Sigma := \mathrm{diag}(\Omega_1, \ldots, \Omega_{N_U}, \Delta_1, \ldots, \Delta_{N_L}),$$

and note that the interconnection matrix is

$$G := \begin{bmatrix} 0 & -R_b^T \\ R_f & 0 \end{bmatrix}, \tag{5.65}$$

since the input to Ω_i is $-q_i$.

To test the stability of the interconnection of Σ with G, we let define $\Pi(j\omega)$ to be composite IQC for the entire system of N_U users using N_L links:

$$\Pi(j\omega) := \Pi_{1,\frac{1}{\beta}} \otimes P_1 + \Pi_{2,\frac{1}{\beta}}(j\omega) \otimes P_2 + \Pi_{1,\gamma_1}(j\omega) \otimes P_3,$$

where P_1, P_2 and P_3 are diagonal $(N_U + N_L) \times (N_U + N_L)$ matrices, $[P_1]_{ii} = \alpha_{i1}$, $[P_2]_{ii} = \alpha_{i2}$, $[P_3]_{ii} = 0$ for $i = 1, \ldots, N_U$, and $[P_1]_{ii} = 0$, $[P_2]_{ii} = 0$, $[P_3]_{ii} = \alpha_{i1}$ for $i = N_U{+}1, \ldots, N_U{+}N_L$. We then search for $\alpha_{i1} \geq 0$ for $i = 1, \ldots N_U + N_L$ and $\alpha_{j2} \geq$ for $j = 1, \ldots, N_U$ such that (5.6) holds for all $\omega \in \mathbb{R}$.

5.9 Example

As an example application to the Internet congestion control problem, we consider the interconnection in Figure 5.5 with one link serving N_U users. We combine the forward and backward delays to yield the round trip time

$$T_i = T_i^f + T_i^b.$$

In this special case, we can equivalently test the stability for an interconnection where all of the delay is in the forward routing matrix R_f.

Lemma 5.7. *Let $G(j\omega)$ and $\tilde{G}(j\omega)$ represent two different interconnections as in* (5.65) *where R_f, R_b, $\tilde{R}_f$, $\tilde{R}_b$ are $r \times 1$ such that* (5.69) *holds.*

There exists a diagonal nonsingular $D(j\omega) \in \mathbb{C}^{n\times n}$ such that

$$D(j\omega)^{-1}G(j\omega)D(j\omega) = \tilde{G}(j\omega). \tag{5.66}$$

Proof. Since (5.69) holds, choosing

$$d_1(j\omega) := 1, \tag{5.67}$$

$$d_i(j\omega) := d_{i-1}\frac{[R_b(j\omega)]_i}{[\tilde{R}_b(j\omega)]_i} \quad i = 2, \ldots, n, \tag{5.68}$$

and $D(j\omega) = \operatorname{diag}(d_1(j\omega), \ldots, d_n(j\omega))$ provides a nonsingular, diagonal $D(j\omega)$ such that (5.66) holds. □

Theorem 5.5. *Let $G(j\omega)$ and $\tilde{G}(j\omega)$ represent two different interconnections as in* (5.65) *where R_f, R_b, $\tilde{R}_f$, $\tilde{R}_b$ are $r \times 1$ and*

$$[R_f(s)]_i[R_b(s)]_i = [\tilde{R}_f(s)]_i[\tilde{R}_b(s)]_i \quad \forall i = 1, \ldots, r. \tag{5.69}$$

Let $\{\Pi_k(j\omega)\}_{k=1}^{p}$ represent an arbitrary set of IQCs.

Define

$$\Pi(j\omega) = \sum_{k=1}^{p} \Pi_k(j\omega) \otimes P_k$$

and

$$\tilde{\Pi}(j\omega) = \sum_{k=1}^{p} \Pi_k(j\omega) \otimes \tilde{P}_k,$$

where $P_k = diag(\alpha_{ik}), \tilde{P}_k = diag(\tilde{\alpha}_{ik})$ *for* $i = 1, \dots, r+1$. *There exist constants* $\alpha_{ik} \geq 0$ *and* $\epsilon > 0$ *such that for all* $\omega \in \mathbb{R}$

