AN ALTERNATIVE APPROACH TO THE PROBLEM OF THE ASYMPTOTIC MAKESPAN OF SERIALLY MULTI-TASKED JOBS

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AN ALTERNATIVE APPROACH TO THE PROBLEM OF THE ASYMPTOTIC MAKESPAN OF SERIALLY MULTI-TASKED JOBS. †

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ABSTRACT

In this technical report we present an alternative proof of the main result proven in [2]. The proof presented here is more complex and technical than that in [2], but is direct and self-contained, in contradiction to the one in [2], which is indirect and relies heavily on results from the stability theory of tandem queueing networks.

The problem is the following. Consider an ordered set of processes, each consisting of $K \in \mathbb{Z}_+$ tasks, to be processed in the specified order by a system of K processors in tandem. Each process, immediately after having its i-th $(1 \le i < K)$ task served by the i-th processor, is queued up in a first-come-first-served basis for processing of its $(i+1)$-st task by the $(i+1)$-st processor, until all its tasks have been served. Define the makespan (execution time) to be the time to serve all the processes in the set.

Given that the processes form a stationary and ergodic sequence, as far as the processing times of the tasks are concerned, the asymptotic makespan is first explicitly computed, as the number of processes tends to infinity. This is done by directly computing the asymptotics of a known analytic formula for this makespan.

1. Introduction.

The purpose of this technical report is to present a direct and self-contained proof of the main result in [2], in contradiction to the proof of the same result in [2], which is indirect and relies heavily on results from the stability theory of tandem queueing networks. The proof given here is technically more complex than that in [2]. However, it is based on a completely different method, which requires some new interesting techniques with wider potential applications, making it worth documenting as a technical report.

Consider an infinite random sequence of multi-tasked processes of the form $A = \{\alpha_j = (\sigma_j^1, \sigma_j^2, \cdots \sigma_j^K), j \in \mathbb{Z}\}$ and a set of $K \in \mathbb{Z}_+$ processors, indexed by $i \in \{0, 1, 2, \cdots K\}$. Let σ_j^i be the processing time of the i-th task in the j-th process. In each process, the i-th task has to be served by the i-th processor.

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For $m, n \in \mathbb{Z}, m < n$, define $A_{m,n} = \{\alpha_j = (\sigma_j^1, \sigma_j^2, \cdots \sigma_j^K), \quad j \in \{m+1, m+2, \cdots n-1, n\}\}$. The processes in $A_{m,n}$ are served in the order specified by the index j. Each process, immediately after having its i-th ($1 \le i \le K-1$) task processed by the i-th processor, is queued up on a first-come-first-served basis for servicing of its $(i+1)$-st task by the $(i+1)$-st processor, until all its tasks have been processed. Therefore, each processor is specialized in serving just the corresponding task in each process.

Define the **makespan** (execution time) $T(A_{m,n}) \in \mathbb{R}_+$ to be the time to process all the processes in the set $A_{m,n}$.

It is assumed that the random sequence A, defined on some probability space (Ω, F, P), is **stationary** and **ergodic** under the transformation $\theta_l \mathrm{A} = \{\alpha_{j-l} = (\sigma_{j-l}^1, \sigma_{j-l}^2, \cdots \sigma_{j-l}^K), j \in \mathbb{Z}\}, l \in \mathbb{Z}$. Recall that stationarity means that $\theta_l \mathrm{A} \overset{D}{=} \mathrm{A}$, for any $l \in \mathbb{Z}$, the symbol $\overset{D}{=}$ indicating equality of the finite dimensional distributions of the two random sequences. Also, ergodicity means that every θ-invariant set of realizations of A has probability 0 or 1. A θ-invariant set of realizations of A is any measurable set U of realizations of A, such that $\mathrm{A} \in U$ if and only if $\theta_l \mathrm{A} \in U$, for all $l \in \mathbb{Z}$. Since A is θ-stationary, define $E[\sigma^i] = E[\sigma_j^i]$, for every $j \in \mathbb{Z}$, $i \in \{1, 2, \cdots K\}$. It is also assumed that $\max\{E[\sigma^i], i \in \{1, 2, \cdots K\}\} < \infty$.

Given the above setup, we are interested in computing the quantities $\lim_{n \to \infty} [\frac{T(A_{m,n})}{n-m}]$ and $\lim_{m \to -\infty} [\frac{T(A_{m,n})}{n-m}]$, that is, the asymptotic behavior of the makespan, as the number of processes goes to infinity, provided that the multi-tasked processes form a stationary and ergodic sequence. For motivation and practical impact of this problem see the paper [2].

At this point we need to define various constructions that will be of use later.

First, define the **grid** $G_{m,n}$, $m, n \in \mathbb{Z}, m < n$, as the directed graph $G = (V, E)$ with node set $V = \{(i, j), i \in \{1, 2, \cdots K\}, j \in \{m+1, m+2, \cdots n-1, n\}\}$ and edge set $E = \{((i, j), (i+1), j) \text{ and } ((i, j), (i, j+1)), i \in \{1, 2, \cdots K\}, j \in \{m+1, m+2, \cdots n-1\}\}$. Then, define a **traversing chain** $c(m, n)$, $m, n \in \mathbb{Z}, m < n$ of the grid $G_{m,n}$ to be a subset of V, such that:

1) $(1, m+1)$ and (K, n) both belong to $c(m, n) \subset V$ (are the end-points) and

2) if $(i, j) \in c(m, n)$, then either $(i+1, j)$ or $(i, j+1)$ (but not both) belongs to $c(m, n)$.

Finally, define $C_{m,n}$ to be the set of all the traversing chains of the grid $G_{m,n}$.

For any $m, n \in \mathbb{Z}, m < n$, the execution time of the set $A_{m,n} = (\sigma_j^1, \sigma_j^2, \cdots \sigma_j^K), j \in \{m+1, m+2, \cdots n-1, n\}\}$ of multi-tasked processes, has been shown to be

$$T(A_{m,n}) = \max_{c \in C_{m,n}} \{ \sum_{(i,j) \in c} \sigma_j^i \}. \tag{1}$$

This is proven by a double induction argument on the indices i, j (see Bellman et al [1], pg. 141).

