The Fixed-Point Theory of Strictly Causal Functions

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Abstract

We ask whether strictly causal components form well defined systems when arranged in feedback configurations. The standard interpretation for such configurations induces a fixed-point constraint on the function modelling the component involved. We define strictly causal functions formally, and show that the corresponding fixed-point problem does not always have a well defined solution. We examine the relationship between these functions and the functions that are strictly contracting with respect to a generalized distance function on signals, and argue that these strictly contracting functions are actually the functions that one ought to be interested in. We prove a constructive fixed-point theorem for these functions, introduce a corresponding induction principle, and study the related convergence process.

1 Introduction

This work is part of a larger effort aimed at the construction of well defined mathematical models that will inform the design of programming languages and model-based design tools for timed systems. We use the term "timed" rather liberally here to refer to any system that will determinately order its events relative to some physical or logical clock. But our emphasis is on timed computation, with examples ranging from concurrent and distributed real-time software to hardware design, and from discrete-event simulation to continuous-time and hybrid modelling, spanning the entire development process of what we would nowadays refer to as cyber-physical systems. Our hope is that our work will lend insight into the design and application of the many languages and tools that have and will increasingly come into use for the design, simulation, and analysis of such systems. Existing languages and tools to which this work applies, to varying degrees, include hardware description languages such as VHDL (see [1]) and SystemC (see [2]), modeling and simulation tools such as Simulink and LabVIEW, network simulation tools such as ns-2/ns-3 and OPNET, and general-purpose simulation formalisms such as DEVS (see [63], [64]), or even emerging standards such as OMG's SysML (see [3]) and SAE's AADL (see [16]).

Considering the breadth of our informal definition for timed systems, we cannot hope for a comprehensive formalism or syntax for such systems at a granularity finer than that of a network of components. We will thus ignore any internal structure or state, and think of any particular component as an opaque flow transformer. Formally, we will model such components as functions, and use a suitably generalized concept of signal as flow (see Definition 2.2). This point of view is consistent with the one presented by most of the languages and tools mentioned above.

The greatest challenge in the construction of such a model is, by and large, the interpretation of feedback. Feedback is an extremely useful control mechanism, present in all but the most trivial systems. But it makes systems self-referential, with one signal depending on another, and vice versa (see Figure 1).

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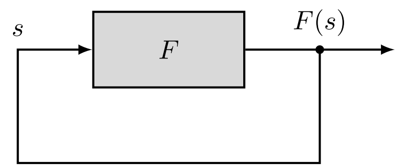

Figure 1. Block-diagram of a functional component F in feedback. The input signal s and the output signal $F(s)$ are but the same signal; that is, $s = F(s)$.

Mathematically, this notion of self-reference manifests itself in the form of a fixed-point problem, as illustrated by the simple block-diagram of Figure 1: the input signal s and the output signal $F(s)$ are but the same signal transmitted over the feedback wire of the system; unless F has a fixed point, the system has no model; unless F has a unique or otherwise canonically chosen fixed point, the model is not uniquely determined; unless we can construct the unique or otherwise canonically chosen fixed point of F, we cannot know what the model is. This imposes constraints on the functions that one may use to model components, and thus, ultimately, on components themselves.

From both a programming and a modelling point of view, the functions of primary interest to the study of timed systems are the causal functions. Causal functions model components that are non-anticipative, meaning that the output of the component does not depend on future values of its input. But non-anticipative components can still react instantaneously to input stimuli, refusing to assume a well defined behavior when arranged in a feedback configuration (see Example 3.4). For this reason, causal functions must be constrained further.

One idea is to impose a positive lower bound on the reaction time of the component. This was successfully carried out, first by Zeigler in [63], then by Yates and Gao in [61] and [60], then by Müller and Scholz in [43], and later again by one of us and colleagues in [28], [27], and [30]. The same idea had also been used in the context of timed systems by Reed and Roscoe in [52] and [53] under the rubric of realism, but there are also good technical reasons for it. The bounded reaction-time constraint can be used to preclude what is known as the real-time programming version of *Zeno's paradox*, according to which, infinitely many events take place in a finite interval of time (see [4]). This can, and generally does, prevent the use of classical results from fixed-point theory, such as the *Banach contraction principle* [33], which has undoubtedly been the most successful tool in the treatment of recursion and feedback in timed systems (e.g., see [52], [53], [60], [43], [28], [27], [30]). But even so, the constraint is excessive. For, even if not physically realizable, components that violate it are perfectly viable and extremely common in modelling and simulation, where time is represented as an ordinary program variable. And after the recently proposed extension of modelling and simulation techniques with the capability to relate logical and physical time (see [65] and [14]), such components may even find their way into programming models for embedded and distributed real-time systems. The question is how much one can relax the bounded reaction-time constraint.

The first, natural step in this line of inquiry is to dispose of any bound, and simply rule out what has caused trouble in the first place: instantaneous reaction. What we are left with is the class of strictly causal functions. And the first question to ask about strictly causal functions is whether every such function has a fixed point. But in order to answer this question, we need a formal, mathematical definition of what a strictly causal function is.

In [28], [27], and [30], strictly causal functions were defined to be the functions that are strictly contracting with respect to the *Cantor metric* (also called the *Baire distance*) on signals over non-negative real time. This turned out to be rather limiting, not only with respect to what we might think of as a strictly causal function, but also with respect to what we might think of as time (see [33]). In [44], an alternative definition was put forward, better fit to intuition, using only that one aspect of time truly relevant to causality: order. In [33], this definition was formalized using a generalized distance function, according to which, the distance between two signals is the largest segment of time closed under time precedence, and over which the two signals agree. This once more identified “strictly causal” with “strictly contracting”. But in all [28], [27], [44], [30], and [33], the precise relationship between the proposed definition and the classical notion of strict causality, as established within the physics and engineering communities, was never formally examined, only informally presumed.

From a classical standpoint, a component is strictly causal if and only if its output at any time depends only on past values of the input. This is probably a folklore definition, but one that is universally accepted. After a careful, precise formalization of it, we show the following:

- There is a strictly causal endofunction that has no fixed point (see Example 3.8).

Therefore, the class of strictly causal functions is, in its entire generality, too large. In fact, even the class of strictly causal functions that do have a fixed point is too large. In particular, we show the following:

- There is a strictly causal endofunction that has more than one fixed point, among which there is no canonical, or otherwise sensible choice (see Example 3.10).

An immediate consequence is that both classes are actually different from the class of strictly contracting functions of [33]. This is because every strictly contracting endofunction of [33] has exactly one fixed point (see [33, thm. 3]).

Stimulated by the latter fact, we begin to probe the exact relationship between strictly causal functions and the strictly contracting functions of [33] (henceforth referred to simply as strictly contracting functions). We prove the following:

- Every strictly contracting function is strictly causal (see Theorem 4.8).

A pleasing development would be that every strictly causal function that has a unique fixed point be strictly contracting. This is too much to hope for though, and we show the following:

- There is a strictly causal endofunction that has a unique fixed point, but is not strictly contracting (see Example 4.6).

However, we prove the following:

- A function from one set of signals to another is strictly contracting if and only if for every causal function from the latter set to the former, the composition of the two functions has a fixed point (see Theorem 4.7).

