RECENT RESULTS CONCERNING THE INPUT - OUTPUT PROPERTIES OF LINEAR TIME - INVARIANT SYSTEMS

by

C. A. Desoer and F. L. Lam

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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Introduction.

The purpose of this report is to collect in a single document a number of recent results concerning input-output stabilitity theory. Sufficient conditions for the L^p-stability of multiple-input, multiple-output linear time-invariant systems were given in [1] for the continuous-time case. Better sufficient conditions were given for the discrete-time case in [2] and [3]. For the single-input single-output continuous-time case, Baker and Vakharia showed how to take care of multiple poles in the closed right half plane, [4]. Theorems 1, 2 and 3 and Corollarys 2.1 and 3.1 improve upon the results in [1], [2], [3] and [4]. Some of the techniques used were stimulated by Vidyasagar's recent work [5]. For completeness, we include two theorems from [6]: these theorems, numbered 4 and 5, show for the multiple-input multiple-output case that the sufficient conditions of Desoer and Wu in [1] and [3] are indeed necessary in a much more general setting. Theorem 4 is followed by comments which give an intuitive understanding of the mechanism whereby these conditions are necessary.

Notations.

In the following, $\mathbb{R}$ ($\mathbb{C}$) denotes the field of real (complex) numbers. $\mathbb{R}_+$ denotes the nonnegative real numbers. $\mathbb{R}^n$ ($\mathbb{R}^{nxn}$) denotes the set of all n-vectors (nxn matrices) with elements in $\mathbb{R}$. $\mathbb{C}^n$ and $\mathbb{C}^{nxn}$ are similarly defined. For any $\sigma \in \mathbb{R}$, $\mathcal{A}(\sigma)$ denotes the Banach algebra, [1], (where "+" is the pointwise addition and product is the convolution) of generalized functions of the form:

$$f(t) = \begin{cases} f_a(t) + \sum_{i=0}^{\infty} f_i \, \delta(t-t_i) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{1}$$

where $t \mapsto f_a(t)e^{-\sigma t}$ is in L^1; with $0 = t_0 < t_1 < \cdots$, $f_i \in \mathbb{R}$, $\forall_i$, and $\sum_{i=0}^{\infty} |f_i| e^{-\sigma t_i} < \infty$. $\mathcal{A}^n(\sigma)$ ($\mathcal{A}^{nxn}(\sigma)$) denotes the set of all n-vectors (nxn matrices) with components in $\mathcal{A}(\sigma)$. If $\sigma = 0$, we write $\mathcal{A}$ instead of $\mathcal{A}(0)$.

The superscript $\hat{}$ ($\tilde{}$) denotes Laplace transforms: $\hat{f} = \mathcal{L}[f]$. (z-transforms: $\tilde{f} = \mathcal{Z}[f]$). For a treatment of analytic functions taking values in $\mathbb{C}^{nxn}$ see [7].

<u>Results</u>.

We consider below an n-input, n-output, linear, time-invariant feedback system: it has unity feedback and its open-loop gain is the nxn matrix transfer function $\hat{G}(s)$ in the continuous-time case and $\tilde{G}(z)$ in the discrete-time case.

It is important to note that for the case where $\hat{G}(s)$ is a <u>proper rational-function</u> matrix, the necessary and sufficient conditions for stability are known (Theor. 9-10 of [8]).

I. <u>Sufficient Conditions</u>.

<u>Theorem 1</u>. (Continuous-time) Suppose that

$$\hat{G}(s) = \hat{G}_a(s) + \sum_{i=0}^{\infty} G_i \, e^{-st_i} + \sum_{\alpha=1}^{k} \sum_{\beta=1}^{m_\alpha} \frac{R_{\alpha\beta}}{(s-p_\alpha)^\beta} \tag{2}$$

$$\triangleq \hat{G}_{\ell}(s) + \sum_{\alpha=1}^{k}\sum_{\beta=1}^{m_\alpha} \frac{R_{\alpha\beta}}{(s-p_\alpha)^\beta} \tag{3}$$

where

(a) $\hat{G}_{\ell}(\cdot) \in \hat{\mathcal{A}}^{n\times n}(\sigma)$ for some $\sigma \in \mathbb{R}$;

(b) $R_{\alpha\beta} \in \mathbb{C}^{n\times n}$ for $\beta = 1, 2, \cdots, m_\alpha$, $\alpha = 1, 2, \cdots, k$

(c) for $\alpha = 1, 2, \cdots, k$, $\mathrm{Re}[p_\alpha] \geq \sigma$; and $p_\alpha \neq p_{\alpha'}$ for $\alpha \neq \alpha'$.