In this technical report, provided that A is stationary and ergodic, we prove that

$$\lim_{n \to \infty} [\frac{T(A_{m,n})}{n-m}] = \lim_{m \to -\infty} [\frac{T(A_{m,n})}{n-m}] = \max\{E[\sigma^i], i \in \{1, 2, \cdots K\}\}, \tag{2}$$

by directly exploiting relation (1).

2. The Asymptotic Makespan.

In this section we explicitly compute the asymptotic execution time by working on the relation (1). In order to do that we need to study carefully the asymptotic structure of the set of traversing chains used ultimately in this relation. Through a series of definitions, observations and lemmas the result is finally presented in Theorem 1.

First observe that application of Birkoff's Individual Ergodic Theorem on the function σ_j^i under the measure preserving transformations θ_l and θ_{-l}, $l \in \mathbb{Z}$, yields

$$\lim_{n \to \infty} [\frac{1}{n-m} \sum_{j=m+1}^{n} \sigma_j^i] = \lim_{m \to -\infty} [\frac{1}{n-m} \sum_{j=m+1}^{n} \sigma_j^i] = E[\sigma^i], \quad i \in \{1, 2, \cdots K\}, \tag{3}$$

almost surely. Also, by (3), we have $\lim_{n \to \infty} [\frac{\sigma_n^i}{n}] = 0$, for every $i \in \{1, 2, \cdots K\}$.

Now, define $I_* \subset \{1, 2, ...K\}$ to be the set of indices, on which the quantity $E[\sigma^i], i \in \{1, 2, \cdots K\}$ attains its maximum value. That is

$$\max\{E[\sigma^i], i \in \{1, 2, ...K\}\} = E[\sigma^i], \quad i \in I_* \tag{4}$$

and

$$\max\{E[\sigma^i], i \in \{1, 2, ...K\}\} > E[\sigma^i], \quad i \in \{1, 2, \cdots K\} - I_*. \tag{5}$$

Let $I_* = \{i_1, i_2, \cdots i_a, i_{a+1}, \cdots i_L\}$, $L = |I_*|$ ($|\ |$ denoting the cardinality of the set) and $i_a \leq i_{a+1}$, for every $a \in \{1, 2, \cdots L-1\}$. Observe that the set I_* is never empty.
Define also the sets $I_0 = \{i \in \{1, 2, \cdots K\} : i < i_1\}$, $I_a = \{i \in \{1, 2, \cdots K\} : i_a < i < i_{i+1}\}$, for every $a \in \{1, 2, \cdots L-1\}$, and $I_L = \{i \in \{1, 2, \cdots K\} : i_L < i\}$. Observe that the previously defined sets are pairwise disjoint and each one may be empty. Also,

$$I_* \cup (\bigcup_{a=0}^{L} I_a) = \{1, 2, \cdots K\}. \tag{6}$$

In view of (1), for any fixed $m, n \in \mathbb{Z}, m < n$ define a **maximal traversing chain** $c_*(m,n) \in C_{m,n}$ to be a traversing chain of the grid $G_{m,n}$, on which the quantity $\sum_{(i,j) \in c} \sigma_j^i, c \in C_{m,n}$ achieves its maximum value, over all the traversing chains in $C_{m,n}$. That is,

$$\sum_{(i,j) \in c_*(m,n)} \sigma_j^i = \max_{c \in C_{m,n}} \{ \sum_{(i,j) \in c} \sigma_j^i \} = T(A_{m,n}). \tag{7}$$

Observe that there may be more than one maximal traversing chains.

Reflecting on how a maximal traversing chain is defined above, we construct the following sets on one specific maximal traversing chain $c_*(m,n)$. For any fixed $i \in \{1, 2, \cdots K\}$, define $M_i(m,n) = \{(i,j) : (i,j) \in c_*(m,n)\}$, $m, n \in \mathbb{Z}, m < n$. Observe that the sets $M_i(m,n), i \in \{1, 2, \cdots K\}$ are non-empty and are pairwise disjoint. Furthermore, as easily seen,

$$c_*(m,n) = \bigcup_{i=1}^{K} M_i(m,n), \tag{8}$$

for every $m, n \in \mathbb{Z}, m < n$, and, by (7), we have

$$T(A_{m,n}) = \sum_{i=1}^{K} [\sum_{(i,j) \in M_i(m,n)} \sigma_j^i]. \tag{9}$$

Finally, for any fixed $i \in \{1, 2, \cdots K\}$, define $b_i(m,n) = \inf\{j \in \mathbb{Z} : (i,j) \in M_i(m,n)\}$ and $b_i^* = \sup\{j \in \mathbb{Z} : (i,j) \in M_i(m,n)\}$, $m, n \in \mathbb{Z}, m < n$. Observe that $b_i(m,n) \le b_i^*(m,n)$, $b_i^*(m,n) = b_{i+1}(m,n)$, and $M_i(m,n) = \{(i,j) : b_i(m,n) \le j \le b_i^*(m,n)\}$, for every $i \in \{1, 2, \cdots K\}$.

Recall that all the above quantities are defined on one specific maximal traversing chain.

The following Lemmas 1,2 and 3 provide the necessary asymptotic results on the structure of the maximal traversing chain, to be used in the proof of Theorem 1. The lemmas have been written in a consistent logical order, so that Lemma 1 is used in the proof of Lemma 2. However, it might be a good idea for the reader to first study Lemma 2 and then come back and study Lemma 1, since the proof of the second lemma provides the motivation for the first one.

Lemma 1

For the stationary and ergodic sequence $A = \{\alpha_j = (\sigma_j^1, \sigma_j^2, \cdots \sigma_j^K), j \in \mathbb{Z}\}$, we have

$$\lim_{n \to \infty} b_i^*(m,n) = \infty \Longrightarrow \lim_{n \to \infty} [\frac{b_i(m,n)}{b_i^*(m,n)}] = 1, \tag{10}$$

almost surely, for every $i \in I_a \neq \varnothing$, any $a \in \{1, 2, \cdots L\}$, and any fixed $m \in \mathbb{Z}$.