This is a key result. Besides completely characterizing strictly contracting functions in terms of the classical notion of causality, it identifies the class of all such functions as the largest class of functions that have a fixed point not by some fortuitous coincidence, but as a direct consequence of their causality properties. The implication, we believe, is that the class of strictly contracting functions is the largest class of strictly causal functions that one can reasonably hope to attain a uniform fixed-point theory for.

Interestingly, and rather pleasingly, in the case of computational timed systems, the situation is much simpler. In that case, components are expected to operate not on all signals, but only on discrete-event ones. And once we restrict the domains of the functions to reflect this, the difference between strictly causal functions and strictly contracting ones vanishes. A bit more generally, we prove the following:

- If the domain of every signal in the domain of the function is well ordered under the time precedence relation, then the function is strictly causal if and only if it is strictly contracting (see Corollary 4.11).

In other words, when it comes to timed computation, which includes the case of all languages and tools mentioned in the beginning of this introduction, the fixed-point theory of strictly contracting functions is exactly the fixed-point theory of strictly causal functions. Incidentally, when all signals, input and output, satisfy the above condition, even the definition of [28], [27], and [30] becomes accurate, albeit for different reasons.

Either way, it is, we hope, clear that the fixed-point problem that one ought to be interested in is the one pertaining to the class of strictly contracting functions. And this is a problem that is not typical in computer science. For despite the abundance of fixed-point problems in the field, it is almost invariably the fixed-point theory of order-preserving functions on ordered sets or that of contraction mappings on metric spaces that is applied for their solution. However, neither of those is generally applicable to the problem in hand. The reason is that there is no non-trivial order relation that will render every strictly contracting endofunction order-preserving (see Theorem A.2), and no metric that will render every such endofunction a contraction mapping (see Theorem A.4).

To our knowledge, there are only two results in the existing literature that are generally applicable to the problem in hand. The first is the fixed-point theorem of Priess-Crampe and Ribenboim for strictly contracting functions on spherically complete generalized ultrametric spaces (see [49, thm. 1]). The second is an ad hoc fixed-point theorem proved by Naundorf, specific to the type of functions considered here (see [44, thm. 1]). And although the two have been proved in very different ways, they are both inherently non-constructive, and hence, both inadequate for our purposes.

Our main contribution in this work is a constructive fixed-point theorem for strictly contracting functions on sets of signals. We use the term "constructive" in the stronger sense of [11] here to mean that we characterize fixed points as "limits of stationary transfinite iteration sequences". Specifically, for every suitable set X of signals, and every strictly contracting function F on X, we prove the following:

- The unique fixed point of F is the limit of the transfinite orbit of every post-fixed point of F under the function $\lambda x : X \,.\, F(x) \sqcap F(F(x))$ (see Theorem 5.13).

By "suitable" we mean a non-empty, directed-complete subsemilattice of the complete semilattice of all signals under the signal prefix relation (see Section 2.3). By "limit" of an orbit we mean the least upper bound or join of that orbit in that subsemilattice. And by "$\sqcap$" we denote the greatest lower bound or meet operation of that semilattice.

The reader might of course ask what the practical merits of such a characterization are. We consult Cousot and Cousot for an answer (see [11, p. 44]):

> The advantage of characterizing fixed points by iterative schemes is that they lead to practical computation or approximation procedures. Also the definition of fixed points as limits of stationary iteration sequences allows the use of transfinite induction for proving properties of these fixed points.

Their first point is rather evident from Lemma 5.9.2 and Theorem 5.13 here. And as regards their second point, we prove the following:

- The unique fixed point of F is a member of every non-empty, strictly inductive subset of X that is closed under the function $\lambda x : X \,.\, F(x) \sqcap F(F(x))$ (see Theorem 5.16).

We believe this to be a very promising induction principle, seemingly better a proof rule than the ones afforded by the fixed-point theories of order-preserving functions and contraction mappings (see discussion in Section 5.3).

What is interesting to observe is that our characterization is purely order-theoretic. It also bares a close resemblance to the respective characterization in the classical order-theoretic case (see [11]). This resemblance is most acutely pronounced in the following corollary characterization, which is identical in form to Tarski's characterization of greatest fixed points of order-preserving functions on complete lattices (see [59, thm. 1]):

- The unique fixed point of F is the join of all post-fixed points of F (see Theorem 5.14).

What is there to account for this?

As Davey and Priestley observe in [12, p. 182], "order theory plays a role when X carries an order and when the [fixed-point] can be realized as the join of elements which approximate it". And so, our characterization is just another testament to this empirical observation. But our derivation is in no way a reduction to an order-theoretic fixed-point problem, or more specifically, a fixed-point problem involving an order-preserving function. In fact, we show the following:

- There is a suitable (in the above sense) set X of signals, and a strictly contracting function F on X such that $\lambda x : X \,.\, F(x) \sqcap F(F(x))$ does not preserve the prefix relation (see Example 5.15)

Rather, it is the interplay between the generalized distance function and the prefix relation on signals that validates our construction, and accounts for the above observations. Working out the rules that govern this interplay (see Section 2.4) is the other major contribution of this work. Here, these rules serve to determine the extent of our results, and simplify our proofs. But elsewhere, we prove that clauses 1 and 2 of Proposition 2.15 constitute a complete axiomatization of the relationship between the generalized distance function and the prefix relation in subsemilattices of signals.

The rest of this document is organized into seven sections. In Section 2, we set up the background: we review the concept of signal, define the generalized distance function and prefix relation pertaining to that concept, and study the relationship between the two. In Section 3, we formalize the notions of causality and strict causality, and through a series of examples, demonstrate that these notions are by themselves too weak to accommodate a uniform fixed-point theory suitable for a semantic theory of timed systems. In Section 4, we introduce contracting and strictly contracting functions, and examine their relationship to the causal and strictly causal functions respectively, as defined in Section 3. In particular, we provide evidence to the argument that strictly contracting functions are really the functions that one ought to focus on. The fixed-point theory of these functions is developed in Section 5. Starting from a more structured reworking of Naundorf's fixed-point existence argument, we prove a constructive fixed-point theorem, introduce a corresponding induction principle, and study the related convergence process. In Section 6, we review the developed theory, assessing its practicability, and in Section 7, we discuss related work. We conclude in Section 8 with a few directions for future work. Finally, in Appendix A, we include proof that the standard fixed-point theories of ordered sets and metric spaces are not generally applicable to the problem in hand.

2 Background

In this section, we set the scene for our work. Our basic framework is inspired by the tagged-signal model of [28]. The generalized distance function of Section 2.2 was first introduced and studied in [33], and the prefix relation of Section 2.3 is rather standard, but the analysis of the relationship between the two in Section 2.4 is new.

2.1 Signals

The term "signal" is typically applied to something that conveys information via some form of variation (e.g., see [46], [29]). Mathematically, one commonly represents signals as functions over one or more independent variables. Here, we are concerned with signals that involve a single independent variable standing for some, possibly conceptual, notion of time.

We postulate a non-empty set T of *tags*, and an order relation[1] $\preceq$ on T.

