ROBUST AND ADAPTIVE NONLINEAR OUTPUT REGULATION

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Robust and Adaptive Nonlinear Output Regulation *

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Abstract

The object of this paper is to prove the stability of an adaptive control scheme designed to asymptotically achieve output regulation for a class of nonlinear systems. The solution proposed in [1] to the nonlinear output regulation problem is reviewed and the robustness of the solution to parametric uncertainty is analyzed. A standard adaptive scheme is then applied to the problem and slowly-varying results are employed to achieve asymptotic output regulation.

Keywords. Nonlinear Output Regulation, Adaptive Control, Center Manifold, Slowly-varying.

1 Introduction

The task at hand is to analyze and account for parameter uncertainty in the nonlinear output regulation problem. Recent work by Isidori and Byrnes [1] has produced necessary and sufficient conditions for the solvability of both the state feedback and output feedback regulator problem for a class of nonlinear systems. In their work, the signals to track are restricted to those that can be considered as the output of a Poisson stable exosystem. Their analysis is based on the local properties of center manifolds. Using the work in [1] as a point of reference, this paper will proceed to examine the same problem in the presence of parameter uncertainty. In section 2, we

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review the nonlinear regulator theory and the solution developed in [1]. In section 3 we introduce parametric uncertainty to the problem. In section 4 we lay the ground work for our adaptive scheme by reviewing slowly-varying theory for nonlinear systems. Finally, our adaptive scheme is developed in section 5.

2 Nonlinear Regulator Theory

The subsequent discussion follows closely that of [1]. The class of systems that will be examined is those of the form

$$\begin{aligned} \dot{x} &= f(x,\theta^*) + g(x,\theta^*)u + p(x,\theta^*)w \\ y &= h(x) \end{aligned} \tag{1}$$

where w is the state of an (autonomous) exosystem

$$\dot{w} = s(w,\theta^*) \tag{2}$$

For this system, we will begin by considering $\theta^* \in \mathbb{R}^p$ as a vector of *known* parameters in order to review nonlinear regulator theory in the absence of uncertainty. The control objective is to have the output track a reference signal that is the output of the exosystem and given by $-q(w(t))$. The plant (1) is assumed to have m inputs and o outputs. The state x of the plant is defined on a neighborhood X of the origin in $\mathbb{R}^n$. The state w of the exosystem is defined on a neighborhood W of the origin in $\mathbb{R}^s$. Further, f and the columns of g and p are assumed to be smooth vector fields and $h(x)$ is a smooth mapping on X. Also, s is a smooth vector field and $q(w)$ is a smooth mapping defined on W. The composite system is then

$$\begin{aligned} \dot{x} &= f(x,\theta^*) + g(x,\theta^*)u + p(x,\theta^*)w \\ \dot{w} &= s(w,\theta^*) \\ e &= h(x) + q(w) \end{aligned} \tag{3}$$

Finally, it is assumed that $f(0,\cdot) = 0$, $s(0,\cdot) = 0$, $h(0) = 0$, $q(0) = 0$ so that, for $u = 0$, the composite system (3) has an equilibrium state $(x,w) = (0,0)$ which yields zero error, independent of the value of θ^*. For the state feedback regulator problem, we seek a state feedback of the form

$$u = \alpha(x,w,\theta^*)$$

such that the closed loop system

$$\begin{array}{rcl} \dot{x} & = & f(x,\theta^*) + g(x,\theta^*)\alpha(x,w,\theta^*) + p(x,\theta^*)w \\ \dot{w} & = & s(x,\theta^*) \\ e & = & h(x) + q(w) \end{array} \tag{4}$$

exhibits some stability property and $\lim_{t\to\infty} e(t) = 0$. Following [1], we state the nonlinear state feedback regulator problem formally.