Proof:

Fix some $m \in \mathbb{Z}$, some $a \in \{1, 2, \cdots L\}$, and some $i \in I_a \neq \varnothing$.

We shall first prove that, if $\lim_{n \to \infty} b_i^*(m,n) = \infty$, $i \in I_a \neq \varnothing$, then $\lim_{n \to \infty} b_\kappa^*(m,n) = \infty$, for every $\kappa \in \{i_a, i_a + 1, \cdots i-1, i\}$.

Indeed, first suppose that $\lim_{n \to \infty} b_i^*(m,n) = \infty$ and $\liminf_{n \to \infty} b_{i-1}^*(m,n) = \liminf_{n \to \infty} b_i(m,n) = \delta < \infty$. Then, there is an increasing subsequence $\{n_\mu, \mu \in \mathbb{Z}_+\}$ with $\lim_{\mu \to \infty} n_\mu = \infty$, such that $\lim_{\mu \to \infty} b_i^*(m, n_\mu) = \infty$ and $\lim_{\mu \to \infty} b_i(m, n_\mu) = \delta < \infty$.

Construct now, for each $\mu \in \mathbb{Z}_+$, the traversing chain $c_0(m, n_\mu) \in C_{m, n_\mu}$, by

$$c_0(m,n_\mu) = (\bigcup_{\kappa=1}^{i_a} M_\kappa) \cup (\bigcup_{\lambda = b_{i_a}^*(m,n_\mu)+1}^{b_i^*(m,n_\mu)} \{(i_a, \lambda)\}) \cup (\bigcup_{\kappa = i_a+1}^{i} \{(\kappa, b_i^*(m,n_\mu))\}) \cup (\bigcup_{\kappa=i+1}^{K} M_\kappa(m,n_\mu)).$$

Recall also that $c_*(m,n_\mu) = \bigcup_{\kappa=1}^{K} M_\kappa(m,n_\mu)$ is defined to be a maximal traversing chain, for each $\mu \in \mathbb{Z}_+$.

Then,

$$\lim_{\mu \to \infty} [\frac{X(m,n_\mu)}{b_i^*(m,n_\mu)}] = \lim_{\mu \to \infty} [\frac{1}{b_i^*(m,n_\mu)} \{ \sum_{(\kappa,\lambda) \in c_0(m,n_\mu)} \sigma_\lambda^\kappa - \sum_{(\kappa,\lambda) \in c_*(m,n_\mu)} \sigma_\lambda^\kappa \}] = \tag{12}$$

$$= \lim_{\mu \to \infty} [\frac{1}{b_i^*(m,n_\mu)} \{ \sum_{\lambda = b_i(m,n_\mu)}^{b_i^*(m,n_\mu)} \sigma_\lambda^{i_a} - \sum_{\lambda = b_i(m,n_\mu)}^{b_i^*(m,n_\mu)} \sigma_\lambda^{i} \}] + \tag{13}$$

$$+ \lim_{\mu \to \infty} [\frac{1}{b_i^*(m,n_\mu)} \{ \sum_{\lambda = b_{i_a}^*(m,n_\mu)+1}^{b_i(m,n_\mu)-1} \sigma_\lambda^{i_a} - \sum_{\kappa = i_a+1}^{i-1} (\sum_{\lambda = b_\kappa(m,n_\mu)}^{b_\kappa^*(m,n_\mu)} \sigma_\lambda^\kappa) \}] + \lim_{\mu \to \infty} [\frac{1}{b_i^*(m,n_\mu)} \sum_{\kappa = i_a+1}^{i} \sigma_{b_i^*(m,n_\mu)}^\kappa].$$

The first limit in (14) is zero, because $\lim_{\mu \to \infty} b_i(m,n_\mu) = \delta < \infty$, so the nominator always consists of a finite number of terms and is thus finite. The second limit in (14) is also zero, as easily seen, by use of (3). Finally, for the limit in (13), we have

$$\lim_{\mu\to\infty}\left[\frac{1}{b_i(m,n_\mu)}\sum_{\lambda=b_i(m,n_\mu)}^{b_i^*(m,n_\mu)}(\sigma_\lambda^{i_a}-\sigma_\lambda^{i})\right]=\lim_{\mu\to\infty}\left[\frac{1}{b_i^*(m,n_\mu)}\sum_{\lambda=1}^{b_i^*(m,n_\mu)}\sigma_\lambda^{i_a}\right]-\lim_{\mu\to\infty}\left[\frac{1}{b_i^*(m,n_\mu)}\sum_{\lambda=1}^{b_i^*(m,n_\mu)}\sigma_\lambda^{i}\right]-$$

$$-\lim_{\mu\to\infty}\left[\frac{1}{b_i^*(m,n_\mu)}\sum_{\lambda=1}^{b_i(m,n_\mu)-1}(\sigma_\lambda^{i_a}-\sigma_\lambda^{i})\right]=E[\sigma^{i_a}]-E[\sigma^{i}]>0, \tag{16}$$

since $\lim_{\mu\to\infty} b_i^*(m,n_\mu)=\infty$ and $\lim_{\mu\to\infty} b_i(m,n_\mu)<\infty$, and by using (3). Thus,

$$\lim_{\mu\to\infty}\left[\frac{X(m,n_\mu)}{b_i^*(m,n_\mu)}\right]=E[\sigma^{i_a}]-E[\sigma^{i}]>0, \tag{17}$$

and so there is a finite $\mu_0\in\mathbb{Z}_+$, such that

$$X(m,n_{\mu_0})=\sum_{(\kappa,\lambda)\in c_0(m,n_{\mu_0})}\sigma_\lambda^\kappa-\sum_{(\kappa,\lambda)\in c_*(m,n_{\mu_0})}\sigma_\lambda^\kappa>0, \tag{18}$$

which contradicts the fact that $c_*(m,n_{\mu_0})$ is a maximal traversing chain. So, indeed, $\lim_{n\to\infty} b_i^*(m,n)=\infty$ implies $\lim_{n\to\infty} b_i(m,n)=\lim_{n\to\infty} b_{i-1}^*(m,n)=\infty$. Recursive application of the above arguments yields immediately that, if $\lim_{n\to\infty} b_i^*(m,n)=\infty$, then $\lim_{n\to\infty} b_\kappa^*(m,n)=\infty$, for every $\kappa\in\{i_a, i_a+1,\ldots i-1, i\}$.