We use T to represent our time domain. The order relation $\preceq$ is meant to play the role of a chronological precedence relation, and therefore, it is reasonable to require that $\preceq$ be a total order. However, such a requirement is often unnecessary. For the sake of generality, we shall assume that $\langle \mathrm{T}, \preceq \rangle$ is an arbitrary

[1] A binary relation R is an *order* relation if and only if R is reflexive, transitive, and antisymmetric.

ordered set. But for the sake of simplicity, if and when a stronger assumption is needed, we shall forgo our pursuit of generality, and fall back on the requirement that $\preceq$ be a total order.

We would like to define signals as functions over an independent variable ranging over T. But being primarily concerned with computational systems, we should expect our definition to accommodate the representation of variations that may be undefined for some instances or even periods of time. In fact, we think of such instances and periods of time as part of the variational information. Such considerations lead directly to the concept of partial function.

We postulate a non-empty set V of *values*.

Definition 2.1. An *event* is an ordered pair $\langle \tau, v \rangle \in \mathrm{T} \times \mathrm{V}$.

We write E for the set of all events.

Definition 2.2. A *signal* is a single-valued[2] subset of E.

We write S for the set of all signals.

Notice that the empty set is vacuously single-valued, and hence, by Definition 2.2, a signal.

We call the empty set the *empty signal*.

We adopt common practice in modern set theory and identify a function with its graph. A signal is then a function with domain some subset of T, and range some subset of V, or in other words, a partial function from T to V.

Assume $s_1, s_2 \in \mathrm{S}$ and $\tau \in \mathrm{T}$.

We write $s_1(\tau) \simeq s_2(\tau)$ if and only if one of the following is true:

1. $\tau \notin \mathsf{dom}\, s_1$ and $\tau \notin \mathsf{dom}\, s_2$;
2. $\tau \in \mathsf{dom}\, s_1$, $\tau \in \mathsf{dom}\, s_2$, and $s_1(\tau) = s_2(\tau)$.

In other words, we use $\simeq$ to denote Kleene's equality among partially defined value expressions.

2.2 The generalized distance function

There is a natural, if abstract, notion of distance between any two signals, corresponding to the largest segment of time closed under time precedence, and over which the two signals agree; the larger the segment, the closer the two signals. Under certain conditions, this can be couched in the language of metric spaces (e.g., see [28], [27], [30]). All one needs is a map from such segments of time to non-negative real numbers. But this step of indirection excessively restricts the kind of ordered sets that one can use as models of time (see [33]), and in fact, can be avoided as long as one is willing to think about the notion of distance in more abstract terms, and use the language of generalized ultrametric spaces[3] instead (see [50]).

[2] For every set A and B, and every $S \subseteq A \times B$, S is *single-valued* if and only if for any $\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in S$, if $a_1 = a_2$, then $b_1 = b_2$.

[3] For every set A, every pointed[4] ordered set $\langle P, \leqslant, 0 \rangle$, and every function d from $A \times A$ to P, $\langle A, P, \leqslant, 0, d \rangle$ is a *generalized ultrametric space* if and only if for any $a_1, a_2, a_3 \in A$ and every $p \in P$, the following are true:

1. $d(a_1, a_2) = 0$ if and only if $a_1 = a_2$;
2. $d(a_1, a_2) = d(a_2, a_1)$;
3. if $d(a_1, a_2) \leqslant p$ and $d(a_2, a_3) \leqslant p$, then $d(a_1, a_3) \leqslant p$.

[4] An ordered set is *pointed* if and only if it has a least element. We write $\langle P, \leqslant, 0 \rangle$ for a pointed ordered set $\langle P, \leqslant \rangle$ with least element 0.

We write d for a function from $\mathrm{S} \times \mathrm{S}$ to $\mathscr{L}\langle \mathrm{T}, \preceq \rangle$ such that for every $s_1, s_2 \in \mathrm{S}$,[5]

$$\mathrm{d}(s_1, s_2) = \{\tau \mid \tau \in \mathrm{T}, \text{ and for every } \tau' \preceq \tau,\ s_1(\tau') \simeq s_2(\tau')\}.$$

Proposition 2.3. *$\langle \mathrm{S}, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d} \rangle$ is a generalized ultrametric space.*

Proof. See [34, lem. 1]. □

The following is immediate, and indeed, equivalent:

Proposition 2.4. *For every $s_1, s_2, s_3 \in \mathrm{S}$, the following are true:*

1. $\mathrm{d}(s_1, s_2) = \mathrm{T}$ *if and only if* $s_1 = s_2$;
2. $\mathrm{d}(s_1, s_2) = \mathrm{d}(s_2, s_1)$;
3. $\mathrm{d}(s_1, s_2) \supseteq \mathrm{d}(s_1, s_3) \cap \mathrm{d}(s_3, s_2)$.

We refer to clause 1 as the *identity of indiscernibles*, clause 2 as *symmetry*, and clause 3 as the *generalized ultrametric inequality*.

Proposition 2.5. *$\langle \mathrm{S}, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d} \rangle$ is spherically complete*[7].

Proof. See [34, lem. 2]. □

Spherical completeness implies Cauchy-completeness[9], but the converse is not true in general (see [22, prop. 10]). The following shows that it is not true in the special case of generalized ultrametric spaces of signals either:

Example 2.6. Suppose that $\mathrm{T} = \mathbb{R}$, and $\preceq$ is the standard order on $\mathbb{R}$.

Let $X = \{s \mid s \in \mathrm{S} \text{ and for every } \tau \in \mathrm{T},\ s \restriction \{\tau' \mid \tau' \preceq \tau\} \text{ is finite}\}$.[12]

It is easy to see that $\langle X, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d} \rangle$ is Cauchy-complete.

Let v be a value in V.

Let $C = \{\{s \mid s \in X \text{ and } \mathrm{d}(s, \{\langle 1 - \frac{1}{m+1}, v \rangle \mid m \leq n\}) \supseteq (-\infty, \frac{1}{n+1}]\} \mid n \in \mathbb{N}\}$.[13]

The ordered set $\langle C, \subseteq \rangle$ is a non-empty chain of balls in $\langle X, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d} \rangle$, but $\bigcap C = \emptyset$. Thus, $\langle X, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d} \rangle$ is not spherically complete.

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[5] For every ordered set $\langle P, \leqslant \rangle$, we write $\mathscr{L}\langle P, \leqslant \rangle$ for the set of all lower sets[6] of $\langle P, \leqslant \rangle$.

[6] For every ordered set $\langle P, \leqslant \rangle$, and every $L \subseteq P$, L is a *lower set* (also called a *down-set* or an *order ideal*) of $\langle P, \leqslant \rangle$ if and only if for any $p_1, p_2 \in P$, if $p_1 \leqslant p_2$ and $p_2 \in L$, then $p_1 \in L$.

[7] A generalized ultrametric space $\langle A, P, \leqslant, 0, d \rangle$ is *spherically complete* if and only if for every non-empty chain C of balls[8] in $\langle A, P, \leqslant, 0, d \rangle$, $\bigcap C \neq \emptyset$.