State Feedback Regulator Problem. Given a nonlinear system of the form (3), find, if possible, a feedback $u = \alpha(x,w,\theta^*)$ such that

1. the equilibrium $x = 0$ of

$$\dot{x} = f(x,\theta^*) + g(x,\theta^*)\alpha(x,0,\theta^*) \tag{5}$$

   is asymptotically stable in the first approximation. i.e.

$$\sigma(\frac{d}{dx}[f(x,\theta^*) + g(x,\theta^*)\alpha(x,0,\theta^*)]|_{x=0}) \subset \mathbb{C}^{\circ}_{-}$$

2. there exists a neighborhood $U \subset X \times W$ of (0,0) such that, for each initial condition $(x(0), w(0)) \in U$, the solution of the closed loop system satisfies

$$\lim_{t\to\infty} (h(x(t)) + q(w(t))) = 0$$

Under the following two hypotheses, statements concerning the existence of a solution to the state feedback regulator problem can be formulated:

(H1) the linear approximation of (5) is stabilizable.

(H2) the point $w = 0$ is a stable equilibrium of the exosystem, and there is an open neighborhood of the point $w = 0$ in which every point is Poisson stable. In short, this assumption implies that the eigenvalues of the linear approximation of the exosystem lie on the imaginary axis.

Byrnes and Isidori state necessary and sufficient conditions for the solution of the state feedback regulator problem.

Theorem 2.1 (Byrnes and Isidori) *Under hypotheses (H1) and (H2), the state feedback regulator problem is solvable if and only if there exist $C^k(k \geq 2)$ mappings $x = \pi(w,\theta^*)$, with $\pi(0,\theta^*) = 0$ and $u = c(w,\theta^*)$, with $c(0,\theta^*) = 0$, both defined in a neighborhood $W^\circ \subset W$ of 0, satisfying the conditions*

$$\begin{array}{rcl} \frac{\partial \pi}{\partial w} s(w,\theta^*) & = & f(\pi,\theta^*) + p(\pi,\theta^*)w \\ & & g(\pi,\theta^*)c(w,\theta^*) \\ h(\pi(w,\theta^*)) + q(w) & = & 0 \end{array} \tag{6}$$

Remark. The proof relies on center manifold theory and constructs a state feedback

$$u = \alpha(x, w, \theta^*) = c(w, \theta^*) + K^*[x - \pi(w, \theta^*)] \tag{7}$$

that is shown to be a solution of the state feedback regulator problem. K^* is a matrix of feedback gains such that the eigenvalues of the linear approximation of (5) have negative real part.

3 Parametric Uncertainty

To proceed with the discussion, θ^* is now considered as a vector of *unknown* parameters. Before attempting to handle the uncertainties of the plant and exosystem with adaptation, the question of robustness is addressed. In this setting, a control is formulated based on a model of the composite system, given by

$$\begin{aligned} \dot{x} &= f(x, \theta^\circ) + g(x, \theta^\circ)u + p(x, \theta^\circ)w \\ \dot{w} &= s(w, \theta^\circ) \\ e &= h(x) + q(w) \end{aligned} \tag{8}$$

where θ° is a *fixed* estimate of θ^*. Assume the following:

A 1 *A certainty equivalence feedback law of the form*

$$u = \alpha(x, w, \theta^\circ) = c(w, \theta^\circ) + K^\circ[x - \pi(w, \theta^\circ)] \tag{9}$$

is applied to the actual composite system (3), where

1. $K^\circ \equiv K(\theta^\circ)$ *is such that* $K(\theta^*)$ *asymptotically stabilizes the linear approximation of (5) at* $x = 0$.

2. $x = \pi(w, \theta^*)$ *and* $u = c(w, \theta^*)$ *satisfy the conditions of theorem 2.1 for the system (3).*

The stability of the composite system with (9) as input is now examined.