We shall now prove that, if $\lim_{n\to\infty} b_i^*(m,n)=\infty$, $i\in I_a\neq\emptyset$, then $\lim_{n\to\infty}\left[\frac{b_\kappa(m,n)}{b_\kappa^*(m,n)}\right]=1$, for every $\kappa\in\{i_a+1, i_a+2, \cdots i-1, i\}$.

Given that $\lim_{n\to\infty} b_i^*(m,n)=\infty$, first recall that $\lim_{n\to\infty} b_\kappa^*(m,n)=\lim_{n\to\infty} b_{\kappa+1}(m,n)=\infty$, for every $\kappa\in\{i_a, i_a+1, \cdots i-1, i\}$. Recall also that $\frac{b_\kappa(m,n)}{b_\kappa^*(m,n)}\leq 1$.

Arguing by contradiction, suppose that there exists a $\gamma\in\{i_a+1, i_a+2, \cdots i-1, i\}$, such that $\liminf_{n\to\infty}\left[\frac{b_\gamma(m,n)}{b_\gamma^*(m,n)}\right]=1-\varepsilon<1$, $\varepsilon\in(0,1)$, and $\lim_{n\to\infty}\left[\frac{b_\kappa(m,n)}{b_\kappa^*(m,n)}\right]=1$, for every $\kappa\in\{i_a+1, i_a+2, \cdots \gamma-2, \gamma-1\}$. Then, there exists an increasing subsequence $\{n_\nu, \nu\in\mathbb{Z}_+\}$ with $\lim_{\nu\to\infty} n_\nu=\infty$, such that $\lim_{\nu\to\infty}\left[\frac{b_\gamma(m,n_\nu)}{b_\gamma^*(m,n_\nu)}\right]=1-\varepsilon<1$ and $\lim_{\nu\to\infty}\left[\frac{b_\kappa(m,n_\nu)}{b_\kappa^*(m,n_\nu)}\right]=1$, for every $\kappa\in\{i_a+1, i_a+2, \cdots \gamma-2, \gamma-1\}$.

Construct now, for each $\nu\in\mathbb{Z}_+$, the traversing chain $\bar{c}(m,n_\nu)\in C_{m,n_\nu}$, by

$$\bar{c}(m,n_\nu)=\left(\bigcup_{\kappa=1}^{i_a}M_\kappa\right)\cup\left(\bigcup_{\lambda=b_{i_a}^*(m,n_\nu)+1}^{b_\gamma^*(m,n_\nu)}\{(i_a,\lambda)\}\right)\cup\left(\bigcup_{\kappa=i_a+1}^{i}\{(\kappa,b_\gamma^*(m,n_\nu))\}\right)\cup\left(\bigcup_{\kappa=\gamma+1}^{K}M_\kappa(m,n_\nu)\right).$$

Recall also that $c_*(m,n_\nu)=\bigcup_{\kappa=1}^{K}M_\kappa(m,n_\nu)$ is defined to be a maximal traversing chain, for each $\nu\in\mathbb{Z}_+$.

Then,

$$\lim_{\nu\to\infty}\left[\frac{Y(m,n_\nu)}{b_\gamma^*(m,n_\nu)}\right]=\lim_{\nu\to\infty}\left[\frac{1}{b_\gamma^*(m,n_\nu)}\left\{\sum_{(\kappa,\lambda)\in\bar{c}(m,n_\nu)}\sigma_\lambda^\kappa-\sum_{(\kappa,\lambda)\in c_*(m,n_\nu)}\sigma_\lambda^\kappa\right\}\right]= \tag{20}$$

$$= \lim_{\nu \to \infty} [\frac{1}{b_\gamma^*(m, n_\nu)} \sum_{\kappa = i_a + 1}^{\gamma} \{ \sum_{\lambda = b_\kappa(m, n_\nu)}^{b_\kappa^*(m, n_\nu)} (\sigma_\lambda^{i_a} - \sigma_\lambda^\kappa) \}] + \lim_{\nu \to \infty} [\frac{1}{b_\gamma^*(m, n_\nu)} \sum_{\kappa = i_a + 1}^{\gamma - 1} \sigma_{b_\gamma^*(m, n_\nu)}^\kappa]. \quad (21)$$

Recall that $\lim_{n \to \infty} [\frac{\sigma_n^i}{n}] = 0$, for every $i \in \{1, 2, \cdots K\}$.

But, for each fixed $\kappa \in \{i_a + 1, i_a + 2, \cdots \gamma - 1, \gamma\}$, we have

$$\Phi(\kappa) = \lim_{\nu \to \infty} [\frac{1}{b_\gamma^*(m, n_\nu)} \sum_{\lambda = b_\kappa(m, n_\nu)}^{b_\kappa^*(m, n_\nu)} (\sigma_\lambda^{i_a} - \sigma_\lambda^\kappa)] = \quad (22)$$

$$= \lim_{\nu \to \infty} [\frac{b_\kappa^*(m, n_\nu)}{b_\gamma^*(m, n_\nu)} \{ \frac{\sum_{\lambda = 1}^{b_\kappa^*(m, n_\nu)} (\sigma_\lambda^{i_a} - \sigma_\lambda^\kappa)}{b_\kappa^*(m, n_\nu)} - (\frac{b_\kappa(m, n_\nu)}{b_\kappa^*(m, n_\nu)}) \frac{\sum_{\lambda = 1}^{b_\kappa^*(m, n_\nu) - 1} (\sigma_\lambda^{i_a} - \sigma_\lambda^\kappa)}{b_\kappa(m, n_\nu)} \}]. \quad (23)$$