[8] For every generalized ultrametric space $\langle A, P, \leqslant, 0, d \rangle$, and every $B \subseteq A$, B is a *ball* in $\langle A, P, \leqslant, 0, d \rangle$ if and only if there is $a \in A$ and $p \in P$ such that $B = \{a' \in A \mid d(a', a) \leqslant p\}$.

[9] A generalized ultrametric space $\langle A, P, \leqslant, 0, d \rangle$ is *Cauchy-complete* if and only if for every sequence $\langle a_n \mid n \in \omega \rangle$ over A, if $\langle a_n \mid n \in \omega \rangle$ is Cauchy[10] in $\langle A, P, \leqslant, 0, d \rangle$, then there is $a \in A$ such that for every $p \in P \setminus \{0\}$, there is $n \in \omega$ such that for every $n' > n$, $d(a_{n'}, a) < p$.[11]

[10] For every generalized ultrametric space $\langle A, P, \leqslant, 0, d \rangle$, a sequence $\langle a_n \mid n \in \omega \rangle$ over A is Cauchy in $\langle A, P, \leqslant, 0, d \rangle$ if and only if for every $p \in P \setminus \{0\}$, there is $n \in \omega$ such that for every $n_1, n_2 > n$, $d(a_{n_1}, a_{n_2}) < p$.

[11] We write ω for the least limit ordinal.

[12] For every function f and every set A, we write $f \restriction A$ for the *restriction* of f to A, namely the function $\{\langle a, b \rangle \mid \langle a, b \rangle \in f \text{ and } a \in A\}$.

[13] We write $\mathbb{N}$ for the set of all natural numbers.

The importance of spherical completeness will become clear in Section 4.2 (see Theorem 4.4 and Theorem 4.5).

Finally, notice that if $\langle \mathrm{T}, \preceq \rangle$ is totally ordered, then $\langle \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq \rangle$ is also totally ordered. This is really why more can be proved under the requirement that $\preceq$ be a total order. Proposition 2.7 is a case in point that will come of use.

Assume $s_1, s_2, s_3 \in \mathrm{S}$ and $L \in \mathscr{L}\langle \mathrm{T}, \preceq \rangle$.

Proposition 2.7. *If* $\langle \mathrm{T}, \preceq \rangle$ *is totally ordered, then if* $\mathrm{d}(s_1, s_2) \supset L$ *and* $\mathrm{d}(s_2, s_3) \supset L$*, then* $\mathrm{d}(s_1, s_3) \supset L$*.*

Proof. Suppose that $\langle \mathrm{T}, \preceq \rangle$ is totally ordered.

Then $\langle \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq \rangle$ is totally ordered, and thus, if $\mathrm{d}(s_1, s_2) \supset L$ and $\mathrm{d}(s_2, s_3) \supset L$, then

$$\mathrm{d}(s_1, s_2) \cap \mathrm{d}(s_2, s_3) \supset L. \tag{1}$$

And since

$$\mathrm{d}(s_1, s_2) \supseteq \mathrm{d}(s_1, s_2) \cap \mathrm{d}(s_2, s_3)$$

and

$$\mathrm{d}(s_2, s_3) \supseteq \mathrm{d}(s_1, s_2) \cap \mathrm{d}(s_2, s_3),$$

by the generalized ultrametric inequality,

$$\mathrm{d}(s_1, s_3) \supseteq \mathrm{d}(s_1, s_2) \cap \mathrm{d}(s_2, s_3). \tag{2}$$

Thus, by (1) and (2), $\mathrm{d}(s_1, s_3) \supset L$. □

The above strict variant of the generalized ultrametric inequality cannot be proved in general.

Example 2.8. Suppose that $\mathrm{T} = \{0, 1\}$, and $\preceq$ is the discrete order[14] on $\{0, 1\}$.

Let v be a value in V.

Let $s_1 = \{\langle 0, v \rangle\}$.

Let $s_2 = \emptyset$.

Let $s_3 = \{\langle 1, v \rangle\}$.

Then $\mathrm{d}(s_1, s_2) = \{1\} \supset \emptyset$ and $\mathrm{d}(s_2, s_3) = \{0\} \supset \emptyset$, but $\mathrm{d}(s_1, s_3) = \emptyset$.

2.3 The prefix relation

There is also a natural order relation on signals, namely the prefix relation on signals.

We write $\sqsubseteq$ for a binary relation on S such that for every $s_1, s_2 \in \mathrm{S}$,

$$s_1 \sqsubseteq s_2 \iff \text{for every } \tau, \tau' \in \mathrm{T}, \text{ if } \tau \in \mathsf{dom}\, s_1 \text{ and } \tau' \preceq \tau, \text{ then } s_1(\tau') \simeq s_2(\tau').$$

Assume $s_1, s_2 \in \mathrm{S}$.

[14] For every set A, the *discrete order* on A is the smallest order relation on A, namely the unique order relation on A with respect to which any two distinct members of A are incomparable.

We say that s_1 is a *prefix* of s_2 if and only if $s_1 \sqsubseteq s_2$.

Notice that for every $s \in \mathrm{S}$, $\emptyset \sqsubseteq s$; that is, the empty signal is a prefix of every signal.

The following is straightforward:

Proposition 2.9. *$\langle \mathrm{S}, \sqsubseteq \rangle$ is an ordered set.*

Proposition 2.10. *For every $C \subseteq \mathrm{S}$ such that C is consistent*[15] *in $\langle \mathrm{S}, \sqsubseteq \rangle$, $\bigcup C$ is the least upper bound of C in $\langle \mathrm{S}, \sqsubseteq \rangle$.*

Proof. Assume $C \subseteq \mathrm{S}$ such that C is consistent in $\langle \mathrm{S}, \sqsubseteq \rangle$.

We first prove that $\bigcup C \in \mathrm{S}$.

Suppose, toward contradiction, that $\bigcup C \notin \mathrm{S}$. Then there are $s_1, s_2 \in C$ and τ such that $\tau \in \mathsf{dom}\, s_1$ and $\tau \in \mathsf{dom}\, s_2$, but $s_1(\tau) \neq s_2(\tau)$. Thus, $\{s_1, s_2\}$ cannot have an upper bound in $\langle \mathrm{S}, \sqsubseteq \rangle$, contrary to the hypothesis that C is consistent in $\langle \mathrm{S}, \sqsubseteq \rangle$.

Therefore, $\bigcup C \in \mathrm{S}$.

Assume $s \in C$.

Suppose, toward contradiction, that $s \not\sqsubseteq \bigcup C$. Then there are $\tau, \tau' \in \mathrm{T}$ such that $\tau \in \mathsf{dom}\, s$ and $\tau' \preceq \tau$, but $s(\tau') \not\simeq (\bigcup C)(\tau')$. And since $s \subseteq \bigcup C$, $\tau' \notin \mathsf{dom}\, s$, and there is $s' \in C$ such that $\tau' \in \mathsf{dom}\, s'$. However, $\{s, s'\}$ cannot have an upper bound in $\langle \mathrm{S}, \sqsubseteq \rangle$, contrary to the hypothesis that C is consistent in $\langle \mathrm{S}, \sqsubseteq \rangle$.

Therefore, $s \sqsubseteq \bigcup C$.