Under these conditions, if

(i) $$\det R_{\alpha m_\alpha} \neq 0 \quad \text{for } \alpha = 1, 2, \cdots, k \tag{4}$$

and if

(ii) $$\inf_{\mathrm{Re}\, s \geq \sigma} \left|\det[I + \hat{G}(s)]\right| > 0, \tag{5}$$

then the closed-loop impulse response, $H(\cdot)$, is in $\mathcal{A}^{n\times n}(\sigma)$.

<u>Comments</u>.

(a) For $\sigma = 0$, the conclusion implies that, for any $p \in [1,\infty]$, any input $u \in L_n^p$ produces an output $y \in L_n^p$ and $\|y\|_p \leq \|H\|_a \cdot \|u\|_p$, where $\|\cdot\|_a$ is the norm of H as an element of $\mathcal{A}^{n\times n}$, [1]. It is straightforward to show that similar results hold for $\sigma \neq 0$.

(b) For $\sigma = 0$, suppose that there is only <u>one simple</u> pole in the closed right half plane $\mathrm{Re}\, s \geq 0$ and that this pole is at $s = 0$. Then by the methods of [1], if, as $t \to \infty$, $u(t) \to u_\infty$ (any constant vector), then $y(t) \to u_\infty$. (Again, similar results hold for $\sigma \neq 0$ and the simple pole located at $s = \sigma$.) It is easy to show that if $\det R_1 = 0$, then inputs tending to some constant vectors give rise to nonzero steady-state error. (For the method of proof see [2]).

(c) Assumption (4) is more general than that in [1] and in [4] in that the matrix is only required to have its eigenvalues different from zero and that multiple-input multiple-output systems are considered.

(d) Completely analogous results hold for the discrete-time case and are available in [2].

(e) This theorem can also be derived by the technique of Vidyasagar [5].

<u>Proof</u>. Let $\hat{\phi}$: $\mathbb{C} \mapsto \mathbb{C}$ with
$$\hat{\phi}(s) \triangleq \prod_{\alpha=1}^{k} \left[\frac{s - p_\alpha}{s-\sigma + 1}\right]^{m_\alpha} \tag{6}$$

and note that $\hat{\phi} \in \hat{\mathcal{A}}(\sigma)$. Observe that the closed-loop transfer function is

$$\hat{H}(s) = \hat{G}(s)[I + \hat{G}(s)]^{-1} = \hat{\phi}(s)\ \hat{G}(s)[(I + \hat{G}(s))\hat{\phi}(s)]^{-1} \tag{7}$$

Let $U = \{s \in C \mid \text{Re } s \geq \sigma, |s - p_\alpha| \geq \varepsilon, (\alpha = 1, 2, \cdots, k)$, for some $\varepsilon > 0$ sufficiently small$\}$.

Then (5) implies that

$$\inf_{s \in U} |\det[(I + \hat{G}(s))\hat{\phi}(s)]| > 0 \tag{8}$$

So it remains to check the behavior of the determinant in the neighborhood of the poles p_α's. Now by (6), as $s \to p_\alpha$, $\hat{\phi}(s) \to \hat{\phi}(p_\alpha) = 0$, and by (3) and (6), as $s \to p_\alpha$, $\alpha = 1, 2, \cdots, k$,

$$[I + \hat{G}(s)]\hat{\phi}(s) \to \frac{R_{\alpha m_\alpha}}{(p_\alpha - \sigma + 1)^{m_\alpha}} \prod_{\gamma \neq \alpha} \left(\frac{p_\alpha - p_\gamma}{p_\alpha - \sigma + 1}\right)^{m_\gamma} . \tag{9}$$

By assumption (4), the determinant of $R_{\alpha m_\alpha}$ is nonzero, hence the infimum in (8) can be taken over Re $s \geq \sigma$. Therefore by a standard reasoning, [1], the two factors in the right hand side of (7) are both in $\hat{\mathcal{A}}^{n\times n}(\sigma)$; hence so is $\hat{H}$. ☒

In the next theorem and corollary, we consider the case of simple poles with singular residue matrices. This case is obviously important in practice.