Taking the limits in (23), using (3), and recalling that $\lim_{\nu \to \infty} [\frac{b_\kappa(m, n_\nu)}{b_\kappa^*(m, n_\nu)}] = 1$, for every $\kappa \in \{i_a + 1, i_a + 2, \cdots \gamma - 2, \gamma - 1\}$, we get $\Phi(\kappa) = 0$, for every $\kappa \in \{i_a + 1, i_a + 2, \cdots \gamma - 2, \gamma - 1\}$. But, for $\kappa = \gamma$, since $\lim_{n \to \infty} [\frac{b_\gamma(m, n_\nu)}{b_\gamma^*(m, n_\nu)}] = 1 - \varepsilon < 1$, we get $\Phi(\gamma) = \varepsilon(E[\sigma^{i_a}] - E[\sigma^i]) > 0$. Also, the second limit in the expansion (21), is easily seen to be zero, by using (3). Substituting in (21), we get

$$\lim_{\nu \to \infty} [\frac{Y(m, n_\nu)}{b_\gamma^*(m, n_\nu)}] = \varepsilon(E[\sigma^{i_a}] - E[\sigma^i]) > 0, \quad (24)$$

so there is a finite $\nu_0 \in \mathbb{Z}_+$, such that

$$Y(m, n_{\nu_0}) = \sum_{(\kappa, \lambda) \in \bar{c}(m, n_{\nu_0})} \sigma_\lambda^\kappa - \sum_{(\kappa, \lambda) \in c_*(m, n_{\nu_0})} \sigma_\lambda^\kappa > 0, \quad (25)$$

which contradicts the fact that $c_*(m, n_{\nu_0})$ is a maximal traversing chain.

So, there is no $\gamma \in \{i_a + 1, i_a + 2, \cdots i - 1, i\}$, such that $\liminf_{n \to \infty} [\frac{b_\gamma(m, n)}{b_\gamma^*(m, n)}] < 1$ and $\lim_{n \to \infty} [\frac{b_\kappa(m, n)}{b_\kappa^*(m, n)}] = 1$, for every $\kappa \in \{i_a + 1, i_a + 2, \cdots \gamma - 1\}$. Recursive application of the above implies that, given that $\lim_{n \to \infty} b_i^*(m, n) = \infty$, $i \in I_a \neq \emptyset$, we have that $\lim_{n \to \infty} [\frac{b_\kappa(m, n)}{b_\kappa^*(m, n)}] = 1$, for every $\kappa \in \{i_a + 1, i_a + 2, \cdots i - 1, i\}$, so $\lim_{n \to \infty} [\frac{b_i(m, n)}{b_i^*(m, n)}] = 1$. This completes the proof of the lemma.

□

Lemma 2

For the stationary and ergodic sequence $A = \{\alpha_j = (\sigma_j^1, \sigma_j^2, \cdots \sigma_j^K), j \in \mathbb{Z}\}$, we have

$$\lim_{n \to \infty} [\frac{1}{n - m} \sum_{(i, j) \in M_i(m, n)} \sigma_j^i] = \lim_{n \to \infty} [\frac{1}{n - m} \sum_{j = b_i(m, n) + 1}^{b_i^*(m, n)} \sigma_j^{i_a}] = 0, \quad (26)$$

almost surely, for every $i \in I_a \neq \emptyset$, any $a \in \{1, 2, \cdots L\}$, and any fixed $m \in \mathbb{Z}$.

Proof:

Fix some $m \in \mathbb{Z}$, $a \in \{1, 2, \cdots L\}$, and $i \in I_a \neq \emptyset$. Arguing by contradiction, suppose that

$$\limsup_{n \to \infty} \left[\frac{1}{n-m} \sum_{(i,j) \in M_i(m,n)} \sigma_j^i\right] = \limsup_{n \to \infty} \left[\frac{1}{n-m} \sum_{j = b_i(m,n)}^{b_i^*(m,n)} \sigma_j^i\right] = \delta > 0. \tag{27}$$

Then, expanding the second expression in (27), we see that there is an increasing subsequence $\{n_\mu, \mu \in \mathbb{Z}_+\}$, such that $\lim_{\mu \to \infty} n_\mu = \infty$ and

$$\lim_{\mu \to \infty} \left[\frac{b_i^*(m, n_\mu)}{n_\mu - m} \left\{ \frac{\sum_{j=1}^{b_i^*(m,n_\mu)} \sigma_j^i}{b_i^*(m, n_\mu)} - \left(\frac{b_i(m, n_\mu)}{b_i^*(m, n_\mu)}\right) \frac{\sum_{j=1}^{b_i(m,n_\mu)-1} \sigma_j^i}{b_i(m, n_\mu)} \right\}\right] = \delta . \tag{28}$$

Using (3), it is easily seen that the second term of the product in the limit in (28) is positive and bounded from above independently of $\mu \in \mathbb{Z}_+$. Thus, (28) guarantees that $\limsup_{\mu \to \infty} \left[\frac{b_i^*(m, n_\mu)}{n_\mu - m}\right] = \varepsilon > 0$. This implies that there is an increasing subsequence $\{n_{\mu_\nu}, \nu \in \mathbb{Z}_+\}$, such that $\lim_{\nu \to \infty} n_{\mu_\nu} = \infty$ and

$$\lim_{\nu \to \infty} \left[\frac{b_i^*(m, n_{\mu_\nu})}{n_{\mu_\nu} - m}\right] = \varepsilon > 0, \tag{29}$$

thus, $\lim_{\nu \to \infty} b_i^*(m, n_{\mu_\nu}) = \infty$.