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \preceq -\epsilon I, \tag{5.70}$$

if and only there if exist constants $\tilde{\alpha}_{ik} \geq 0$ *and* $\tilde{\epsilon} > 0$ *such that for all* $\omega \in \mathbb{R}$

$$\begin{bmatrix} \tilde{G}(j\omega) \\ I \end{bmatrix}^* \tilde{\Pi}(j\omega) \begin{bmatrix} \tilde{G}(j\omega) \\ I \end{bmatrix} \preceq -\tilde{\epsilon} I. \tag{5.71}$$

Proof. The proof of Theorem 5.5 is identical to the proof of Theorem 5.1, by defining G and $\tilde{G}$ as in (5.65) such that (5.69) holds, and by defining D such that (5.67) and (5.68) hold. As in Theorem 5.1, the $|d_i|$ terms will be constant scalars, since (5.61), (5.62), (5.67), and (5.68) are assumed in the statement of Theorem 5.5.

□

For simplicity of the presentation, we assume that $T_i := T$ for $i = 1, \dots, N_U$. By Theorem 5.5, we combine all of the forward and backward delay into R_f such that $R_f = e^{-sT}\mathbf{1}^T$ and $R_b = \mathbf{1}$ and test the stability of the new interconnection. The user algorithms Ω_i are defined by (5.63) for $i = 1, \dots, N_U$ with $U_i''(x_i) \leq -\beta$ for some $\beta > 0$ and for all $x_i \in \mathbb{R}$. By Claim 5.3, $\bar{\Omega}_i$ satisfies $\Pi_{1,\frac{1}{\beta}}$ and $\Pi_{2,\frac{1}{\beta}}$ with $\omega_c > \beta$ for $i = 1, \dots, N_U$. The link price algorithm Δ_1 is defined by (5.64), where $h_l(\cdot)$ is a monotone penalty function with a slope less than or equal to γ_1. Thus, Δ_1 satisfies Π_{1,γ_1}. Since there is only one link and the link only satisfies one IQC, we incorporate the IQC gain γ_1 in the feedback loop in Figure 5.5 and assume henceforth that Δ_1 satisfies $\Pi_{1,1}$.

The stability test for this interconnection can be equivalently expressed as the stability test for an interconnection of two subsystems.

Lemma 5.8. *Let* $\Pi(j\omega)$ *represent an arbitrary IQC multiplier* $\Pi_{22}(j\omega) \preceq 0$. *If bounded, causal operators* $\{\Omega_i\}_{i=1}^K$ *on* $\mathbf{L}_{2e}^l \to \mathbf{L}_{2e}^m$ *satisfy the IQC defined by* Π, *then, for all* $\theta_i \geq 0$, *such that* $\sum_{i=1}^{K} \theta_i = 1$, *the bounded, causal operator*

$$\Delta := \sum_{i=1}^{K} \theta_i \Omega_i$$

also satisfies the IQC defined by Π.

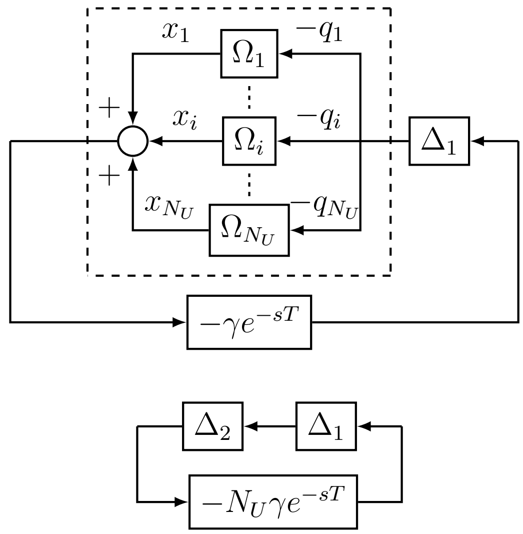

Figure 5.5: Simplified network with one link and N_U users with round trip time T for each user. The stability of this interconnection can be tested on the two system interconnection.