Theorem 3.1 (Bounded-error manifold) *Under the assumptions (A1) and (H2), there exists neighborhoods* $V \subset \mathbb{R}^p$ *of* $\theta^\circ = \theta^*$ *and* $W^\circ \subset W$ *of* $w = 0$ *such that the composite system (3) with (9) as input has an invariant manifold, the graph of a* $C^k (k \geq 2)$ *mapping*

$$x = \Psi(w, \theta^\circ, \theta^*)$$

defined on $W^\circ \times V$ satisfying the condition

$$\begin{array}{rcl} \frac{\partial \Psi}{\partial w} s(w,\theta^*) & = & f(\Psi,\theta^*) + p(\Psi,\theta^*)w \\ & & g(\Psi,\theta^*)\left[c(w,\theta^\circ) + K^\circ(\Psi - \pi)\right] \end{array} \tag{10}$$

As a consequence, the solution (9) to the state feedback regulator problem based on (8) yields bounded tracking error when applied to (3).

Sketch of Proof. First defined $\phi = \theta^\circ - \theta^*$ and replace θ° by $\phi + \theta^*$. Next augment the exosystem with $\dot{\phi} = 0$. From the triangular structure and the assumptions concerning the eigenvalues of the plant disconnected from the exosystem, it follows that the closed loop composite system can be transformed into coordinates in which center manifold theory directly applies. In the original coordinates, and replacing ϕ by $\theta^\circ - \theta^*$, this manifold is the graph of mapping $x = \Psi(w, \theta^\circ, \theta^*)$ satisfying the condition (10). (see [2] for details of center manifold theory.)

Finally, by assumption the point $(x, w) = (0, 0)$ is a stable equilibrium of the closed loop composite system. Under this condition, for sufficiently small $(x(0), w(0))$, bounded tracking follows from center manifold theory and the continuity of h. □

Remarks:

1. The neighborhood V is at least as small as the largest open set such that, with θ° in that set, $K(\theta^\circ)$ also stabilizes the linear approximation of (5) at $x = 0$.

2. The manifold $\Psi(w, \theta^\circ, \theta^*)$ is conceptual and will not need to be calculated.

3. The preceding argument extends naturally to the output feedback regulator problem also described in [1].

4 Slowly-Varying Parameters

The question of robustness is now addressed, under the added assumption that the parameters are allowed to vary slowly. Consider any fixed θ° in a compact set $\Gamma \subset V$. Define $z = x - \Psi(w, \theta^\circ, \theta^*)$ on a neighborhood $Z^\circ \subset \mathbf{R}^n$ with Ψ defined by the previous theorem.

The dynamics of the state z are then

$$\begin{aligned} \dot{z} &= \dot{x} - \dot{\Psi}(w,\theta^\circ,\theta^*) \\ &= f(x,\theta^*) + g(x,\theta^*)\alpha(x,w,\theta^\circ) + p(x,\theta^*)w \\ &\quad -f(\Psi,\theta^*) - g(\Psi,\theta^*)\alpha(\Psi,w,\theta^\circ) - p(\Psi,\theta^*)w \\ &= F(z,w,\theta^\circ,\theta^*) \end{aligned} \tag{11}$$

From Theorem 3.1, for every $\theta^\circ \in \Gamma$, the following three conditions hold

1. F, F_z, F_{θ° are continuous on $Z^\circ \times W^\circ \times \Gamma$
2. For each $\theta^\circ \in \Gamma$, $z = 0$ is a twice continuously differentiable isolated root of $F(z,w,\theta^\circ,\theta^*) = 0$.
3. the equilibrium point $z = 0$ of (11) is uniformly asymptotically stable, uniformly in the parameter θ° with some set $B_r = \{z \in \mathbb{R}^n : |z| \le r\}$ contained in the domain of attraction.

These conditions are the requirements of the following useful lemma formulated by Hoppensteadt [3] and recently restated by Khalil,Kokotovic [4].