Then, by Lemma 1, since $i \in I_a \neq \emptyset$, $a \in \{1, 2, \cdots L\}$, we have

$$\lim_{\nu \to \infty} \left[\frac{b_i(m, n_{\mu_\nu})}{b_i^*(m, n_{\mu_\nu})}\right] = 1, \tag{30}$$

so $\lim_{\nu \to \infty} b_i(m, n_{\mu_\nu}) = \infty$. Using (30) and (3), we get

$$\lim_{\nu \to \infty} \left[\frac{\sum_{j=1}^{b_i^*(m,n_{\mu_\nu})} \sigma_j^i}{b_i^*(m, n_{\mu_\nu})} - \left(\frac{b_i(m, n_{\mu_\nu})}{b_i^*(m, n_{\mu_\nu})}\right) \frac{\sum_{j=1}^{b_i(m,n_{\mu_\nu})-1} \sigma_j^i}{b_i(m, n_{\mu_\nu})}\right] = E[\sigma^i] - 1E[\sigma^i] = 0. \tag{31}$$

The above, together with (29), contradict (28). Thus,

$$\lim_{n \to \infty} \left[\frac{1}{n-m} \sum_{(i,j) \in M_i(m,n)} \sigma_j^i\right] = 0. \tag{32}$$

Also, arguing in exactly the same way as above, we get

$$\lim_{n \to \infty} \left[\frac{1}{n-m} \sum_{j = b_i(m,n)+1}^{b_i^*(m,n)} \sigma_j^{i_a}\right] = 0. \tag{33}$$

This completes the proof of the lemma.

□

Lemma 3

For the stationary and ergodic sequence $A = \{\alpha_j = \{\sigma_j^1, \sigma_j^2, \cdots \sigma_j^K\}, j \in \mathbb{Z}\}$, we have

$$\lim_{n \to \infty} [\frac{1}{n-m} \sum_{(i,j) \in M_i(m,n)} \sigma_j^i] = \lim_{n \to \infty} [\frac{1}{n-m} \sum_{j = b_i(m,n)+1}^{b_i^*(m,n)} \sigma_j^{i_1}] = 0, \tag{34}$$

almost surely, for every $i \in I_0 \neq \emptyset$, and any fixed $m \in \mathbb{Z}$.

Proof:

We shall first prove that $\limsup_{n \to \infty} b_{i_1-1}^*(m,n) < \infty$ ($I_0 \neq \emptyset$). Indeed, arguing by contradiction, suppose that $\limsup_{n \to \infty} b_{i_1-1}^*(m,n) = \infty$. Then, there is an increasing subsequence $\{n_\mu, \mu \in \mathbb{Z}_+\}$ with $\lim_{\mu \to \infty} n_\mu = \infty$, such that $\lim_{\mu \to \infty} b_{i_1-1}^*(m, n_\mu) = \infty$

As will be proven just below (in the second paragraph), this implies that $\lim_{\mu \to \infty} b_i(m, n_\mu) = \infty$, for every $i \in I_0$, including $i = 1$, which contradicts the fact that $b_1(m, n_\mu) = m + 1 < \infty$, for every $\mu \in \mathbb{Z}$. Thus, $\limsup_{n \to \infty} b_{i_1-1}^*(m,n) < \infty$, and so $\limsup_{n \to \infty} b_i(m,n) < \limsup_{n \to \infty} b_i^*(m,n) < \infty$, for every $i \in I_0 \neq \emptyset$. Then, recalling the definition of M_i, the result follows immediately.

As mentioned above, we still have to prove that, if $\lim_{\mu \to \infty} b_{i_1-1}^*(m, n_\mu) = \infty$, then $\lim_{\mu \to \infty} b_i(m, n_\mu) = \infty$, for every $i \in I_0 \neq \emptyset$. For this, it is enough to prove that, if $\lim_{n \to \infty} b_{i_1-1}^*(m,n) = \infty$, then $\lim_{n \to \infty} [\frac{b_i(m,n)}{b_i^*(m,n)}] = 1$, for every $i \in \{1, 2, \cdots i_1-2, i_1-1\}$. Recall that $b_i(m,n) \le b_i^*(m,n) = b_{i+1}(m,n)$.

Arguing by contradiction, suppose that, given that $\lim_{n \to \infty} b_{i_1-1}^*(m,n) = \infty$, there exists a $\gamma \in \{1, 2, \cdots \cdots i_1-2, i_1-1\}$, such that $\liminf_{n \to \infty} [\frac{b_\gamma(m,n)}{b_\gamma^*(m,n)}] = 1 - \varepsilon < 1$, $\varepsilon \in (0,1)$, and $\lim_{n \to \infty} [\frac{b_\kappa(m,n)}{b_\kappa^*(m,n)}] = 1$, for every $\kappa \in \{\gamma+1, \gamma+2, \cdots i_1-2, i_1-1\}$. Then, there exists an increasing subsequence $\{n_\nu, \nu \in \mathbb{Z}_+\}$ with $\lim_{\nu \to \infty} n_\nu = \infty$, such that $\lim_{\nu \to \infty} [\frac{b_\gamma(m,n_\nu)}{b_\gamma^*(m,n_\nu)}] = 1 - \varepsilon < 1$ and $\lim_{\nu \to \infty} [\frac{b_\kappa(m,n_\nu)}{b_\kappa^*(m,n_\nu)}] = 1$, for every $\kappa \in \{\gamma+1, \gamma+2, \cdots i_1-2, i_1-1\}$.

Construct now, for each $\nu \in \mathbb{Z}_+$, the traversing chain $\bar{c}(m, n_\nu) \in C_{m, n_\nu}$, by

$$\bar{c}(m,n_\nu) = (\bigcup_{\kappa=1}^{\gamma-1} M_\kappa(m,n_\nu)) \cup (\bigcup_{\kappa=\gamma}^{i_1} \{(\kappa, b_\gamma(m,n_\nu))\}) \cup (\bigcup_{\lambda = b_\gamma(m,n_\nu)}^{b_{i_1}-1} \{(i_1, \lambda)\}) \cup (\bigcup_{\kappa=i_1}^{K} M_\kappa(m,n_\nu)).$$

Recall also that $c_*(m,n_\nu) = \bigcup_{\kappa=1}^{K} M_\kappa(m,n_\nu)$ is defined to be a maximal traversing chain, for each $\nu \in \mathbb{Z}_+$.