Assume $u \in \mathrm{S}$ such that u is an upper bound of C in $\langle \mathrm{S}, \sqsubseteq \rangle$.

Suppose, toward contradiction, that $\bigcup C \not\sqsubseteq u$. Then there are $\tau, \tau' \in \mathrm{T}$ such that $\tau \in \mathsf{dom} \bigcup C$ and $\tau' \preceq \tau$, but $(\bigcup C)(\tau') \not\simeq u(\tau')$. Thus, there is $s \in C$ such that $\tau \in \mathsf{dom}\, s$, but $s(\tau') \not\simeq u(\tau')$, and hence, $s \not\sqsubseteq u$, contrary to the assumption that u is an upper bound of C in $\langle \mathrm{S}, \sqsubseteq \rangle$.

Therefore, $\bigcup C \sqsubseteq u$.

Thus, by generalization, $\bigcup C$ is the least upper bound of C in $\langle \mathrm{S}, \sqsubseteq \rangle$. □

Assume $C \subseteq \mathrm{S}$ such that C is consistent in $\langle \mathrm{S}, \sqsubseteq \rangle$.

We write $\bigsqcup C$ for the least upper bound of C in $\langle \mathrm{S}, \sqsubseteq \rangle$.

The following is immediate:

Proposition 2.11. *$\langle \mathrm{S}, \sqsubseteq \rangle$ is a complete semilattice*[16]*.*

Assume non-empty $X \subseteq \mathrm{S}$.

We write $\bigsqcap X$ for the greatest lower bound of X in $\langle \mathrm{S}, \sqsubseteq \rangle$.

The next proposition provides an alternative, and arguably, more intuitive definition of the prefix relation on signals, that will be useful in relating the latter with the generalized distance function of Section 2.2.

[15] For every ordered set $\langle P, \leqslant \rangle$, and every $C \subseteq P$, C is *consistent* in $\langle P, \leqslant \rangle$ if and only if $C \neq \emptyset$, and every finite subset of C has an upper bound in $\langle P, \leqslant \rangle$.

[16] An ordered set $\langle P, \leqslant \rangle$ is a *complete semilattice* if and only if every non-empty subset of P has a greatest lower bound in $\langle P, \leqslant \rangle$, and every subset of P that is directed[17] in $\langle P, \leqslant \rangle$ has a least upper bound in $\langle P, \leqslant \rangle$.

[17] For every ordered set $\langle P, \leqslant \rangle$, and every $D \subseteq P$, D is *directed* in $\langle P, \leqslant \rangle$ if and only if $D \neq \emptyset$, and every finite subset of D has an upper bound in $\langle D, \leqslant \rangle$.

Proposition 2.12. *$s_1 \sqsubseteq s_2$ if and only if there is $L \in \mathscr{L}\langle \mathrm{T}, \preceq \rangle$ such that $s_1 = s_2 \restriction L$.*

Proof. Suppose that $s_1 \sqsubseteq s_2$.

Suppose, toward contradiction, that

$$s_1 \neq s_2 \restriction \mathrm{d}(s_1, s_2).$$

Then there is τ such that

$$s_1(\tau) \not\simeq (s_2 \restriction \mathrm{d}(s_1, s_2))(\tau).$$

Suppose that $\tau \in \mathsf{dom}\, s_1$. Then, since $s_1 \sqsubseteq s_2$, $s_1(\tau) = s_2(\tau)$, and thus, $\tau \notin \mathrm{d}(s_1, s_2)$. Thus, there is $\tau' \preceq \tau$ such that $s_1(\tau') \not\simeq s_2(\tau')$, contrary to the hypothesis that $s_1 \sqsubseteq s_2$.

Otherwise, $\tau \notin \mathsf{dom}\, s_1$. Then $\tau \in \mathsf{dom}\, s_2 \restriction \mathrm{d}(s_1, s_2)$. Thus, $\tau \in \mathrm{d}(s_1, s_2)$, but $s_1(\tau) \simeq s_2(\tau)$, obtaining a contradiction.

Therefore,

$$s_1 = s_2 \restriction \mathrm{d}(s_1, s_2),$$

and thus, there is $L \in \mathscr{L}\langle \mathrm{T}, \preceq \rangle$, namely $\mathrm{d}(s_1, s_2)$, such that $s_1 = s_2 \restriction L$.

Conversely, suppose that there is $L \in \mathscr{L}\langle \mathrm{T}, \preceq \rangle$ such that $s_1 = s_2 \restriction L$. Then for every $\tau, \tau' \in \mathrm{T}$, if $\tau \in \mathsf{dom}\, s_1$ and $\tau' \preceq \tau$, then, since L is a lower set of $\langle \mathrm{T}, \preceq \rangle$, $\tau' \in L$, and thus, $s_1(\tau') \simeq s_2(\tau')$. Thus, $s_1 \sqsubseteq s_2$. □

2.4 The relationship between the generalized distance function and the prefix relation

Looking more closely at the proof of Proposition 2.12, we see that there is actually a canonical choice for the witness L, namely $\mathrm{d}(s_1, s_2)$. The next theorem is a powerful generalization of this observation.

Theorem 2.13. *For every non-empty $X \subseteq \mathrm{S}$ and every $s \in X$,*

$$\bigsqcap X = s \restriction \bigcap \{\mathrm{d}(s_1, s_2) \mid s_1, s_2 \in X\}.$$

Proof. Assume non-empty $X \subseteq \mathrm{S}$.

Assume $s \in X$.

Assume $s' \in X$.

Suppose, toward contradiction, that

$$s \restriction \mathrm{d}(s, s') \neq s' \restriction \mathrm{d}(s, s').$$

Then there is τ such that

$$(s \restriction \mathrm{d}(s, s'))(\tau) \not\simeq (s' \restriction \mathrm{d}(s, s'))(\tau).$$

Without loss of generality, assume that $\tau \in \mathsf{dom}(s \restriction \mathrm{d}(s, s'))$. Then $\tau \in \mathrm{d}(s, s')$, and thus, $s(\tau) \simeq s'(\tau)$. Hence,

$$(s \restriction \mathrm{d}(s, s'))(\tau) \simeq (s' \restriction \mathrm{d}(s, s'))(\tau),$$

obtaining a contradiction.

Therefore,

$$s \restriction \mathrm{d}(s, s') = s' \restriction \mathrm{d}(s, s').$$

Since $s, s' \in X$,

$$\mathrm{d}(s, s') \supseteq \bigcap\{\mathrm{d}(s_1, s_2) \mid s_1, s_2 \in X\},$$

and thus,

$$\begin{aligned} s \restriction \bigcap\{\mathrm{d}(s_1, s_2) \mid s_1, s_2 \in X\} &= (s \restriction \mathrm{d}(s, s')) \restriction \bigcap\{\mathrm{d}(s_1, s_2) \mid s_1, s_2 \in X\} \\ &= (s' \restriction \mathrm{d}(s, s')) \restriction \bigcap\{\mathrm{d}(s_1, s_2) \mid s_1, s_2 \in X\} \\ &= s' \restriction \bigcap\{\mathrm{d}(s_1, s_2) \mid s_1, s_2 \in X\}. \end{aligned}$$

Hence, by Proposition 2.12,

$$s \restriction \bigcap\{\mathrm{d}(s_1, s_2) \mid s_1, s_2 \in X\} \sqsubseteq s'.$$

Thus, by generalization, $s \restriction \bigcap\{\mathrm{d}(s_1, s_2) \mid s_1, s_2 \in X\}$ is a lower bound of X in $\langle \mathrm{S}, \sqsubseteq \rangle$.