Lemma 4.1 (Hoppensteadt) *Under conditions (1-3), there exists a Lyapunov function $W(z,\theta^\circ,\theta^*)$ such that*

$$\begin{aligned} \kappa_1(|z|) \le W(z,\theta^\circ,\theta^*) &\le \kappa_2(|z|) \\ W_z(z,\theta^\circ,\theta^*)F(z,w,\theta^\circ,\theta^*) &\le -\kappa_3(|z|) \\ |W_{\theta^\circ}(z,\theta^\circ,\theta^*)| &\le c_1 \\ |W_z(z,\theta^\circ,\theta^*)| &\le c_2 \end{aligned} \tag{12}$$

for all $z \in B_r$ and $\theta^\circ \in \Gamma$, where $\kappa_1(\cdot)$, $\kappa_2(\cdot)$, $\kappa_3(\cdot)$ are strictly increasing functions and c_1 and c_2 are nonnegative constants.

With this Lyapunov function in hand, the slowly-varying analysis proceeds in the following way. Allow θ° to vary. The dynamics of the state z are now

$$\begin{aligned} \dot{z} &= \dot{x} - \tfrac{\partial\Psi}{\partial w}s(w,\theta^*) - \tfrac{\partial\Psi}{\partial\theta^\circ}\dot{\theta}^\circ \\ &= F(z,w,\theta^\circ,\theta^*) - \tfrac{\partial\Psi}{\partial\theta^\circ}\dot{\theta}^\circ \end{aligned} \tag{13}$$

Consider now the Lyapunov function of Lemma 4.1 and take its derivative along the trajectories of the system (13).

$$\begin{aligned} \dot{W} &= W_z\dot{z} + W_{\theta^\circ}\dot{\theta}^\circ \\ &= W_zF(z,w,\theta^\circ,\theta^*) + [W_{\theta^\circ} - W_z\tfrac{\partial\Psi}{\partial\theta^\circ}]\dot{\theta}^\circ \\ \dot{W} &\le -\kappa_3(|z|) + d_1|\dot{\theta}^\circ| \\ &\le -\kappa(W) + d_1|\dot{\theta}^\circ| \end{aligned} \tag{14}$$

where $\kappa = \kappa_3 \circ \kappa_2^{-1}$ and $d_1 = c_1 + c_2 sup_\Gamma(\frac{\partial \Psi}{\partial \theta^\circ})$. To show that z is stable for small $|z(t_0)|$ and sufficiently small $|\dot{\theta}^\circ|$ observe that the set $D = \{W \leq \kappa_1(q)\}$ is an invariant set under the condition

$$|\dot{\theta}^\circ| \leq \kappa(\kappa_1(q))/d_1$$

If $|z(t_0)| \leq \kappa_2^{-1}(\kappa_1(q))$ for any $q \leq r$, then, from (12), $W(t_0) \in D$. Since D is invariant, (12) implies that $|z(t)| \leq q$, $\forall t \geq t_0$. In addition, if $\dot{\theta}^\circ \to 0$ as $t \to \infty$ then $|z(t)| \to 0$ as $t \to \infty$ since $W(z(t), \theta^\circ(t), \theta^*) \to 0$ as $t \to \infty$.

The analysis above is formulated to allow for slow variations in the control parameter θ°. Note, however, that the analysis readily extends to incorporate slow variations in plant and exosystem parameters $\tilde{\theta}$ in a compact set about the nominal value θ^*. To retain stability unformly in the parameters we assume

A 2 *For all $\tilde{\theta} \in \Gamma$, $w = 0$ is a stable equilibrium of the exosystem, and there is an open neighborhood of the point $w = 0$ in which every point is Poisson stable.*

Corollary 4.1 *Under the assumptions (A1) and (A2), for sufficiently small initial conditions $(x(0), w(0))$, the stability of the composite system (3) under the input (9) is robust to plant and exosystem parameter variations that are sufficiently slow and stay in a neighborhood of the nominal parameter value θ^*.*

Proof. Follows immediately from the previous lemma and discussion. □

The previous discussion is now applied to a generic indirect adaptive control scheme. Consider the composite adaptive system,

$$\begin{array}{rcl} \dot{x} & = & f(x,\theta^*) + g(x,\theta^*)\alpha(x,w,\theta^\circ) + p(x,\theta^*)w \\ \dot{w} & = & s(w,\theta^*) \\ \dot{\theta}^\circ & = & G(x,w,\theta^\circ,t) \\ e & = & h(x) + q(w) \end{array} \tag{15}$$

