ASYMPTOTIC STABILITY IN THE LARGE OF A CLASS OF SINGLE-LOOP FEEDBACK SYSTEMS

by

R. A. Baker and C. A. Desoer

Memorandum No. ERL-M201

31 January 1967

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

Manuscript submitted: 15 December 1966

The research reported herein was supported in part by the National Aeronautics and Space Administration under Grant NsG-354, Supplement 3, Ford Foundation support through the Forgivable Predoctoral Loan to Future Engineering Teachers.

ASYMPTOTIC STABILITY IN THE LARGE OF A CLASS OF SINGLE-LOOP FEEDBACK SYSTEMS*

R. A. Baker†‡ and C. A. Desoer†

The purpose of this paper is to obtain some sufficient conditions for asymptotic stability in the large of a large class of systems. The basic idea is due to O'Shea [1]. We consider a system whose block diagram representation is shown in Fig. 1. Our results extend those of O'Shea in several directions: (a) the linear time-invariant subsystem, denoted by G in Fig. 1, is required to belong to a much broader class. By describing G by a convolution operator we allow in the class not only systems described by differential equations but also systems discussed by difference differential equations [2, p. 189; 3]. Also allowed are systems whose internal dynamics require partial differential equations, say, because of diffusion process or wave

---

* Received by the editors ____________________.

† Department of Electrical Engineering and Electronics Research Laboratory, University of California, Berkeley, California.
The research reported herein was supported in part by the National Aeronautics and Space Administration under Grant NsG-354, Supplement 3, Ford Foundation support through the Forgivable Predoctoral Loan to Future Engineering Teachers.

‡ On leave from Washington State University, Pullman, Washington.

propagation. (b) The conditions on the nonlinearity φ are less restrictive. (c) The results are stated more sharply in terms of the disturbance η.

The input-output relation of the linear time-invariant subsystem is

(1) $$\sigma_e(t) = \int_0^t g(t - \tau)\, e(\tau)\, d\tau \qquad t \geq 0$$

and that of the nonlinearity is

(2) $$c(t) = \phi[\sigma(t)] .$$

The specific assumptions which apply throughout are the following.

(N1) (N): $\phi : R \rightarrow R;\ \phi(0) = 0$

For some finite k,

(N2) $$|\phi(\sigma)| < |k\sigma| \quad \text{for all } \sigma \neq 0$$

(N3) $$0 \leq \frac{\phi(\sigma_1) - \phi(\sigma_2)}{\sigma_1 - \sigma_2} \leq k \quad \text{for all } \sigma_1, \sigma_2 \text{ and } \sigma_1 \neq \sigma_2.$$

(G1) $$g \in L_2(0, \infty)$$

The distributional derivative $\dot{g}$ of g is of the form

(G2) $$\dot{g} = \dot{g}_1 + \sum_{i=1}^{\infty} a_i \, \delta(t - t_i)$$

where

(G3) $$\dot{g}_1 \in L_1(0, \infty), \qquad \Sigma |a_i| < \infty$$

(E1) $$\eta \in L_1(0, \infty)$$

(E2) η is differentiable and $\dot{\eta} \in L'(0, \infty)$.

Observe that (G2) and (G3) imply that g is bounded on $[0, \infty)$, and that $g(t) \to 0$ as $t \to \infty$. The same holds for η. Call

$$\eta_M = \sup_{t \geq 0} |\eta(t)|, \quad g_M = \sup_{t \geq 0} |g(t)|.$$

In some manipulations to follow, it is useful to consider the functions g, e, σ, η and c to be defined for all t, all of them being identical to zero for $t < 0$. We use ^ to denote Fourier transforms: e.g.,

$$\hat{g}(i\omega) = \int_0^{\infty} g(t) \, e^{-i\omega t} \, dt .$$

We use $\|\cdot\|$ to denote L_1 norms: e.g.,

$$\|\eta\| = \int_0^{\infty} |\eta(t)| \, dt .$$

We come now to the main result of the paper.