Theorem 2. (Continuous-time) Suppose that $\hat{G}(s)$ is given by (3) and that $k = 1$ and $m_1 = 1$ (i.e. $\hat{G}$ has only a simple pole, p_1, in the closed half plane $\text{Re } s \geq \sigma$). Suppose also that the residue matrix R_{11} is singular. Under these conditions, if[†]

(i) $\det[\hat{M}_{22}(p_1)] \neq 0,$ (10)

and if

(ii) $\inf_{\text{Re } s \geq \sigma} |\det[I + \hat{G}(s)]| > 0$ (11)

then the closed-loop impulse response $H(\cdot)$ is in $\mathcal{A}^{nxn}(\sigma)$.

Proof. What we have to establish is equivalent to proving that

$$[I + \hat{G}(\cdot)]^{-1} \in \hat{\mathcal{A}}^{nxn}(\sigma). \tag{12}$$

If P_1 and Q_1 are nxn nonsingular constant matrices (with complex elements), then (12) is equivalent to

$$[Q_1(I + \hat{G}(\cdot))P_1]^{-1} \in \hat{\mathcal{A}}^{nxn}(\sigma) \tag{13}$$

Let rank $R_{11} = r$, so $r < n$; then select Q_1 and P_1 so that

$$Q_1 R_{11} P_1 = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] \tag{14}$$

where I_r is the rxr unit matrix [11]. The constant matrices Q_1 and P_1 are

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[†]$\hat{M}_{22}(s)$ is defined in the proof; see equation (15) below.

easily determined in terms of elementary row and column operations. Thus

$$Q_1(I + \hat{G}(s))P_1 = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] \frac{1}{s - p_1} + \left[\begin{array}{c|c} \hat{M}_{11}(s) & \hat{M}_{12}(s) \\ \hline \hat{M}_{21}(s) & \hat{M}_{22}(s) \end{array}\right] \tag{15}$$

where all the elements of the second matrix are in $\hat{\mathcal{A}}(\sigma)$. Let $\hat{\phi}_1$: $\mathbb{C} \to \mathbb{C}$ with

$$\hat{\phi}_1(s) \triangleq \frac{s - p_1}{s - \sigma + 1} \tag{16}$$

and
$$\hat{D}_1(s) \triangleq \mathrm{diag}\{\hat{\phi}_1(s), \hat{\phi}_1(s), \cdots, \hat{\phi}_1(s), 1, 1, \cdots, 1\} \tag{17}$$

where $\hat{D}_1(s)$ contains exactly n-r diagonal elements equal to 1. Note that $\hat{D}_1(\cdot) \in \hat{\mathcal{A}}^{n\times n}(\sigma)$. Observe that

$$[Q_1(I + \hat{G}(s))P_1]^{-1} = \hat{D}_1(s)[Q_1(I + \hat{G}(s))P_1\hat{D}_1(s)]^{-1}. \tag{18}$$

The theorem will be proved (or equivalently, (13) will be established) if we prove that

$$[Q_1(I + \hat{G}(\cdot))P_1\hat{D}_1(\cdot)]^{-1} \in \hat{\mathcal{A}}^{n\times n}(\sigma). \tag{19}$$

Now assumption (11) implies that

$$\inf_{\mathrm{Re}\, s \geq \sigma} |\det[Q_1(I + \hat{G}(s))P_1]| > 0.$$

Consequently,
$$\inf_{s \in N_1} |\det[Q_1(I + \hat{G}(s))P_1\hat{D}_1(s)]| > 0 \tag{20}$$

where N_1 is the closed half plane Re s $\geq \sigma$ with a small neighborhood of p_1 deleted. So we study the behavior around p_1. As $s \to p_1$, since $\hat{\phi}(p_1) = 0$,