Corollary 4.2 *Under the assumptions (A1) and (H2), for sufficiently small initial conditions $(x(0), w(0))$, the stability of the composite system (3) under the input (9) is robust to parameter variations in the control law that are sufficiently slow and stay in a neighborhood of the nominal plant parameter value θ^*. Namely, the stability of (15) is achieved if $sup_{t\geq t_0}|G(x,w,\theta^\circ,t)|$ is sufficiently small and G is such that θ° stays in Γ.*

Remark. Because the parameter update law is a function of x and θ°, some additional analysis will be required to guarantee a sufficiently small bound on $sup_{t \geq t_0}|G(x, w, \theta^{\circ}, t)|$.

Corollary 4.3 *Under assumptions (A1) and (H2), for (15), for sufficiently small initial conditions* $(x(0), w(0))$ *and* $sup_{t \geq t_0}|G(x, w, \theta^{\circ}, t)|$ *sufficiently small and* $\theta^{\circ}(t) \in \Gamma$, *if* θ° *converges to some* $\tilde{\theta}$ *then* x *converges to* $\Psi(w, \tilde{\theta}, \theta^*)$ *and the steady state error,* $e(t)$, *of system (15) is bounded and given by*

$$h(\Psi(w, \tilde{\theta}, \theta^*)) - h(\pi(w, \theta^*))$$

Proof. This is the case of $\dot{\theta}^{\circ} \to 0$ as $t \to \infty$ and $\dot{\theta}^* = 0$ so that $z(t) \to 0$ as $t \to \infty$. By definition of z, x converges to $\Psi(w, \tilde{\theta}, \theta^*)$. Then by the continuity of h and the stability of the composite system, the steady state error is bounded and given by $h(\Psi(w, \tilde{\theta}, \theta^*)) - h(\pi(w, \theta^*))$. □

Corollary 4.4 *Under assumptions (A1) and (H2), for (15), for sufficiently small initial conditions* $(x(0), w(0))$ *and* $sup_{t \geq t_0}|G(x, w, \theta^{\circ}, t)|$ *sufficiently small and* $\theta^{\circ}(t) \in \Gamma$, *if* θ° *converges to* θ^* *then*

$$lim_{t \to \infty} e(t) = 0$$

Proof. Here x converges to $\Psi(w, \theta^*, \theta^*)$. Observe that $\Psi(w, \theta^*, \theta^*)$ satisfies the same partial differential equation as $\pi(w, \theta^*)$ since $\Psi(w, \theta^*, \theta^*)$ is the manifold made invariant by the input (9) with $\theta^{\circ} = \theta^*$. Thus, from the properties of center manifolds, x converges to the $\pi(w, \theta^*)$ of Theorem 2.1. Then Theorem 2.1 implies that $lim_{t \to \infty} e(t) = 0$. □

Remark. Typically, it is not possible to guarantee correct parameter convergence a priori without additional assumptions.

5 Adaptive Nonlinear Output Regulation

The last result of the previous section suggested that if an identifier could be constructed that guaranteed θ° converges to θ^* then asymptotic tracking would be guaranteed as well. However, as is known in the adaptive literature, guaranteeing parameter convergence a priori requires additional assumptions. Rather than take that approach here, a specific identifier will be suggested that will result in asymptotic tracking. This identifier is formulated in the mind-set of indirect adaptive control. Namely, an identifier is constructed to estimate plant parameters and then these parameters are used in a certainty equivalence control law. The identifier used here is analogous to the observer-based identifier found in [5].

We prepare by making two additional assumptions.