Then,

$$\lim_{\nu \to \infty} [\frac{Z(m,n_\nu)}{b_\gamma^*(m,n_\nu)}] = \lim_{\nu \to \infty} [\frac{1}{b_\gamma^*(m,n_\nu)} \{ \sum_{(\kappa,\lambda) \in \bar{c}(m,n_\nu)} \sigma_\lambda^\kappa - \sum_{(\kappa,\lambda) \in c_*(m,n_\nu)} \sigma_\lambda^\kappa \}] = \tag{36}$$

$$= \lim_{\nu \to \infty} [\frac{1}{b_\gamma^*(m, n_\nu)} \sum_{\kappa=\gamma}^{i_1-1} \{ \sum_{\lambda = b_\kappa(m,n_\nu)}^{b_\kappa^*(m,n_\nu)} (\sigma_\lambda^{i_1} - \sigma_\lambda^\kappa)\}] + \lim_{\nu \to \infty} [\frac{1}{b_\gamma^*(m, n_\nu)} \sum_{\kappa=\gamma+1}^{i_1-1} \sigma_{b_\gamma^*(m,n_\nu)}^\kappa] \quad (37)$$

Recall that $\lim_{n \to \infty} [\frac{\sigma_n^i}{n}] = 0$, for every $i \in \{1, 2, \cdots K\}$. But, for each fixed $\kappa \in \{\gamma, \gamma+1, \cdots i_1-2, i_1-1\}$, we have

$$\Psi(\kappa) = \lim_{\nu \to \infty} [\frac{1}{b_\gamma^*(m, n_\nu)} \sum_{\lambda = b_\kappa(m,n_\nu)}^{b_\kappa^*(m,n_\nu)} (\sigma_\lambda^{i_1} - \sigma_\lambda^\kappa)] = \quad (38)$$

$$= \lim_{\nu \to \infty} [\frac{b_\kappa^*(m, n_\nu)}{b_\gamma^*(m, n_\nu)} \{ \frac{\sum_{\lambda=1}^{b_\kappa^*(m,n_\nu)} (\sigma_\lambda^{i_1} - \sigma_\lambda^\kappa)}{b_\kappa^*(m, n_\nu)} - (\frac{b_\kappa(m, n_\nu)}{b_\kappa^*(m, n_\nu)}) \frac{\sum_{\lambda=1}^{b_\kappa^*(m,n_\nu)-1} (\sigma_\lambda^{i_1} - \sigma_\lambda^\kappa)}{b_\kappa(m, n_\nu)} \}]. \quad (39)$$

Taking the limits in (39), using (3), and recalling that $\lim_{\nu \to \infty} [\frac{b_\kappa(m, n_\nu)}{b_\kappa^*(m, n_\nu)}] = 1$, for every $\kappa \in \{\gamma+1, \gamma+2, \cdots i_1-2, i_1-1\}$, we get $\Psi(\kappa) = 0$, for every $\kappa \in \{\gamma+1, \gamma+2, \cdots i_1-2, i_1-1\}$. But, for $\kappa = \gamma$, since $\lim_{n \to \infty} [\frac{b_\gamma(m, n_\nu)}{b_\gamma^*(m, n_\nu)}] = 1 - \varepsilon < 1$, we get $\Psi(\gamma) = \varepsilon(E[\sigma^{i_1}] - E[\sigma^i]) > 0$. Also, the second limit in the expansion (37), is easily seen to be zero, by using (3). Substituting in ( ), we get

$$\lim_{\nu \to \infty} [\frac{Z(m, n_\nu)}{b_\gamma^*(m, n_\nu)}] = \varepsilon(E[\sigma^{i_1}] - E[\sigma^i]) > 0, \quad (40)$$

so there is a finite $\nu_0 \in \mathbb{Z}_+$, such that

$$Z(m, n_{\nu_0}) = \sum_{(\kappa,\lambda) \in c^-(m, n_{\nu_0})} \sigma_\lambda^\kappa - \sum_{(\kappa,\lambda) \in c_*(m, n_{\nu_0})} \sigma_\lambda^\kappa > 0, \quad (41)$$

which contradicts the fact that $c_*(m, n_{\nu_0})$ is a maximal traversing chain.

So, there is no $\gamma \in \{1, 2, \cdots i_1-2, i_1-1\}$, such that $\liminf_{n \to \infty} [\frac{b_\gamma(m, n)}{b_\gamma^*(m, n)}] < 1$ and $\lim_{n \to \infty} [\frac{b_\kappa(m, n)}{b_\kappa^*(m, n)}] = 1$, for every $\kappa \in \{\gamma+1, \gamma+2, \cdots i_1-2, i_1-1\}$. Recursive application of the above implies that, given that $\lim_{n \to \infty} b_{i_1-1}^*(m, n) = \infty$, we have that $\lim_{n \to \infty} [\frac{b_i(m, n)}{b_i^*(m, n)}] = 1$, for every $i \in \{1, 2, \cdots i_1-2, i_1-1\}$. As easily seen, this forces $\lim_{n \to \infty} b_i(m, n) = \infty$, for every $i \in \{1, 2, \cdots i_1-2, i_1-1\}$, including $i = 1$, which leads to the contradiction explained in the first paragraph of this proof. This completes the proof of this lemma.

□

Theorem 1

For the stationary and ergodic sequence $A = \{\alpha_j = (\sigma_j^1, \sigma_j^2, \cdots \sigma_j^K), j \in \mathbb{Z}\}$, we have

$$\lim_{n \to \infty} [\frac{T(A_{m,n})}{n-m}] = \lim_{m \to -\infty} [\frac{T(A_{m,n})}{n-m}] = \max\{E[\sigma^i], i \in \{1, 2, \cdots K\}\}, \quad (42)$$

almost surely.