(15) gives

$$Q_1[I + \hat{G}(s)]P_1\hat{D}_1(s) \to \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] \frac{1}{p_1 - \sigma + 1} + \left[\begin{array}{c|c} 0 & \hat{M}_{12}(p_1) \\ \hline 0 & \hat{M}_{22}(p_1) \end{array}\right] \tag{21}$$

Hence

$$\det[Q_1(I + \hat{G}(s))P_1\hat{D}_1(s)] \to \left(\frac{1}{p_1 - \sigma + 1}\right)^r \det \hat{M}_{22}(p_1) \quad \text{as } s \to p_1 \tag{22}$$

By assumption (10), this limit is different from zero. Therefore, the continuity of the $\hat{M}_{ij}(s)$ in Re $s \geq \sigma$ and (20) imply, by a standard reasoning, [1], that (19) holds. This establishes the theorem. ⊠

Remark. The proof of Theorem 2 shows that for all s in the closed half plane Re $s \geq \sigma$,

$$[I + \hat{G}(s)]^{-1} = P_1\hat{D}_1(s)[Q_1(I + \hat{G}(s))P_1\hat{D}_1(s)]^{-1}Q_1 \tag{23}$$

where the right hand side expression is an analytic function mapping $\{s | \text{Re } s > \sigma\}$ into $\mathbb{C}^{nxn}$ and is in $\hat{\mathcal{A}}^{nxn}(\sigma)$.

Corollary 2.1. Suppose that $\hat{G}(s)$ is given by (3) but that $k > 1$ and $m_\alpha = 1$ for $\alpha = 1, 2, \cdots, k$ (i.e. $\hat{G}(s)$ has only simple poles in Re $s \geq \sigma$). Suppose also that

(i) either $\det R_{\alpha 1} \neq 0$ (24)

or, whenever $\det R_{\alpha 1} = 0$ we have

$$\det[\hat{M}_{22}(p_\alpha)] \neq 0, \tag{25}$$

and

(ii) $$\inf_{\mathrm{Re}\, s \geq \sigma} \left|\det[I + \hat{G}(s)]\right| > 0 \tag{26}$$

Then the closed-loop impulse response H is in $\mathcal{A}^{n\times n}(\sigma)$.

Proof. Consider a covering of the closed half plane Re $s \geq \sigma$ with k open subsets S_α such that for $\alpha = 1, 2, \ldots, k$, S_α includes one and only one pole of $\hat{G}(s)$, namely p_α. Since each S_α is open, it includes an open neighborhood about p_α. By Theorems 1 and 2, in view of assumptions (24), (25) and (26), on each S_α, $[I + \hat{G}]^{-1}$ is equal to an analytic function which is in $\hat{\mathcal{A}}^{n\times n}(\sigma)$. Hence $[I + \hat{G}]^{-1}$ is in $\hat{\mathcal{A}}^{n\times n}(\sigma)$ and hence $\hat{H}(\cdot) \in \hat{\mathcal{A}}^{n\times n}(\sigma)$. ☒

In the discrete-time case, the impulse response is specified as a sequence of matrices in $\mathbb{C}^{n\times n}$(or $\mathbb{R}^{n\times n}$) say, $(G_0, G_1, G_2, \cdots\cdots)$. We say that a sequence belongs to $\ell^1_{n\times n}(\rho)$ for some positive real number ρ iff $\sum_{k=0}^{\infty} \|G_k\| \rho^{-k} < \infty$, and we say that its corresponding z-transform $\tilde{G}(z) = \sum_{0}^{\infty} G_k z^{-k}$ is in $\tilde{\ell}^1_{n\times n}(\rho)$. The analogous results of Theorem 1 for the discrete-time case can be found in [2]. We state below in Theorem 3 and Corollary 3.1 the discrete-time analogs to Theorem 2 and Corollary 2.1.

Theorem 3. (Discrete-time) Suppose that $\tilde{G}(z)$ is given by

$$\tilde{G}(z) = \sum_{0}^{\infty} G_i z^{-i} + \frac{R_{11}}{(z-p_1)} \tag{27}$$

$$\triangleq \tilde{G}_\ell(z) + z^{-1}(1 - p_1 z^{-1})^{-1} R_{11} \tag{28}$$

where

(a) $\tilde{G}_\ell(\cdot) \in \tilde{\ell}^1_{n\times n}(\rho)$ for some positive real ρ,

(b) $p_1 \in \mathbb{C}$ and $|p_1| \geq \rho$

(c) $R_{11} \in \mathbb{C}^{nxn}$ is <u>singular</u>.