A 3 *For all $\theta^\circ \in \Gamma$, $x = \pi(w, \theta^\circ)$ and $u = c(w, \theta^\circ)$ satisfy the conditions*

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w, \theta^\circ) &= f(\pi, \theta^\circ) + p(\pi, \theta^\circ) w \\ & \quad g(\pi, \theta^\circ) c(w, \theta^\circ) \\ h(\pi(w, \theta^\circ)) + q(w) &= 0 \end{aligned}$$

Consider again the composite system (3) where θ^* is considered a constant but unknown parameter vector. The following standard assumption for adaptive systems is made.

A 4 *The vector fields $f(x, \theta^*)$ and $s(w, \theta^*)$ and the columns of $g(x, \theta^*)$ and $p(x, \theta^*)$ have the following linear parameter dependence:*

$$\begin{aligned} f(x, \theta^*) &= \sum_{i=1}^{p} \theta_i^* f_i(x) \\ g_j(x, \theta^*) &= \sum_{i=1}^{p} \theta_i^* g_{i,j}(x) \\ p_j(x, \theta^*) &= \sum_{i=1}^{p} \theta_i^* p_{i,j}(x) \\ s(w, \theta^*) &= \sum_{i=1}^{p} \theta_i^* s_i(w) \end{aligned}$$

where θ_i^*, $i = 1, \ldots, p$ are unknown parameters, which appear linearly, and the smooth vector fields $f_i(x)$, $g_{i,j}(x)$, $p_{i,j}(x)$, $s_i(w)$ are known.

Regressors are formed as

$$\begin{aligned} \chi_x^T(x, w, u) &= [f_1(x) + g_{1,j}(x) u_j + p_{1,k}(x) w_k, \ldots, \\ & \quad f_p(x) + g_{p,j}(x) u_j + p_{p,k}(x) w_k] \\ \chi_w^T(w) &= [s_1(w), \ldots, s_p(w)] \end{aligned}$$

where summation over j, k is implied. Consequently, $\chi_x^T(x, w, u) \in \mathbb{R}^{n \times p}$ and $\chi_w^T(w) \in \mathbb{R}^{s \times p}$ contain all of the nonlinearities of the system. Now the composite system can be written as

$$\begin{aligned} \dot{x} &= \chi_x^T(x, w, u) \theta^* \\ \dot{w} &= \chi_w^T(w) \theta^*. \end{aligned}$$

In what follows, the conventional notation for estimates of unknown parameters, $\hat{\theta}$, will replace the previously used θ°. To estimate the unknown parameters, the following identifier system is used.

$$\begin{aligned} \dot{\hat{x}} &= \Omega_x(\hat{x} - x) + \chi_x^T(x, w, u) \hat{\theta} \\ \dot{\hat{w}} &= \Omega_w(\hat{w} - w) + \chi_w^T(w) \hat{\theta} \\ \dot{\hat{\theta}} &= -\rho \chi_x(x, w, u) P_x(\hat{x} - x) - \rho \chi_w(w) P_w(\hat{w} - w) \end{aligned} \tag{16}$$

Here $\Omega_x \in \mathbb{R}^{n \times n}$, $\Omega_w \in \mathbb{R}^{s \times s}$ are Hurwitz matrices and $P_x \in \mathbb{R}^{n \times n}$, $P_w \in \mathbb{R}^{s \times s}$ are positive definite symmetric solutions to the Lyapunov equations

$$\begin{aligned} \Omega_x^T P_x + P_x \Omega_x &= -I_{n \times n} \\ \Omega_w^T P_w + P_w \Omega_w &= -I_{s \times s}. \end{aligned}$$

Finally, ρ is a small positive constant. Now, define $\varepsilon_x = \hat{x} - x$, $\varepsilon_w = \hat{w} - w$, and $\phi = \hat{\theta} - \theta^*$. Then the identifier error system becomes

$$\begin{aligned} \dot{\varepsilon}_x &= \Omega_x \varepsilon_x + \chi_x^T(x, w, u)\phi \\ \dot{\varepsilon}_w &= \Omega_w \varepsilon_w + \chi_w^T(w)\phi \\ \dot{\phi} &= -\rho \chi_x(x, w, u) P_x \varepsilon_x - \rho \chi_w(w) P_w \varepsilon_w. \end{aligned} \tag{17}$$