Proof. Let

$$\Pi(j\omega) = \begin{bmatrix} \Pi_a(j\omega) & \Pi_b(j\omega) \\ \Pi_b^*(j\omega) & -\Pi_c(j\omega) \end{bmatrix},$$

where, $\Pi_c(j\omega) \succeq 0$ for all ω. Assume $\{\Omega_i\}_{i=1}^K$ are bounded causal operators that satisfy the IQC defined by Π. For any $u \in \mathbf{L}_2$, define $y_i := \Omega_i(u)$, and note that for each i,

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{y}_i(j\omega) \end{bmatrix}^* \begin{bmatrix} \Pi_a(j\omega) & \Pi_b(j\omega) \\ \Pi_b^*(j\omega) & -\Pi_c(j\omega) \end{bmatrix} \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{y}_i(j\omega) \end{bmatrix} \, d\omega \geq 0. \tag{5.72}$$

If $\sum_{i=1}^{K} \theta_i = 1$ and $\theta_i \geq 0$ for all i, then

$$\sum_{i=1}^{K} \theta_i \left(\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{y}_i(j\omega) \end{bmatrix}^* \begin{bmatrix} \Pi_a(j\omega) & \Pi_b(j\omega) \\ \Pi_b^*(j\omega) & -\Pi_c(j\omega) \end{bmatrix} \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{y}_i(j\omega) \end{bmatrix} \, d\omega \right) \geq 0.$$

Distributing the sum of θ_i yields

$$\begin{aligned} \int_{-\infty}^{\infty} \sum_{i=1}^{K} \theta_i \widehat{u}(j\omega)^* \Pi_a(j\omega) \widehat{u}(j\omega) + \widehat{u}(j\omega)^* \Pi_b(j\omega) \left(\sum_{i=1}^{K} \theta_i \widehat{y}_i(j\omega) \right) \\ + \widehat{u}(j\omega)^* \left(\sum_{i=1}^{K} \theta_i \widehat{y}_i(j\omega) \right)^* \Pi_b^*(j\omega) \widehat{u}(j\omega) - \sum_{i=1}^{K} \theta_i \widehat{y}_i(j\omega)^* \Pi_c(j\omega) \widehat{y}_i(j\omega) \, d\omega \geq 0. \end{aligned} \tag{5.73}$$

Since $\Pi_c(j\omega) \succeq 0$, there exist $L(j\omega)$ such that $\Pi_c(j\omega) = L(j\omega)^* L(j\omega)$. By definition $\Delta := \sum_{i=1}^{K} \theta_i \Omega_i$, so

$$\widehat{\Delta(u)}(j\omega) = \sum_{i=1}^{K} \theta_i \widehat{y}_i(j\omega),$$

and (5.73) is equivalent to

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{\Delta(u)}(j\omega) \end{bmatrix}^* \begin{bmatrix} \Pi_a(j\omega) & \Pi_b(j\omega) \\ \Pi_b^*(j\omega) & 0 \end{bmatrix} \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{\Delta(u)}(j\omega) \end{bmatrix} - \sum_{i=1}^{K} \theta_i \|L(j\omega)\widehat{y}_i(j\omega)\|^2 \, d\omega \geq 0,$$

Let $g : \mathbb{C}^n \to \mathbb{R}_+$, $A \in \mathbb{C}^{m\times n}$, $g(x) := \|Ax\|$, and $h : \mathbb{R} \to \mathbb{R}$, $h(z) := z^2$. The composition $h \circ g$ is convex on $\mathbb{C}^n$ since g is convex and h is convex and nondecreasing on the range of g. Hence, Jensen's inequality [13] implies

$$\left\| \sum_{i=1}^{K} \theta_i L(j\omega)\widehat{y}_i(j\omega) \right\|^2 \leq \sum_{i=1}^{K} \theta_i \|L(j\omega)\widehat{y}_i(j\omega)\|^2 \quad \forall \omega$$

and, thus:

$$\begin{aligned} 0 \leq \int_{-\infty}^{\infty} & \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{\Delta(u)}(j\omega) \end{bmatrix}^* \begin{bmatrix} \Pi_a(j\omega) & \Pi_b(j\omega) \\ \Pi_b^*(j\omega) & 0 \end{bmatrix} \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{\Delta(u)}(j\omega) \end{bmatrix} \\ & - \left\| \sum_{i=1}^{K} \theta_i L(j\omega)\widehat{y}_i(j\omega) \right\|^2 d\omega = \int_{-\infty}^{\infty} \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{\Delta(u)}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{\Delta(u)}(j\omega) \end{bmatrix} d\omega, \end{aligned} \tag{5.74}$$

since $\widehat{\Delta(u)}^*(j\omega)\Pi_c(j\omega)\widehat{\Delta(u)}(j\omega) = \| \sum_{i=1}^{K} \theta_i L(j\omega)\widehat{y}_i(j\omega)\|^2$ pointwise in ω. Thus, Δ satisfies the IQC defined by Π.