Assume $l \in \mathrm{S}$ such that l is a lower bound of X in $\langle \mathrm{S}, \sqsubseteq \rangle$.

Since l is a lower bound of X in $\langle \mathrm{S}, \sqsubseteq \rangle$, $l \sqsubseteq s$.

Suppose, toward contradiction, that

$$l \not\sqsubseteq s \restriction \bigcap\{\mathrm{d}(s_1, s_2) \mid s_1, s_2 \in X\}.$$

Then there are $\tau, \tau' \in \mathrm{T}$ such that $\tau \in \mathsf{dom}\, l$ and $\tau' \preceq \tau$, but

$$l(\tau') \not\simeq (s \restriction \bigcap\{\mathrm{d}(s_1, s_2) \mid s_1, s_2 \in X\})(\tau').$$

If $\tau' \notin \mathsf{dom}\, l$, then $\tau' \in \mathsf{dom}(s \restriction \bigcap\{\mathrm{d}(s_1, s_2) \mid s_1, s_2 \in X\})$. Thus, $\tau' \in \mathsf{dom}\, s$, and hence, $l(\tau') \not\simeq s(\tau')$. Thus, $l \not\sqsubseteq s$, contrary to the assumption that l is a lower bound of X in $\langle \mathrm{S}, \sqsubseteq \rangle$.

Otherwise, $\tau' \in \mathsf{dom}\, l$, and since $l \sqsubseteq s$, $\tau' \in \mathsf{dom}\, s$ and $l(\tau') = s(\tau')$.

If $\tau' \in \bigcap\{\mathrm{d}(s_1, s_2) \mid s_1, s_2 \in X\}$, then

$$\begin{aligned} (s \restriction \bigcap\{\mathrm{d}(s_1, s_2) \mid s_1, s_2 \in X\})(\tau') &\simeq s(\tau') \\ &= l(\tau'), \end{aligned}$$

obtaining a contradiction.

Otherwise, $\tau' \notin \bigcap\{\mathrm{d}(s_1, s_2) \mid s_1, s_2 \in X\}$, and thus, there are $s_1, s_2 \in X$ such that $\tau' \notin \mathrm{d}(s_1, s_2)$. Then there is $\tau'' \preceq \tau'$ such that $s_1(\tau'') \not\simeq s_2(\tau'')$, and thus, $l(\tau'') \not\simeq s_1(\tau'')$ or $l(\tau'') \not\simeq s_2(\tau'')$. Without loss of generality, assume that $l(\tau'') \not\simeq s_1(\tau'')$. Then, since $\tau \in \mathsf{dom}\, l$ and $\tau'' \preceq \tau$, $l \not\sqsubseteq s_1$, contrary to the assumption that l is a lower bound of X in $\langle \mathrm{S}, \sqsubseteq \rangle$.

Therefore,

$$l \sqsubseteq s \restriction \bigcap\{\mathrm{d}(s_1, s_2) \mid s_1, s_2 \in X\}.$$

Thus, by generalization, $s \restriction \bigcap\{\mathrm{d}(s_1, s_2) \mid s_1, s_2 \in X\}$ is the greatest lower bound of X in $\langle \mathrm{S}, \sqsubseteq \rangle$. $\square$

The following is immediate from the generalized ultrametric inequality:

Corollary 2.14. *For every non-empty $X \subseteq \mathrm{S}$ and every $s \in X$,*

$$\bigsqcap X = s \restriction \bigcap \{\mathrm{d}(s', s) \mid s' \in X\}.$$

Theorem 2.13 is a fine portrait of the relationship between d and $\sqsubseteq$. The only problem is that it is too concrete. Being expressed in the language of set theory, it is closely tied to the low-level representation of signals. In practice, one would rather work at a higher level of abstraction, and ignore the low-level representation details. The next proposition aims at distilling the essence of Theorem 2.13 (at least with respect to the needs of this work) into a couple of simple properties expressed in a language that only references d and $\sqsubseteq$.[18]

Proposition 2.15. *For every $s_1, s_2, s_3 \in \mathrm{S}$, the following are true:*

1. *if* $\mathrm{d}(s_1, s_2) \supseteq \mathrm{d}(s_1, s_3)$, *then* $s_1 \sqcap s_3 \sqsubseteq s_1 \sqcap s_2$;
2. $\mathrm{d}(s_1 \sqcap s_2, s_1 \sqcap s_3) \supseteq \mathrm{d}(s_2, s_3)$.

Proof. Assume $s_1, s_2, s_3 \in \mathrm{S}$.

Suppose that

$$\mathrm{d}(s_1, s_2) \supseteq \mathrm{d}(s_1, s_3). \tag{3}$$

Then, by Theorem 2.13 and (3),

$$\begin{aligned} s_1 \sqcap s_3 &= s_1 \restriction \mathrm{d}(s_1, s_3) \\ &= (s_1 \restriction \mathrm{d}(s_1, s_2)) \restriction \mathrm{d}(s_1, s_3) \\ &= (s_1 \sqcap s_2) \restriction \mathrm{d}(s_1, s_3). \end{aligned}$$

and thus, by Proposition 2.12,

$$s_1 \sqcap s_3 \sqsubseteq s_1 \sqcap s_2.$$

Thus, 1 is true.

Suppose, toward contradiction, that

$$\mathrm{d}(s_1 \sqcap s_2, s_1 \sqcap s_3) \not\supseteq \mathrm{d}(s_2, s_3).$$

Then there is τ such that $\tau \in \mathrm{d}(s_2, s_3)$, but $\tau \notin \mathrm{d}(s_1 \sqcap s_2, s_1 \sqcap s_3)$. Thus, there is $\tau' \preceq \tau$ such that

$$(s_1 \sqcap s_2)(\tau') \not\simeq (s_1 \sqcap s_3)(\tau'). \tag{4}$$

Without loss of generality, assume that

$$\tau' \in \mathsf{dom}(s_1 \sqcap s_2). \tag{5}$$

Then, by Theorem 2.13,

$$\tau' \in \mathrm{d}(s_1, s_2). \tag{6}$$

[18] Notice that $\sqcap$ is definable in $\langle \mathrm{S}, \sqsubseteq \rangle$, and conversely, $\sqsubseteq$ is definable in $\langle \mathrm{S}, \sqcap \rangle$.

And since $\tau \in \mathrm{d}(s_2, s_3)$ and $\tau' \preceq \tau$,

$$\tau' \in \mathrm{d}(s_2, s_3). \tag{7}$$

By (6), (7), and the generalized ultrametric inequality,

$$\tau' \in \mathrm{d}(s_1, s_3) \tag{8}$$

And by Theorem 2.13, (5), and (8),

$$\begin{aligned}(s_1 \sqcap s_2)(\tau') &= (s_1 \restriction \mathrm{d}(s_1, s_2))(\tau') \\ &= s_1(\tau') \\ &= (s_1 \restriction \mathrm{d}(s_1, s_3))(\tau') \\ &= (s_1 \sqcap s_3)(\tau').\end{aligned}$$

in contradiction to (4).