Theorem 5.1 *Under the assumptions (A1), (H2), (A3) and (A4), for sufficiently small initial conditions $(x(0), w(0))$ and $(\varepsilon_x(0), \varepsilon_w(0), \phi(0))$, for the composite system (3) under (adaptive) input (9), $\exists \rho > 0$ of the identifier (16) such that*

1. $\phi \in L_\infty$,
2. $\varepsilon_x, \varepsilon_w \in L_\infty \cap L_2$,
3. $(x, w) \in L_\infty$,
4. $\dot{\varepsilon}_x, \dot{\varepsilon}_w \in L_\infty$,
5. $\lim_{t \to \infty} \varepsilon_x(t) = \lim_{t \to \infty} \varepsilon_w(t) = 0$,
6. $\lim_{t \to \infty} e(t) = h(x(t)) + q(w(t)) = 0$.

Proof. Consider the Lyapunov function

$$V(\varepsilon_x, \varepsilon_w, \phi) = \rho \varepsilon_x^T P_x \varepsilon_x + \rho \varepsilon_w^T P_w \varepsilon_w + \phi^T \phi \tag{18}$$

Taking the derivative of V along the trajectories of (17) yields

$$\dot{V} = -\rho \varepsilon_x^T \varepsilon_x - \rho \varepsilon_w^T \varepsilon_w \leq 0$$

Hence $0 \leq V(t) \leq V(0)$ for all $t \geq 0$, so that $V, \phi, \varepsilon_x, \varepsilon_w \in L_\infty$. Since V is a positive, monotonically decreasing function, the limit $V(\infty)$ is well-defined and

$$-\int_0^\infty \dot{V}\, d\tau = \rho \int_0^\infty (\varepsilon_x^T \varepsilon_x + \varepsilon_w^T \varepsilon_w)\, d\tau < \infty$$

so that ε_x,$\varepsilon_w \in L_2$.

It is now shown that ρ can be chosen so that the analysis of section 4 holds. This will imply that x remains bounded. Consider the parameter update law

$$\dot{\hat{\theta}} = \dot{\phi} = -\rho\chi_x(x, w, u)P_x\varepsilon_x - \rho\chi_w(w)P_w\varepsilon_w$$

Since χ_x,χ_w are smooth, w is bounded and ε_x, $\varepsilon_w \in L_\infty$, it follows that $\exists m > 0$ and a strictly increasing function $\kappa_4(.)$ such that

$$|\dot{\hat{\theta}}| \leq \rho(m + \kappa_4(|z|))$$

for all $z \in B_r = \{z \in \mathbf{R}^n : |z| \leq r\}$. Then for the Lyapunov function of section (4), equation (14) becomes

$$\begin{aligned} \dot{W} &\leq -\kappa_3(|z|) + d_1|\dot{\hat{\theta}}| \\ &\leq -\kappa_3(|z|) + d_1\rho(m + \kappa_4(|z|)) \\ &\leq -\kappa(W) + \rho d_1\kappa_5(W) + \rho d_1 m \end{aligned}$$

where $\kappa = \kappa_3 \circ \kappa_2^{-1}$ and $\kappa_5 = \kappa_4 \circ \kappa_1^{-1}$. Now pick ρ_o sufficiently small such that $(\kappa - \rho_o d_1\kappa_5)(.)$ is a strictly increasing function of W. Define $\kappa_6 = (\kappa - \rho_o d_1\kappa_5)$. Then

$$\dot{W} \leq -\kappa_6(W) + \rho d_1 m$$

for all $\rho \leq \rho_o$. Now observe that the set $D = \{W \leq \kappa_1(q)\}$ for any $q \leq r$ is an invariant set if

$$\rho \leq \kappa_6(\kappa_1(q))/(d_1 m) \equiv \rho_1$$

Hence, if ρ is chosen such that $\rho \leq \min\{\rho_o, \rho_1\}$ then D is an invariant set. Finally, if $|z(t_0)| \leq \kappa_2^{-1}(\kappa_1(q))$, then from (12) $W(t_0) \in D$. Since D is invariant (12) implies that $|z(t)| \leq q$ for all $t \geq t_0$.