□

Assume each $\{\Omega_i\}_{i=1}^{N_U}$ in Figure 5.5 satisfies the IQCs defined by Π_{1,γ_1}, Π_{2,γ_2} with ω_c, and, therefore, also satisfies Π_{2,τ,γ_2} with ω_c for any $\tau > 0$. Then, by Lemma 5.8, we can replace the sum of the users Ω_i in the dashed box by the subsystem $N_U \Delta_2$, where

$$\Delta_2 := \sum_{i=1}^{N_U} \frac{1}{N_U} \Omega_i,$$

and Δ_2 satisfies Π_{1,γ_1} and Π_{2,τ,γ_2} with ω_c for any $\tau > 0$. This yields an interconnection of Δ_1 and Δ_2 in negative feedback with gain $N_U\gamma_1$ and delay T, seen in Figure 5.5. If there exist $\epsilon > 0$, $\alpha_{i1}, \alpha_{i2} \geq 0$ such that $H_\Pi \preceq -\epsilon I$ for $\omega \in \theta = \left[-\frac{\pi}{T}, \frac{\pi}{T}\right]$, then from Lemma 5.2, there exist $\bar{\tau} > 0$, $\bar{\epsilon} > 0$, $\bar{\alpha_{i1}}, \bar{\alpha_{i2}} \geq 0$ such that $H_{\Pi^{\bar{\tau}}} \preceq -\bar{\epsilon} I$. Since Δ_2 satisfies $\Pi_{2,\bar{\tau},\gamma_2}$, in particular, stability is achieved.

As a numerical example, we performed the robustness test in (5.6) on the interconnection in Figure 5.5 with Δ_1 satisfying $\Pi_{1,1}$ and Δ_2 satisfying $\Pi_{1,\frac{1}{\beta}}$ and $\Pi_{2,\frac{1}{\beta}}$ at $\omega_c = 1$. We gridded the frequency $\omega \in \left[-\frac{\pi}{T}, \frac{\pi}{T}\right]$ with 300 points and varied $T \in (10^{-3}, 10^{3})$, which is illustrated in Figure 5.6. Since the delays are commensurate, we evaluate (5.6) over this subset of $\mathbb{R}$. Note that the curve is parametrized by N_U, and we need only run the test once to obtain the result for any N_U. If the number of sources N_U increases, the stability bound will decrease.

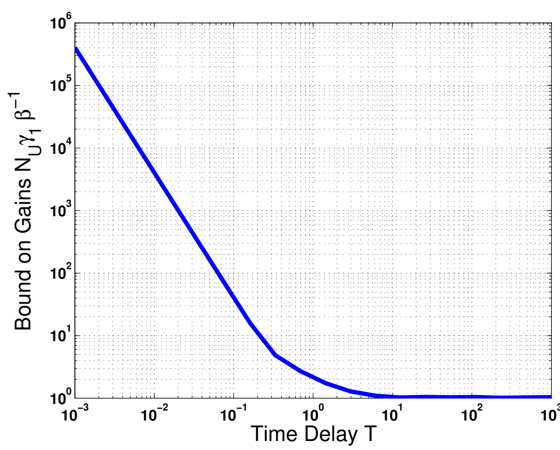

Figure 5.6: Robustness test for the Internet congestion control problem with a static link (5.64) and dynamic user algorithm in (5.63). The bound on the gain $\gamma_1\beta^{-1}$ approaches $\frac{1}{N_U}$ as the time delay T approaches infinity.