Under these conditions, if[††]

$$\text{(i)} \quad \det[\tilde{M}_{22}(p_1)] \neq 0. \tag{29}$$

and if

$$\text{(ii)} \quad \inf_{|z| \geq \rho} |\det[I + \tilde{G}(z)]| > 0 \tag{30}$$

Then the closed-loop impulse response $H \in \ell^1_{nxn}(\rho)$.

Corollary 3.1. Suppose that $\hat{G}(z)$ is given by

$$\tilde{G}(z) = \sum_0^{\infty} G_i z^{-i} + \sum_{\alpha=1}^{k} \frac{R_{\alpha 1}}{(z - p_\alpha)} \tag{31}$$

$$\triangleq \tilde{G}_\ell(z) + \sum_{\alpha=1}^{k} z^{-1}(1 - p_\alpha z^{-1})^{-1} R_{\alpha 1} \tag{32}$$

where

(a) $\tilde{G}_\ell(\cdot) \in \tilde{\ell}^1_{nxn}(\rho)$ for some positive real ρ,

(b) for $\alpha = 1, 2, \cdots, k$, $p_\alpha \in \mathbb{C}$, $|p_\alpha| \geq \rho$, and for $\alpha \neq \alpha'$, $p_\alpha \neq p_{\alpha'}$.

Under these conditions, if

$$\text{(i)} \quad \text{either } \det R_{\alpha 1} \neq 0 \tag{33}$$

or, whenever $\det R_{\alpha 1} = 0$, we have

$$\det[\tilde{M}_{22}(p_\alpha)] \neq 0, \tag{34}$$

[††] $\tilde{M}_{22}(z)$ is defined similarly as in Theorem 2. See equation (15).

and if

(ii) $$\inf_{|z| \geq \rho} |\det[I + \tilde{G}(z)]| > 0 \tag{35}$$

Then the closed-loop impulse response H is in $\ell^1_{nxn}(\rho)$.

Remark. The proofs of Theorem 3 and Corollary 3.1 are exact duplicates of those of Theorem 2 and Corollary 2.1 except that we define $\tilde{\phi}(z) \triangleq (1 - p_1 z^{-1})$ in this case. (See (16)), and we replace $\{s \in \mathbb{C} | \text{Re } s \geq \sigma\}$ by $\{z \in \mathbb{C} | \; |z| \geq \rho\}$

II. Necessary Conditions.

Theorem 4. (Continuous-time) Let $G(\cdot)$ be an nxn matrix whose elements are distributions on $\mathbb{R}_+$, [9]. Assume that these n^2 distributions are Laplace transformable and let $\hat{G} = \mathcal{L}[G]$. If, for some $\sigma \in \mathbb{R}$,

$$\hat{G}(\cdot)[I + \hat{G}(\cdot)]^{-1} \in \hat{\mathcal{A}}^{nxn}(\sigma) \tag{36}$$

then $$\inf_{\text{Re } s \geq \sigma} |\det[I + \hat{G}(s)]| > 0 . \tag{37}$$

Proof. Assumption (36) and the fact that $I \in \hat{\mathcal{A}}^{nxn}(\sigma)$ imply that

$$I - \hat{G}[I + \hat{G}]^{-1} = [I + \hat{G}]^{-1} \in \hat{\mathcal{A}}^{nxn}(\sigma). \tag{38}$$

Hence the function $s \mapsto \det\{[I + \hat{G}(s)]^{-1}\}$ is in $\hat{\mathcal{A}}$ and, consequently, is bounded in the closed half plane Re $s \geq \sigma$. If (37) does not hold, then there is a sequence $\{s_k\}_1^\infty$ in Re $s \geq \sigma$ such that $\det[I + \hat{G}(s_k)] \to 0$. Hence

$$\det\{[I + \hat{G}(s_k)]^{-1}\} = \frac{1}{\det[I + \hat{G}(s_k)]} \to \infty \tag{39}$$

as $k \to \infty$,

which contradicts the previous fact. Hence $|\det[I + \hat{G}(s)]|$ is bounded away from zero in the closed half plane Re $s \geq \sigma$. ☒