Therefore,

$$\mathrm{d}(s_1 \sqcap s_2, s_1 \sqcap s_3) \supseteq \mathrm{d}(s_2, s_3).$$

Thus, 2 is true. □

Looking more closely at the proof of Proposition 2.15.1, we see that Proposition 2.15.1 is actually true in every semilattice[19] of signals. This is not the case for Proposition 2.15.2.

Example 2.16. Suppose that $\mathrm{T} = \{0, 1, 2\}$, and $\preceq$ is the standard order on $\{0, 1, 2\}$.

Let v be a value in V.

Let $s_1 = \{\langle 0, v\rangle, \langle 1, v\rangle\}$.

Let $s_2 = \{\langle 0, v\rangle, \langle 2, v\rangle\}$.

Let $s_3 = \emptyset$.

Clearly, $\langle\{s_1, s_2, s_3\}, \sqsubseteq\rangle$ is a semilattice. However,

$$\begin{aligned}\mathrm{d}(s_1 \sqcap_{\{s_1,s_2,s_3\}} s_1, s_1 \sqcap_{\{s_1,s_2,s_3\}} s_2) &= \mathrm{d}(s_1, s_3) \\ &= \emptyset \\ &\not\supseteq \{0\} \\ &= \mathrm{d}(s_1, s_2).\end{aligned}$$

However, for every semilattice of signals, if that semilattice is a subsemilattice[20] of $\langle \mathrm{S}, \sqsubseteq\rangle$, then both clauses of Proposition 2.15 are true in it. Rather pleasingly, the converse is also true.

Proposition 2.17. *If $\langle X, \sqsubseteq\rangle$ is a semilattice, then the following are equivalent:*

1. *for every $s_1, s_2, s_3 \in X$, the following are true:*
   (a) *if $\mathrm{d}(s_1, s_2) \supseteq \mathrm{d}(s_1, s_3)$, then $s_1 \sqcap_X s_3 \sqsubseteq s_1 \sqcap_X s_2$;*

[19] An ordered set $\langle P, \leqslant\rangle$ is a *semilattice* if and only if every non-empty finite subset of P has a greatest lower bound in $\langle P, \leqslant\rangle$.

[20] For every semilattice $\langle P, \leqslant\rangle$, and every $S \subseteq P$, $\langle S, \leqslant\rangle$ is a *subsemilattice* of $\langle P, \leqslant\rangle$ if and only if every non-empty finite subset of S has a greatest lower bound in $\langle S, \leqslant\rangle$, and that greatest lower bound is the greatest lower bound of that subset in $\langle P, \leqslant\rangle$.

(b) $\mathrm{d}(s_1 \sqcap_X s_2, s_1 \sqcap_X s_3) \supseteq \mathrm{d}(s_2, s_3)$;

2. $\langle X, \sqsubseteq \rangle$ *is a subsemilattice of* $\langle \mathrm{S}, \sqsubseteq \rangle$.

Proof. Suppose that $\langle X, \sqsubseteq \rangle$ is a semilattice.

Suppose that 1 is true.

Suppose, toward contradiction, that 2 is not true. Then there are $s_1, s_2 \in X$ such that

$$s_1 \sqcap_X s_2 \sqsubset s_1 \sqcap s_2. \tag{9}$$

However, by 1b,

$$\begin{aligned} \mathrm{d}(s_1, s_1 \sqcap_X s_2) &= \mathrm{d}(s_1 \sqcap_X s_1, s_1 \sqcap_X s_2) \\ &\supseteq \mathrm{d}(s_1, s_2), \end{aligned}$$

and thus, by Proposition 2.15.1,

$$\begin{aligned} s_1 \sqcap s_2 &\sqsubseteq s_1 \sqcap (s_1 \sqcap_X s_2) \\ &= s_1 \sqcap_X s_2, \end{aligned}$$

in contradiction to (9).

Therefore, 2 is true.

Conversely, if 2 is true, then for every $s_1, s_2 \in X$,

$$s_1 \sqcap_X s_2 = s_1 \sqcap s_2,$$

and thus, by Proposition 2.15, 1 is true. □

Elsewhere, we prove that, under the hypothesis of $\langle \mathrm{T}, \preceq \rangle$ being totally ordered, clauses 1 and 2 of Proposition 2.15 constitute a complete axiomatization of the relationship between the generalized distance function and the prefix relation in subsemilattices of signals. It is then natural to wonder how these "axioms" came about. The simple answer is that they presented themselves while first proving our main fixed-point theorem; they emerged as a minimal set of postulates sufficient to eliminate any reference to individual tags and values.

The entire fixed-point theory of Section 5 is essentially built on Proposition 2.15. Its use has allowed for a much simpler theory, abstract enough to potentially interest other branches of computer science involving similar structures, such as, for example, programming logic (e.g., see [51], [21]).

Finally, it is instructive to contradistinguish between the two concepts of completeness associated with the generalized distance function and the prefix relation respectively, namely the concept of spherical completeness and that of directed-completeness[21].

Example 2.18. Suppose that $\mathrm{T} = \mathbb{Q}$, and $\preceq$ is the standard order on $\mathbb{Q}$.[22]

Suppose that V is a singleton set.

Let $D = \{\{\frac{1}{2}\} \times \mathrm{V}, \{\frac{1}{2}, \frac{2}{3}\} \times \mathrm{V}, \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}\} \times \mathrm{V}, \ldots\}$.

Let $X = D \cup \{(\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\} \cup \{1\}) \times \mathrm{V}, (\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\} \cup \{2\}) \times \mathrm{V}\}$.

It is not hard to verify that $\langle X, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d} \rangle$ is spherically complete. However, $\langle X, \sqsubseteq \rangle$ is not directed-complete: D is directed in $\langle X, \sqsubseteq \rangle$, but has no least upper bound in $\langle X, \sqsubseteq \rangle$.

[21] An ordered set $\langle P, \leqslant \rangle$ is *directed-complete* if and only if every subset of P that is directed in $\langle P, \leqslant \rangle$ has a least upper bound in $\langle P, \leqslant \rangle$.

[22] We write $\mathbb{Q}$ for the set of all rational numbers.

Example 2.19. Suppose that $\mathrm{T} = \mathbb{N}$, and $\preceq$ is the standard order on $\mathbb{N}$.

Suppose that V is a singleton set.

Let $X = \{\emptyset\} \cup \{(\mathbb{N} - \{1\}) \times \mathrm{V}, (\mathbb{N} - \{2\}) \times \mathrm{V}, (\mathbb{N} - \{3\}) \times \mathrm{V}, \ldots\}$.

For every $s_1, s_2 \in X$, $s_1 \sqsubseteq s_2$ if and only if $s_1 = \emptyset$ or $s_1 = s_2$. Thus, trivially, $\langle X, \sqsubseteq \rangle$ is directed-complete.