The global nonlinear results on the stability of the Internet congestion control problem in [68], [98], [63] are not comparable to this study, as they assumed a dynamic link price. Under similar dynamic user and static link algorithms assumptions, [31] analyzed the global asymptotic stability of the Internet congestion control problem with delays by using a small gain technique. However, in [31] the second derivative of the utility function is bounded below, which was not a restriction in our example. Thus, we achieve a less conservative stability estimate by removing this restriction. Other significant results for nonlinear models of Internet congestion control with delay include [74] and [102], which present stability tests that account for arbitrarily large delays, and, therefore, may be conservative. Results from [64] are delay dependent; however, they study a different congestion control algorithm than the one described in equation (5.63).

Chapter 6

Conclusions

In Chapter 2 we developed theoretical and numerical tools for quantitative local analysis of nonlinear systems. Specifically, sufficient conditions for the bounds on the reachable set and $\mathbf{L}_2$ gain of the nonlinear system subject to norm bounded disturbance inputs were derived. We outlined an approach for verifying the dissipation inequalities describing the $\mathbf{L}_2$ gain and reachability using SOS techniques. These techniques were also used in Chapter 3 to establish local IQCs and in the example in Chapter 4 to compare to the frequency-domain analysis.

In Chapter 3 we defined the notion of a local IQC and established a method for generating a particular local IQCs using linear offsets, linear weights and estimates of local $\mathbf{L}_2$ gains. We presented several important technical lemmas and theorems involving extensions of local operators to global operators and positive combinations of IQC multipliers which were used in Chapter 4. We established invariants for IQCs based on an input-output scaling and a time scaling with a variable change. Through dimensional analysis, the number of free parameters in a ODE model is reduced, which reduces the dimensionality of a parameterized family of IQCs describing the system. For a simple four parameter first order system, we reduced the dimensionality to zero, allowing us to express all parameterized IQCs in terms of IQCs for one, specific system. For a general problem, if the desired, parameterized IQCs must be generated via numerical means, such as in [82], then the efficiency gain by performing dimensional analysis is worthwhile.

In Chapter 4 established a frequency-domain stability and performance test for interconnections of systems who satisfy local IQCs. We presented a state-space condition for the case when the performance metric defined by an IQC describing an input-output gain. An example of an interconnection of three systems, which each satisfy many IQCs is presented. In this case, the frequency domain IQC analysis outperformed the direct SOS approach. Future work will include a state-space test with a general performance IQC, development of large libraries of small (2-3 state) nonlinear systems and collections of IQCs which they satisfy, and analysis of larger interconnections of locally stable systems.

In Chapter 5 we presented an IQC approach to analyze the stability of interconnected passive systems with time delay. This approach employs a roll-off IQC to complement passivity with time scale information, and yields stability estimates that depend on the duration of delay. In the case of a cyclic interconnection, the exact stability bound was derived in Theorem 5.4. For other interconnection structures where such closed-form stability bounds are not available, the proposed test can be applied numerically, by constructing a frequency grid and by evaluating the IQC stability condition for each frequency. However, an important problem that arises when applying the test numerically is how to ensure that no crucial frequencies will be missed. In structured singular value (μ) analysis, the exact meaning of the finite-frequency grid analysis is known (for all values of uncertainty, no poles cross the imaginary axis at the frequency grid points), and can often aid in interpreting a finite-frequency computation. Moreover, adaptive methods to certify that a finite-frequency grid upper-bound μ analysis is in fact valid over the entire frequency axis have been developed, [38]. Related, but different approaches to avoid frequency gridding in specific non-rational applications of IQC theory have also been developed, [48, 50]. A general understanding of finite frequency gridding in IQC tests remains to be developed.

A further contribution was the equilibrium-independent verification of the OSP and roll-off IQCs, given a state space model of the subsystems. This problem is important because in many of our motivating applications, such as biological networks and the Internet congestion control problem, the equilibrium of an interconnection is not known a priori. Equilibrium-independent verification of the output strict passivity IQC can be achieved through methods outlined in [41]. The roll-off IQC was verified here by cascading the system of interest with a system whose output “rolls-up”, and by estimating the bound on the gain for the cascaded system. If the state model is described by polynomial vector functions, then elementary $\mathbf{L}_2$ gain methods [83], [82], combined with sum-of-squares (SOS) optimization [67], [8], provide one method to verify the output strictly passivity or roll-off IQC. A future research topic will be to investigate whether stability bounds similar to the ones obtained here with an input-output approach can be derived using pure state-space methods. A particularly interesting problem is whether it may be possible to construct Lyapunov-Krasovskii functionals from storage functions describing the OSP and roll-off properties the subsystems.