Theorem. Consider the system shown in Fig. 1. Suppose that assumptions (N1) to (N3), (G1) to (G3), (E1) and (E2) hold. Let y be any real-valued function which has a Fourier transform $\hat{y}$ and such that $y(t) = 0$ for $t < 0$, $y(t) \leq 0$ for $t \geq 0$ and $\|y\| < 1$. Under these conditions, if for some $\alpha > 0$

$$\text{(3)} \qquad \mathrm{Re}\left\{[1 + i\omega\alpha + \hat{y}(i\omega)]\,[\hat{g}(i\omega) + 1/k]\right\} \geq 0 \quad \text{for all } \omega \in (-\infty, \infty)\,,$$

then,

(i) $\sup_{t\geq 0} |\sigma(t)| < \infty$,

(ii) $\sigma(t) \to 0$ as $t \to \infty$,

(iii) as $\|\eta\| + \|\dot{\eta}\| \to 0$, the corresponding σ has the property that $\sup_{t\geq 0} |\sigma(t)| \to 0$.

Notes. 1) By (N3), the output c has the same properties.

2) If ϕ is identically zero, the conclusions are immediate consequences of (E1) and (E2). From now on, ϕ is assumed not identically zero.

The proof of this theorem is somewhat involved. In order to simplify it we quote a lemma (see Ref. [5]).

Lemma. Let x and y be in $L_2(-\infty, \infty)$. Let, for each $t \in R$, $\big(x(t), y(t)\big) \in \varphi$ where φ is a monotonically increasing relation $\{$i.e., $\xi_1, \xi_2 \in R$ implies $[\varphi(\xi_1) - \varphi(\xi_2)]\,(\xi_1 - \xi_2) \geq 0\}$, then for all $t \in R$

$$\int_{-\infty}^{\infty} x(t)\, y(t)\, dt \ge \int_{-\infty}^{\infty} x(t)\, y(t-\tau)\, dt .$$

If, in addition, φ is odd (i.e., $(\xi, \eta) \in \varphi$ implies $(-\xi, -\eta) \in \varphi$), then the inequality above holds with absolute value signs on both integrands.

Proof.

I. The system shown in Fig. 1 is characterized by the equation

$$\sigma(t) = \eta(t) - \int_0^t g(t-t')\, \varphi[\sigma(t')]\, dt' \qquad t \ge 0 . \tag{4}$$

The given function η is continuous and bounded, g is bounded and, by (N3), φ satisfies a Lipschitz condition; then solving (4) by iteration we can apply the standard arguments to show that the resulting sequence coverges uniformly on every bounded interval, and that (4) has a unique solution which is continuous. For brevity, let $L_{\infty e}$ be the class of all measurable functions which are bounded on every bounded interval. Thus $\sigma \in L_{\infty e}$; clearly $c \in L_{\infty e}$ and $e \in L_{\infty e}$.

II. Let T be an arbitrary positive number. Let $\sigma_m = \sigma + \sigma * y$, $c_m = c + c * y$, and, in general, given any function x, we define $x_m = x + x * y$. Then

$$\int_0^T (\sigma_m - c_m/k)\, c\, dt = \int_0^T (\sigma - c/k)\, c\, dt + \int_0^T (y * (\sigma - c/k))(t)\, c(t)\, dt , \tag{5}$$

where all integrals are finite since $y \in L_1$, $\sigma \in L_{\infty e}$, and $c \in L_{\infty e}$. Now let the subscript T denote the truncation of a function to the interval $[0, T]$: thus, $f_T(t) = f(t)$ on $[0, T]$, and $f_T(t) = 0$ elsewhere. Considering the second integral in (5), we define

$$\text{(5a)} \qquad R(t) = \int_0^{\infty} [\sigma_T(t - \tau) - c_T(t - \tau)/k]\, c_T(t)\, dt\,,$$

and observe that by Fubini's theorem

$$\text{(6)} \qquad \int_0^T (y * (\sigma - c/k))\,(t)\, c(t)\, dt = \int_0^{\infty} y(\tau)\, R(\tau)\, d\tau\,.$$