Proof:

First define the integer quantities $d_1(m,n)=m+1, d_a(m,n)=b_{i_a}(m,n)$, for every $a\in\{2,\cdots K\}, d_{L+1}(m,n)=n, m,n\in\mathbb{Z}, m<n$. Then consider the quantity,

$$\Delta(m,n)=\frac{1}{n-m}\sum_{a=1}^{L}\{\sum_{j=d_a(m,n)}^{d_{a+1}(m,n)}\sigma_j^{i_a}\} \tag{43}$$

We shall prove that the $\lim_{n\to\infty}\Delta(m,n)=\max\{E[\sigma^i], i\in\{1,2,\cdots K\}\}$. Recall that $E[\sigma^{i_a}]=\max\{E[\sigma^i], i\in\{1,2,\cdots K\}$, for every $a\in\{1,2,\cdots L\}$. First observe that from ( ) we have

$$\Delta(m,n)=\frac{1}{n-m}\sum_{a=1}^{L}\{\sum_{j=m}^{d_{a+1}(m,n)}\sigma_j^{i_a}-\sum_{j=m}^{d_a(m,n)-1}\sigma_j^{i_a}\}= \tag{44}$$

$$\frac{1}{n-m}\sum_{j=m}^{n}\sigma_j^{i_L}+\sum_{a=2}^{L}\{\frac{1}{n-m}(\sum_{j=m}^{d_a(m,n)}\sigma_j^{i_a-1}-\sum_{j=m}^{d_a(m,n)-1}\sigma_j^{i_a})\} \tag{45}$$

We shall now prove that

$$\lim_{n\to\infty}[\frac{1}{n-m}(\sum_{j=m}^{d_a(m,n)}\sigma_j^{i_a-1}-\sum_{j=m}^{d_a(m,n)-1}\sigma_j^{i_a})]=0, \tag{46}$$

for every $a\in\{2,3,\cdots L\}$.
Indeed, arguing by contradiction, suppose that

$$\limsup_{n\to\infty}[\frac{1}{n-m}(\sum_{j=m}^{d_a(m,n)}\sigma_j^{i_a-1}-\sum_{j=m}^{d_a(m,n)-1}\sigma_j^{i_a})]=\varepsilon>0, \tag{47}$$

for some fixed $a\in\{2,3,\cdots L\}$. Then, there is an increasing subsequence $\{n_\mu, \mu\in\mathbb{Z}_+\}$ with $\lim_{\mu\to\infty}n_\mu=\infty$, such that

$$\lim_{\mu\to\infty}[\frac{d_a(m,n_\mu)-m}{n_\mu-m}(\frac{\sum_{j=m}^{d_a(m,n_\mu)}\sigma_j^{i_a-1}}{d_a(m,n_\mu)-m}-\frac{\sum_{j=m}^{d_a(m,n_\mu)-1}\sigma_j^{i_a}}{d_a(m,n_\mu)-m})]=\varepsilon>0. \tag{48}$$

Using (3), it is easily seen that the second term of the product in the limit in (48) is bounded from above and below independently of $\mu\in\mathbb{Z}_+$. Thus, (48) guaranties that there is an increasing subsequence $\{n_{\mu_\nu}, \nu\in\mathbb{Z}_+\}$ with $\lim_{\nu\to\infty}n_{\mu_\nu}=\infty$, such that

$$\lim_{\nu\to\infty}[\frac{d_a(m,n_{\mu_\nu})-m}{n_{\mu_\nu}-m}]=\xi>0, \tag{49}$$

thus $\lim_{n\to\infty}d_a(m,n_{\mu_\nu})=\infty$. Then, using (3), we get

$$\lim_{\nu\to\infty}[\frac{\sum_{j=m}^{d_a(m,n_{\mu_\nu})}\sigma_j^{i_a-1}}{d_a(m,n_{\mu_\nu})-m}-\frac{\sum_{j=m}^{d_a(m,n_{\mu_\nu})-1}\sigma_j^{i_a}}{d_a(m,n_{\mu_\nu})-m})]=E[\sigma^{i_a-1}]-E[\sigma^{i_a}]=0. \tag{50}$$

The above, together with (49), contradict (48). So, (46) is true.

Using now (3) and (46) in (45), we get

$$\lim_{n \to \infty} [\Delta(m,n)] = \lim_{n \to \infty} [\frac{1}{n-m} \sum_{a=1}^{L} \{ \sum_{j=d_a(m,n)}^{d_{a+1}(m,n)} \sigma_j^{i_a} \}] = \max\{E[\sigma^i], i \in \{1,2, \cdots K\}. \quad (51)$$

But, by using Lemmas 2 and 3, and by reflecting on the construction of the M_i's and the d_a's, we see that the only possibly surviving components (their limit is not zero) in (51) are the $\lim_{n \to \infty} [\frac{1}{n-m} \sum_{(i,j) \in M_i(m,n)} \sigma_j^i]$, for $i \in I_* = \{i_1, i_2, \cdots K\}$, so,

$$\lim_{n \to \infty} [\Delta(m,n)] = \lim_{n \to \infty} [\frac{1}{n-m} \sum_{i \in I_*} \{ \sum_{(i,j) \in M_i(m,n)} \sigma_j^i \}] = \max\{E[\sigma^i], i \in \{1,2, \cdots K\}. \quad (52)$$

Again by Lemmas 2 and 3, we have

$$\lim_{n \to \infty} [\frac{1}{n-m} \sum_{i \in \{1,2, \cdots K\} - I_*} \{ \sum_{(i,j) \in M_i(m,n)} \sigma_j^i \}] = 0. \quad (53)$$

Combining (52) and (53), we have

$$\lim_{n \to \infty} [\Delta(m,n)] = \max\{E[\sigma^i], i \in \{1,2, \cdots K\}\} = \lim_{n \to \infty} [\frac{1}{n-m} \sum_{i=1}^{K} \{ \sum_{(i,j) \in M_i(m,n)} \sigma_j^i \}] = \lim_{n \to \infty} [\frac{T(A_{m,n})}{n-m}],$$

completing one part of the theorem.

The other part of the theorem is completely analogously proven, by applying the same constructions and arguments, which have been used so far, but with $m \to -\infty$ and n fixed. This completes the proof of the theorem.

□

3. References.

[1] Bellman, R., Esogbue, A.O., Nabeshima, I. (1982). Mathematical Aspects of Scheduling and Applications. International Series in Modern Applied Mathematics and Computer Science, Volume 4. Pergamon Press.

[2] Bambos, N., Walrand, J. (1988). On the Asymptotic Execution Time of Multi-tasked Processes on Tandem Processors. Submitted for publication. Systems and Control Letters.