Because z is bounded and w is bounded by assumption, x is bounded. Since x, w are bounded, $\chi_x(x, w, u)$, $\chi_w(w)$ are bounded. This implies $\dot{\varepsilon}_x$, $\dot{\varepsilon}_w$ are bounded. Since $\varepsilon_x, \dot{\varepsilon}_x, \varepsilon_w, \dot{\varepsilon}_w \in L_\infty$ and $\varepsilon_x, \varepsilon_w \in L_2$, $\lim_{t\to\infty} \varepsilon_x = \lim_{t\to\infty} \varepsilon_w = 0$.

Finally, the convergence of the tracking error is proved. Return to the Lyapunov function of (18). The nontrivial trajectories corresponding to $\dot{V} = 0$ are given by the set

$$\begin{aligned} &S = \{(\varepsilon_x, \varepsilon_w, \phi) : \\ &\varepsilon_x = 0, \varepsilon_w = 0, \chi_x^T(x, w, u)\phi = 0, \chi_w^T(w)\phi = 0\} \end{aligned}$$

From the definition of ϕ, trajectories in this set are such that

$$\begin{aligned} \chi_x^T(x,w,u)\hat{\theta} &= \chi_x^T(x,w,u)\theta^* \\ \chi_w^T(w)\hat{\theta} &= \chi_w^T(w)\theta^* \end{aligned} \tag{19}$$

From Theorem 3.1, $\Psi(w,\hat{\theta},\theta^*)$ satisfies the condition

$$\frac{\partial \Psi}{\partial w}\chi_w^T(w)\theta^* = \chi_x^T(\Psi(w,\hat{\theta},\theta^*),w,u)\theta^* \tag{20}$$

Further, by assumption (A3), $\pi(w,\hat{\theta})$ satisfies the condition

$$\frac{\partial \pi}{\partial w}\chi_w^T(w)\hat{\theta} = \chi_x^T(\pi(w,\hat{\theta}),w,u)\hat{\theta} \tag{21}$$

From (19), $\pi(w,\hat{\theta})$ also satisfies

$$\frac{\partial \pi}{\partial w}\chi_w^T(w)\theta^* = \chi_x^T(\pi(w,\hat{\theta}),w,u)\theta^* \tag{22}$$

Now, $\lim_{t\to\infty}\varepsilon_x = \lim_{t\to\infty}\varepsilon_w = 0$ implies $\lim_{t\to\infty}\dot{\phi} = 0$. So from Corollary 4.3, x converges to $\Psi(w,\hat{\theta},\theta^*)$. Now since $\pi(w,\hat{\theta})$ satisfies the same manifold equation as $\Psi(w,\hat{\theta},\theta^*)$, the properties of center manifolds imply that x converges to $\pi(w,\hat{\theta})$. From assumption (A3), $q(w) = -h(\pi(w,\hat{\theta}))$. Then, from the continuity of h,

$$\begin{aligned} \lim_{t\to\infty} e(t) &= \lim_{t\to\infty} h(x(t)) + q(w(t)) \\ &= \lim_{t\to\infty} h(x(t)) - h(\pi(w,\hat{\theta})) = 0. \end{aligned}$$

□

6 Conclusion

This paper has analyzed the dynamics of a system with parameter uncertainties in the setting of nonlinear regulation. For small initial conditions, the nonlinear regulator solutions were shown to be robust to parameter uncertainties and to slowly-varying parameters. The adaptive nonlinear regulator solution was cast into this slowly-varying framework. It was shown then that there exists an identifier with sufficiently small gains that, in conjunction with a certainty equivalence control law, yielded zero error tracking in the limit.

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