Observe that, for each t, the real numbers $c_T(t)$ and $\sigma_T(t) - c_T(t)/k$ are monotonically related: indeed, denoting $c_T(t_i)$ by c_i and $\sigma_T(t_i)$ by σ_i,

$$[(\sigma_1 - \sigma_2) - (c_1 - c_2)/k](c_1 - c_2) = (\sigma_1 - \sigma_2)(c_1 - c_2)[k - (c_1 - c_2)/\sigma_1 - \sigma_2)]k \geq 0,$$

where the inequality follows from (N3). Consequently by Lemma 1, $R(\tau) \leq R(0)$; and since $y \leq 0$, (6) gives

$$\text{(6a)} \qquad \int_0^{\infty} y(\tau)\, R(\tau)\, d\tau \geq -R(0)\, \|y\|\,.$$

Thus, the left-side integral of (5) is larger or equal to $(1 - \|y\|)\, R(0) \geq 0$ In other words, for each $T > 0$, there is a finite $b(T) \geq (1 - \|y\|) > 0$ such that

$$\text{(7)} \qquad \int_0^T [\sigma_m(t) - c_m(t)/k]\, c(t)\, dt = b(T)\, R(0) \geq 0\,.$$

III. From the block diagram, $\sigma = \sigma_e + \eta$, hence

$$\int_0^T [\sigma_m(t) + \alpha\dot{\sigma}(t) - c_m(t)/k]\, c(t)\, dt = \int_0^T [\sigma_{em}(t) + \alpha\dot{\sigma}_e(t) - c_m(t)/k]\, c(t)\, dt$$

$$(8) \qquad\qquad + \int_0^T [\eta_m(t) + \alpha\dot{\eta}(t)]\, c(t)\, dt\,.$$

In order to show by Fourier methods that the first integral in (8) is nonpositive, observe that if $\sigma'_e = -g * c_T$ and $\dot{\sigma}'_e = -\dot{g} * c_T$ then, on $[0, T]$, $\sigma_e = \sigma'_e$, $\sigma_{em} = \sigma'_{em}$, and $\dot{\sigma}_e = \dot{\sigma}'_e$. Similarly, c may be replaced by c_T. Now $c \in L_{\infty e}$, hence $c_T \in L_1 \cap L_2$. With $g \in L_2$, this implies $\sigma'_e \in L_2$; hence, since $\dot{g}$ and $y \in L_1$, $\dot{\sigma}_e \in L_2$ and $\sigma'_{em} \in L_2$ [4]. Therefore the first integral in (8) is the product of two L_2-functions. Using Parseval's theorem, and noting that odd functions of ω contribute nothing to the integral, we obtain

$$\int_0^T [\sigma'_{em}(t) + \alpha\dot{\sigma}'_e(t) - c'_m(t)/k]\, c(t)\, dt$$

$$= \frac{-1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{Re}\Big\{[1 + \alpha i\omega + \hat{y}(i\omega)][\hat{g}(i\omega) + 1/k]\Big\}\, \hat{c}_T(i\omega)\, \hat{c}_T^{*}(i\omega)\, d\omega \le 0\,,$$

where the inequality follows by (3). Thus (8) implies that, for all $T > 0$,

$$(9) \qquad \int_0^T [\sigma_m(t) + \alpha\dot{\sigma}(t) - c_m(t)/k]\, c(t)\, dt \le \int_0^T [\eta_m(t) + \alpha\dot{\eta}(t)]\, c(t)\, dt\,.$$

This is the fundamental inequality.

IV. Using (7) in (9), we conclude that, for all $T > 0$,

$$\alpha \int_0^T \dot{\sigma}(t)\, c(t)\, dt \le \int_0^T [\eta_m(t) + \alpha \dot{\eta}(t)]\, c(t)\, dt \,. \tag{10}$$

Let $c_{TM} = \sup_t |c_T(t)|$. Since $\|\eta_m\| \le \|\eta\| + \|y\|\,\|\eta\|$, we conclude that

$$\alpha \int_0^T \dot{\sigma}(t)\, c(t)\, dt \le c_{TM}\left[\left(1 + \|y\|\right) \|\eta\| + \alpha \|\dot{\eta}\|\right] = c_{TM}\, M \,. \tag{11}$$