Chapter 7

Appendix

7.1 Notation

Let $\mathbb{N}$ be the set of natural numbers. Let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers. The set of $m \times n$ matrices whose elements are in $\mathbb{R}$ or $\mathbb{C}$ are denoted as $\mathbb{R}^{m\times n}$ and $\mathbb{C}^{m\times n}$. For $v \in \mathbb{C}^m$, $\|v\|$ denotes the Euclidean norm. A single superscript index denotes vectors, e.g. $\mathbb{R}^m$ is the set of $m \times 1$ vectors whose elements are in $\mathbb{R}$. Let $e_i \in \mathbb{R}^n$ be the unit vector with zeros everywhere except in the *ith* element. Let $\mathbf{1}$ represent the vector whose entries are all one. For a matrix A let $[A]_{ij}$ denote the (i, j)th entry. Given an $m \times n$ matrix A and a $p \times q$ matrix B, their Kronecker product $C = A \otimes B$ is an $(mp) \times (nq)$ matrix with elements defined by $[C]_{\alpha\beta} = [A]_{ij}[B]_{kl}$, where $\alpha = p(i-1)+k$ and $\beta = q(j-1)+l$.

Let $\mathbb{R}\mathbf{H}_\infty$ denote the set of stable, proper systems. The set of $m \times n$ matrices whose elements are stable and proper are denoted as $\mathbb{R}\mathbf{H}_\infty^{m\times n}$. Let $\mathcal{U}_{\mathbb{R}\mathbf{H}_\infty}$ denote the set of systems whose inverses are also in $\mathbb{R}\mathbf{H}_\infty$.

Basic system theory and functional analysis drawn from texts such as [27], [28] and [30] is used without further citation. $\mathbf{L}_2^m$ is the space of $\mathbb{R}^m$-valued functions $f : [0,\infty) \to \mathbb{R}^m$ of finite energy $\|f\|_2^2 = \int_0^\infty f(t)^T f(t)\, dt$. For $u \in \mathbf{L}_2^m$, $\hat{u}$ denotes the Fourier (Plancherel) transform of u. Associated with $\mathbf{L}_2^m$ is the $e$xtended space $\mathbf{L}_{2e}^m$, consisting of functions whose truncation f_T ($f_T(t) := f(t)$ for $t \le T$; $f_T(t) := 0$ for $t > T$) is in $\mathbf{L}_2^m$ for all $T > 0$.

For $u, v \in \mathbf{L}_2$, define

$$\langle u, v\rangle := \int_{-\infty}^{\infty} \hat{u}(\omega)^* \hat{v}(\omega)\ d\omega,$$

which is an inner product associated with the $\mathbf{L}_2$ norm. Let $\|u\|_{2,[a,b]}$ denote the truncated $\mathbf{L}_2$ norm from $\sqrt{\int_a^b |u(t)| dt}$. For notational simplicity, if the bounds of integration are $a = 0$ to $b = 0$, we abbreviate this as $\|u\|_{2,T}$.

7.2 Functional Analysis

Fact 7.1. *If $f, g \in \mathbf{L}^n_{2,[a,b]}$, then $f^T g \in \mathbf{L}_{1,[a,b]}$ and $\left\|f^T g\right\|_1 \leq \|f\|_2 \|g\|_2$*

Proof.

$$\begin{array}{rcl} \left\|f^T g\right\|_1 & = & \int_a^b \left|f^T(t) g(t)\right| dt \\ & \leq & \sum_i \int_a^b |f_i(t)| \cdot |g_i(t)| \, dt \\ & \leq & \sum_i \|f_i\|_2 \cdot \|g_i\|_2 \\ & \leq & \sqrt{\sum_i \|f_i\|_2^2} \sqrt{\sum_i \|g_i\|_2^2} \\ & = & \|f\|_2 \|g\|_2 \end{array}$$

□

Fact 7.2. *If $M(\cdot) \in \mathbf{L}^{n \times m}_{2,[a,b]}$, then $\bar{\sigma}(M(\cdot)) \in \mathbf{L}_{2,[a,b]}$.*