Comments. Perhaps a more illuminating way of understanding Theorem 4 is the following:

(1) From assumption (36), it follows that $[I + \hat{G}(\cdot)]^{-1}$ is bounded in Re $s \geq \sigma$ and is analytic in Re $s > \sigma$. Hence the function $\omega \mapsto \frac{1}{1 + \sigma_1 + j\omega} \cdot [I + \hat{G}(\sigma_1 + j\omega)]^{-1}$, for any $\sigma_1 \geq \sigma$, is in L^2 and converges to zero (uniformly in σ_1, with $\sigma_1 \geq \sigma$) as $|\omega| \to \infty$. Now suppose (37) is false, i.e. suppose that $\inf_{\text{Re } s \geq \sigma} |\det[I + \hat{G}(s)]| = 0$. One possibility is that the determinant function has some finite zeros in Re $s > \sigma$. Let z_1 be one of those which is farthest to the right. Then the standard techniques of L^2 Laplace transform theory ([10]) and of contour integration are available to show the existence of an exponential term $Pe^{z_1 t}$ in the inverse transform of $\frac{1}{1+s}[I + \hat{G}(s)]^{-1}$. It follows immediately that the inverse transform of $[I + \hat{G}(s)]^{-1}$ also has a term $Qe^{z_1 t}$, for multiplication of the transform by $(1+s)$ does not destroy the exponential term. (This assumes $z_1 \neq -1$; if z_1 were -1, we would then use $\frac{1}{2+s}$ instead of $\frac{1}{1+s}$ as a convergence factor). Then, from (38), it follows that $\hat{G}(\cdot)[I + \hat{G}(\cdot)]^{-1}$ has a term $-Qe^{z_1 t}$ and hence $\hat{H}(\cdot) \notin \hat{\mathcal{A}}^{n\times n}(\sigma)$, which is a contradiction.

(2) The second possibility would be that $\det[I + \hat{G}(s)]$ approaches zero along a sequence $\{s_k\}_1^\infty$ in Re $s \geq \sigma$ such that $|s_k| \to \infty$ as $k \to \infty$. To discuss this case let us assume that $\hat{G}(\cdot)$ is given by (3) and that Re $s_k \geq \varepsilon > 0$ for large k. Under these conditions $\hat{G}(s_k) \to G_0$ as $k \to \infty$: indeed $\hat{G}_a(s_k) \to 0$ by the Riemann-Lebesgue lemma, and

$e^{-s_k t_i} \to 0$ for $i = 1, 2, \cdots$, since $t_i > 0$. Thus we have $\det[I + \hat{G}(s_k)] \to \det[I + G_0] = 0$. Now it is well-known that when that last condition is obtained, the closed-loop system is not a dynamical system. Indeed, for some well-behaved inputs, it does not have a well-defined response: e.g. consider an input $u(t) = u\delta(t)$, where $0 \neq u \in \mathbb{C}^n$ and u is outside the range of $(I + G_0)$, then the error is not defined; moreover, even if u is in the range of $(I + G_0)$, the $\delta(t)$ term in the system error, e, is not uniquely defined.

Theorem 5. (Discrete-Time) Suppose $\tilde{G}(z)$ has a positive radius of convergence ρ as a function of z^{-1}. If

$$\tilde{G}(\cdot)[I + \tilde{G}(\cdot)]^{-1} \in \tilde{\ell}^1_{nxn}(\rho) \tag{40}$$

then

$$\inf_{|z| \geq \rho} |\det[I + \tilde{G}(z)]| > 0 \tag{41}$$

Comments.

(1) Note that (41) is equivalent to

$$\det[I + G_0] \neq 0 \tag{42}$$

and

$$\det[I + \tilde{G}(z)] \neq 0 \quad \text{for } |z| \geq \rho \tag{43}$$

(2) The proof of Theorem 5 follows exactly the same line as that of Theorem 4, except that the closed half plane $\{s \in \mathbb{C} | \text{Re } s \geq \sigma\}$ is again replaced by $\{z \in \mathbb{C} \mid |z| \geq \rho\}$.

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