Call $\Phi(x) = \int_0^x \phi(\sigma)\, d\sigma$, then, with $\alpha > 0$, (11) implies that for all $T > 0$

$$\Phi[\sigma(T)] \le \Phi[\sigma(0)] + c_{TM}\, M\, \alpha^{-1} \tag{12}$$

The slope condition (N3) on ϕ implies that $\Phi(x) \ge [\phi(x)]^2/2k$, hence for all $T > 0$

$$\frac{1}{2k} |c(T)|^2 \le \Phi[\sigma(0)] + c_{TM}\, M\, \alpha^{-1} .$$

This inequality implies that $c \in L_\infty$, recalling that c is continuous this is easily shown by contradiction. In fact

$$\sup_{t \ge 0} |c(t)| \le \left\{\left(k M \alpha^{-1}\right)^2 + 2k\Phi[\sigma(0)]\right\} + k M \alpha^{-1} \,. \tag{15}$$

Since $\sigma_e(0) = 0$ by (1), as $\|\eta\| + \|\dot{\eta}\| \to 0$ both M and β tend to zero and so does $\sup_{t\geq 0} |c(t)|$.

V. Let us show that

$$\int_0^T (\sigma - c/k)\, c\, dt$$

is bounded. By (5a), this integral is R(0). If we let $c_M = \sup_{t\geq 0} |c(t)|$ then (7), (9) and (11) give

$$b(T) \int_0^T (\sigma - c/k)\, c\, dt \leq c_M M + \alpha \Phi[\sigma(0)] .$$

Observing that ϕ is monotonic and that $\sigma(0) = \eta(0)$, we obtain

$$\Phi[\sigma(0)] \leq \eta(0)\, c(0) \leq \|\dot{\eta}\|\, c_M \leq M\, c_M .$$

Hence,

$$\int_0^T (\sigma - c/k)\, c\, dt \leq c_M M(1 + \alpha)\left(1 - \|y\|\right)^{-1} .$$

Since the integrand is nonnegative by (N3) and since the right hand side is independent of T, we have

$$\text{(16)} \qquad \int_0^\infty (\sigma - c/k)\, c\, dt \leq c_M M(1 + \alpha)\left(1 - \|y\|\right)^{-1} .$$

Note that as $\|\eta\| + \|\dot{\eta}\| \to 0$, the bound on the right-hand side of (16) tends to zero, because both $c_M \to 0$ and $M \to 0$.

VI. To complete the proof we must consider the several possible behaviors of the nonlinear characteristic in the neighborhood of the origin.

Case 1. $\phi(\sigma) = 0$ implies $\sigma = 0$.

We know that $\sigma = \sigma_e + \eta$ and that $\eta \to 0$ as $t \to \infty$. Since $\dot{\sigma}_e = -\dot{g} * c$, where $\dot{g} \in L_1$ and $c \in L_\infty$, it follows that $\dot{\sigma}_e \in L_\infty$; hence σ_e is uniformly continuous on $[0, \infty)$. If σ_e did not $\to 0$ as $t \to \infty$, then σ does not go to zero; using the uniform continuity of σ_e we can easily show that the area under the function

$$\left(\sigma(t) - \frac{\phi[\sigma(t)]}{k}\right) \phi[\sigma(t)]$$

would then be infinite. This contradicts (16). Hence $\sigma \to 0$ as $t \to \infty$. This, together with $\sigma \in L_{\infty e}$, implies that $\sigma \in L_\infty$. Thus (i) and (ii) are established, and (iii) follows by contradiction: if $\|\eta\| + \|\dot{\eta}\| \to 0$ and $\sup_{t \geq 0} |\sigma(t)|$ does not go to zero then, because of the uniform continuity of σ_e, the bound on the integral in (16) could not go to zero.

Case 2. $\phi(\sigma) = 0$ implies $\sigma \in [-\sigma_1, \sigma_2]$ with $\sigma_1 > 0$, $\sigma_2 > 0$.