Proof. $\bar{\sigma}$ is a continuous function, hence $\bar{\sigma}(M(\cdot))$ is measurable. Pointwise,

$$\bar{\sigma}^2(M(t)) \leq \sum_i \sum_j |M_{i,j}(t)|^2$$

which completes the proof. □

Fact 7.3. *If $M(\cdot) \in \mathbf{L}^{n \times m}_{2,[a,b]}$, and $x \in \mathbb{R}^m$, then $Mx \in \mathbf{L}^n_{2,[a,b]}$, and $\|Mx\|_2 \leq \|x\|_2 \|\bar{\sigma}(M(\cdot))\|_2$.*

Proof.

$$\|Mx\|_2^2 = \int_a^b \|M(t)x\|_2^2 \, dt \leq \int_a^b \bar{\sigma}(M(t))^2 \|x\|_2^2 \, dt = \|x\|_2^2 \|\bar{\sigma}(M(\cdot))\|_2^2$$

□

Fact 7.4. *If $M(\cdot) \in \mathbf{L}^{n \times m}_{2,[a,b]}$, and $x \in \mathbf{L}^m_{2,[a,b]}$ then*

$$\begin{array}{rcl} \left\|\int_a^b M(t)x(t)dt\right\|_2^2 & = & \sum_i \left|\int_a^b \sum_j M_{ij}(t)x_j(t)dt\right|^2 \\ & \leq & \sum_i \left(\int_a^b \left|\sum_j M_{ij}(t)x_j(t)\right| dt\right)^2 \\ & \leq & \sum_i \left(\int_a^b \sum_j |M_{ij}(t)x_j(t)| \, dt\right)^2 \\ & = & \sum_i \left(\sum_j \int_a^b |M_{ij}(t)x_j(t)| \, dt\right)^2 \\ & \leq & \sum_i \left(\sum_j \|M_{ij}\|_2 \|x_j\|_2\right)^2 \\ & \leq & \sum_i \left(\sum_j \|M_{ij}\|_2 \|x\|_2\right)^2 \\ & \leq & \sum_i \|x\|_2^2 \left(\sum_j \|M_{ij}\|_2\right)^2 \\ & \leq & \|x\|_2^2 \sum_i \left(\sum_j \|M_{ij}\|_2\right)^2 \end{array}$$

Result we want is the bound between initial and final expressions, namely

$$\left\| \int_a^b M(t)x(t)dt \right\|_2 \leq \|x\|_2 \sqrt{\sum_i \left(\sum_j \|M_{ij}\|_2 \right)^2}$$

Lemma 7.1. *Let F be causal and $c \in \mathbb{R}$, $c > 0$. If $\|Fu\|_{2,T} \leq c\|u\|_2$ for all $u \in \mathbf{L}_2$, then $\|Fu\|_{2,T} \leq c\|u\|_{2,T}$ for all of $u \in \mathbf{L}_2$.*

Proof. Let $T > 0$, $u \in \mathbf{L}_2$. Define

$$u_T(t) = \begin{cases} u(t) & t \leq T \\ 0 & t > T \end{cases}.$$

Clearly, $u_T \in \mathbf{L}_2$. Assume $\|Fu_T\|_{2,T} \leq c\|u_T\|_2$. Since $\|u_T\|_2 = \|u\|_{2,T}$,

$$\|Fu_T\|_{2,T} = \|Fu\|_{2,T} \leq c\|u_T\|_2 = c\|u\|_{2,T}. \tag{7.1}$$

□

7.3 Lipschitz Extensions

Locally Lipschitz continuous functions can be extended to globally Lipschitz continuous functions as follows [53, 93].

Lemma 7.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz continuous on $\mathcal{B} \subseteq \mathbb{R}^n$ with $\mathcal{B} \neq \emptyset$ and Lipschitz constant L. For each $x \in \mathbb{R}^n$, define $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$*

$$\tilde{f}(x) := \min_{y \in \mathcal{B}} \quad f(y) + L\|x - y\|. \tag{7.2}$$

Then $\tilde{f}(x) = f(x)$ $\forall x \in \mathcal{B}$ and $\tilde{f}$ is globally Lipschitz continuous (with Lipschitz constant L).