Using the monotonicity of ϕ, and inequality (16), we obtain

$$\text{(18)} \quad c_M M(1+\alpha)(1 - \|y\|)^{-1} \geq \int_0^\infty (\sigma - c/k)\, c\, dt$$

$$\geq \sigma_1 \int_0^\infty c^-(t)\, dt + \sigma_2 \int_0^\infty c^+(t)\, dt.$$

Hence $c \in L^1$. But already we know $c \in L^\infty$ hence $c \in L^2$. Now $\sigma_e = -g * c$ hence $\hat{\sigma}_e = \hat{g}\hat{c}$ where $\hat{g}, \hat{c} \in L^2$. Consequently $\hat{\sigma}_e \in L^1$. Now by the Rieman Labesgue lemma $\sigma_e(t) \to 0$ as $t \to \infty$, hence $\sigma(t) \to 0$ as $t \to \infty$. Now as $\|\eta\| + \|\dot{\eta}\| \to 0$ (18) implies that $\|c\| \to 0$; but

$$|\sigma_e(t)| \leq g_M \|c\|$$

consequently $\sup_{t \geq 0} |\sigma_e(t)| \to 0$, and so does $\sup_{t \geq 0} |\sigma(t)|$. Therefore (i), (ii) and (iii) have been established.

Case 3. $\phi(\sigma) = 0$ implies $\sigma \in [0, \sigma_2]$ with $\sigma_2 > 0$.

Combining the techniques of Cases 1 and 2, we first observe that

$$\infty > \int_0^\infty (\sigma - c/k)\, c\, dt \geq \int_0^\infty \left\{ \sigma^-(t) - \phi[\sigma^-(t)]/k \right\} \phi[\sigma^-(t)]\, dt$$

$$+ \sigma_2 \int_0^\infty c^+(t)\, dt .$$

Each of the integrals is finite and its value $\to 0$ as $\|\eta\| + \|\dot{\eta}\| \to 0$. By the reasoning of Case 1, σ^- satisfies (i), (ii) and (iii), and by that of Case 2 so does σ^+. Clearly, $\sigma = \sigma^+ - \sigma^-$ does so too.

Corollary.

If ϕ, the characteristic of the nonlinearity, is an odd function, then the theorem still holds without the requirement that $y(t) \leq 0$ for $t \geq 0$.

Proof. By the lemma, since ϕ is odd, $R(0) \geq |R(\tau)|$ for all τ. The previous assumption that $y(t) \leq 0$ was used only in the derivation of (7) from (6). Under the present conditions,

$$\left|\int_0^\infty y(\tau)\, R(\tau)\, d\tau\right| \leq \int_0^\infty |y(\tau)|\, |R(\tau)|\, d\tau \leq R(0)\, \|y\| ,$$

hence, the previous equation (6a) still holds. The remainder of the proof requires no modifications.

REFERENCES

[1] R. P. O'Shea, A combined frequency time-domain stability criterion for autonomous continuous systems, Trans. IEEE on Automatic Control, AC-11, 3, (July 1966), pp. 477-484.

[2] R. Bellman and K. L. Cooke, Differential-Difference Equations, Academic Press, New York, 1963.

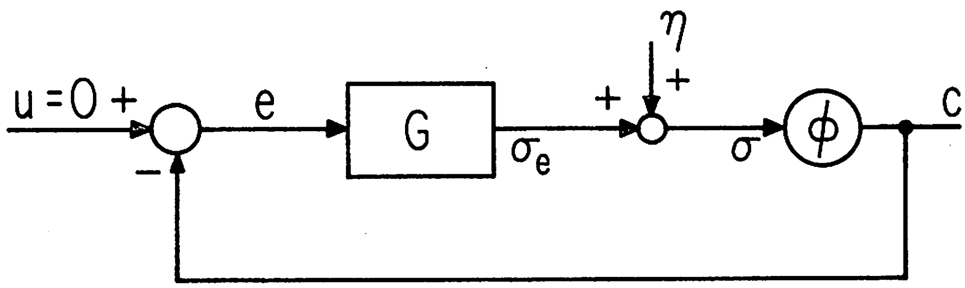
[3] A. Halonoy, Differential Equations: Stability; Oscillations, Time Logs, Academic Press, New York, 1966.

[4] S. Bochner and K. Chandrasekharan, Fourier Transforms, Princeton University Press, Princeton, 1949.

[5] R. A. Baker and C. A. Desoer, "On scalar products of signals passing through memoryless nonlinearities with delay," Memorandum No. ERL-M198, 22 Dec. 1966, Electronics Research Laboratory, College of Engineering, University of California (to be published).

FIGURE CAPTION

Fig. 1. Feedback system under consideration.



u = 0 +
−
e
G
σ_e
+
η
+
σ
ϕ
c