Proof. Consider first the case $x \in \mathcal{B}$. Clearly $\tilde{f}(x) \leq f(x)$, because $\tilde{f}$ involves a minimum over all $y \in \mathcal{B}$ and the value obtained at $y = x \in \mathcal{B}$ is $f(x)$. Next, since f is Lipschitz continuous on $\mathcal{B}$ it follows that $f(x) \leq f(y) + L\|x - y\|$ for all $y \in \mathcal{B}$. Minimizing the right-hand side over $y \in \mathcal{B}$ (which gives $\tilde{f}(x)$) preserves the inequality, hence $f(x) \leq \tilde{f}(x)$. Together these imply that for $x \in \mathcal{B}$, $\tilde{f}(x) = f(x)$.

For global Lipschitz continuity of $\tilde{f}$, take any $x_1 \in \mathbb{R}^n, x_2 \in \mathbb{R}^n$. For all $z \in \mathbb{R}^n$,

$$f(z) + L\|x_1 - z\| \leq f(z) + L\|x_2 - z\| + L\|x_1 - x_2\|.$$

Minimize both sides of this expression over $z \in \mathcal{B}$ to get $\tilde{f}(x_1) \leq \tilde{f}(x_2) + L\|x_1 - x_2\|$. Reversing the role of x_1 and x_2 gives $\tilde{f}(x_2) \leq \tilde{f}(x_1) + L\|x_2 - x_1\|$. Combining these gives $\left|\tilde{f}(x_1) - \tilde{f}(x_2)\right| \leq L\|x_1 - x_2\|$ as desired. □

7.4 Polynomials, Sum-of-squares and S-procedure

A *monomial* m_α in n variables is a function defined as $m_\alpha(x) = x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for $\alpha \in \mathbb{Z}_+^n$. The degree of a monomial is defined, $\deg m_\alpha := \sum_{i=1}^n \alpha_i$. A *polynomial* f in n variables is a finite linear combination of monomials, with $c_\alpha \in \mathbb{R}$:

$$f := \sum_\alpha c_\alpha m_\alpha = \sum_\alpha c_\alpha x^\alpha$$

The set of all polynomials in n indeterminate variables is denoted $\mathcal{R}_n$. The particular variables are not noted, and usually there is an obvious $n-$dimensional variable present in the discussion. The degree of f is defined as $\deg f := \max_\alpha \deg m_\alpha$ (provided the associated c_α is non-zero).

The notation Σ_n denotes the set of sum of squares (SOS) polynomials in n variables,

$$\Sigma_n := \left\{ p \in \mathcal{R}_n : p = \sum_{i=1}^t f_i^2 \quad , \, t > 0, f_i \in \mathcal{R}_n, \, i = 1, \ldots, t \right\}.$$

If $p \in \Sigma_n$, then $p(x) \geq 0 \ \forall x \in \mathbb{R}^n$. The notation Σ_{n+m} also appears, referring to SOS polynomials in $n+m$ real variables, where, again, the particular variables are clear in the context of the discussion. The following lemma is a trivial generalization of the well known $\mathcal{S}$-procedure [12], and is a special case of the Positivstellensatz Theorem [11, Theorem 4.2.2].

Lemma 7.3 (Generalized $\mathcal{S}$-procedure)**.** *Given* $\{p_i\}_{i=0}^m \in \mathcal{R}_n$. *If there exist* $\{s_k\}_{i=1}^m \in \Sigma_n$ *such that* $p_0 - \sum_{i=1}^m s_i p_i \in \Sigma_n$, *then*

$$\bigcap_{i=1}^m \{x \in \mathbb{R}^n : p_i(x) \geq 0\} \subseteq \{x \in \mathbb{R}^n \,|\, p_0(x) \geq 0\}.$$

Proof. Since $p_0 - \sum_{i=1}^m s_i p_i \in \Sigma_n$, so $p_0 \geq \sum_{i=1}^m s_i p_i \ \forall x$. For any $\bar{x} \in \bigcap_{i=1}^m \{x \in \mathbb{R}^n \,|\, p_i(x) \geq 0\}$, since $s_i(\bar{x}) \geq 0$, so $\sum_{i=1}^m s_i p_i \geq 0$, hence $p_0(\bar{x}) \geq 0$. $\square$

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