Let $C = \{\{s \mid s \in X \text{ and } \mathrm{d}(s, (\mathbb{N} - \{n+1\}) \times \mathrm{V}) \supseteq \{m \mid m \leq n\}\} \mid n \in \mathbb{N}\}$.

The ordered set $\langle C, \subseteq \rangle$ is non-empty a chain of balls in $\langle X, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d} \rangle$, but $\bigcap C = \emptyset$. Thus, $\langle X, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d} \rangle$ is not spherically complete.

Notice that the ordered set of Example 2.19 is actually a semilattice. We defer the case of a subsemilattice to Section 5.1 (see Corollary 5.6 and Example 5.8).

3 Causal and strictly causal functions

In this section, we formalize the folklore, but well established, notions of causality and strict causality, and through a series of examples, demonstrate that these notions are by themselves too weak to accommodate a uniform fixed-point theory suitable for a semantic theory of timed systems.

3.1 Causal functions

Causality is a concept of fundamental importance in the study of timed systems. Informally, it represents the constraint that, at any time instance, the output events of a component do not depend on its future input events. This is only natural for components that model or simulate physical processes, or realize online algorithms; an effect cannot precede its cause.

Assume a partial function F on S.

We say that F is *causal* if and only if there is a partial function f such that for any $s \in \operatorname{dom} F$ and every $\tau \in \mathrm{T}$,

$$F(s)(\tau) \simeq f(s \upharpoonright \{\tau' \mid \tau' \preceq \tau\}, \tau).$$

Notice that since s is, in general, a partial function, $s(\tau)$ need not be defined, and thus, τ cannot, in general, be inferred from $s \upharpoonright \{\tau' \mid \tau' \preceq \tau\}$, and must be provided as a separate argument.

Example 3.1. Suppose that $\mathrm{T} = \mathbb{N}$, and $\preceq$ is the standard order on $\mathbb{N}$.

Suppose that $\mathrm{V} = \mathbb{R}$.[23]

Let F be a function on S such that for every $s \in \mathrm{S}$ and every $\tau \in \mathrm{T}$,

$$F(s)(\tau) = \sum \{s(n) \mid 0 \preceq n \preceq \tau \text{ and } n \in \operatorname{dom} s\}.$$

Clearly, F is causal.

The function of Example 3.1 models a component that, at each time instance, produces an event whose value is the running total of the values of all input events occurring before or at that time instance. The function of our next example models a simple sampling process, and substantiates our claim that τ must be provided as a separate argument.

[23] We write $\mathbb{R}$ for the set of all real numbers.

Example 3.2. Suppose that $\mathrm{T} = \mathbb{R}$, and $\preceq$ is the standard order on $\mathbb{R}$.

Let p be a positive real number.

Let F be a function on S such that for every $s \in \mathrm{S}$ and every $\tau \in \mathrm{T}$,

$$F(s)(\tau) \simeq \begin{cases} s(\tau) & \text{if there is } i \in \mathbb{Z} \text{ such that } \tau = p \cdot i; \\ \text{undefined} & \text{otherwise.} \end{cases}$$ [24]

Clearly, F is causal.

Of course, unless T is a singleton, not every function on S is causal. The function of our next example models a constant look-ahead process, and is a simple instance of a function on S that is not causal.

Example 3.3. Suppose that $\mathbb{R}$, and $\preceq$ is the standard order on $\mathbb{R}$.

Let F be a function on S such that for every $s \in \mathrm{S}$ and every $\tau \in \mathrm{T}$,

$$F(s)(\tau) \simeq s(\tau + 1).$$

Clearly, F is not causal.

Now, as explained in Section 1, due to its relevance to the interpretation of feedback, of special interest is whether any particular partial function F on S has a fixed point, that is, whether there is $s \in \mathrm{S}$ such that

$$s = F(s)$$

(see Figure 1), and whether that fixed point is unique. The function of Example 3.1 has exactly one fixed point, namely $\mathbb{N} \times \{0\}$, whereas that of Example 3.2 has uncountably many fixed points, namely every $s \in \mathrm{S}$ such that

$$\mathsf{dom}\, s \subseteq \{\tau \mid \text{there is } i \in \mathbb{Z} \text{ such that } \tau = p \cdot i\}.$$

Even the non-causal function of Example 3.3 has uncountably many fixed points, namely every $s \in \mathrm{S}$ such that for every $\tau \in \mathbb{R}$,

$$s(\tau) \simeq s(\tau + 1).$$

However, it is easy to construct a causal function that does not have a fixed point.

Example 3.4. Let τ be a tag in T.

Let v be a value in V.

Let F be a function on S such that for every $s \in \mathrm{S}$,

$$F(s) = \begin{cases} \emptyset & \text{if } \tau \in \mathsf{dom}\, s; \\ \{\langle \tau, v \rangle\} & \text{otherwise.} \end{cases}$$

It is easy to verify that F is causal. However, F has no fixed point; for $F(\{\langle \tau, v \rangle\}) = \emptyset$, whereas $F(\emptyset) = \{\langle \tau, v \rangle\}$.

The function of Example 3.4 models a component whose behaviour at τ resembles a logical inverter, turning presence of event into absence of event, and vice versa.

Finally, we note that causal functions are closed under function composition.

[24] We write $\mathbb{Z}$ for the set of all integers.

3.2 Strictly causal functions

Strict causality is causality bar instantaneous reaction. Informally, it represents the constraint that, at any time instance, the output events of a component do not depend on its present or future input events. This operational definition has its origins in natural philosophy, and is of course inspired by physical reality: every physical system is a strictly causal system.[25]

We say that F is *strictly causal* if and only if there is a partial function f such that for any $s \in \mathsf{dom}\, F$ and every $\tau \in \mathrm{T}$,

$$F(s)(\tau) \simeq f(s \restriction \{\tau' \mid \tau' \prec \tau\}, \tau).$$

The following is immediate:

Proposition 3.5. *If F is strictly causal, then F is causal.*

Of course, the converse is false. For example, the sampling function of Example 3.2 is causal but not strictly causal.

Example 3.6. Suppose that $\mathrm{T} = \mathbb{R}$, and $\preceq$ is the standard order on $\mathbb{R}$.

Let F be a function on S such that for every $s \in \mathrm{S}$ and every $\tau \in \mathrm{T}$,

$$F(s)(\tau) \simeq s(\tau - 1).$$

Clearly, F is stricty causal.

The function of Example 3.6 models a simple constant-delay component. It is in fact a "delta causal" function, as defined in [28] and [27], and it is not hard to see that every such function is strictly causal (as is every "Δ-causal" function, as defined in [61] and [60]). The function of our next example models a variable reaction-time component, and is a strictly causal function that is not "delta causal" (nor "Δ-causal").

Example 3.7. Suppose that $\mathrm{T} = [0, \infty)$, and $\preceq$ is the standard order on $[0, \infty)$.[26]

Suppose that $\mathrm{V} = (0, \infty)$.[27]

Let F be a function on S such that for every $s \in \mathrm{S}$ and any $\tau \in \mathrm{T}$,

$$F(s)(\tau) \simeq \begin{cases} 1 & \text{if there is } \tau' \in \mathsf{dom}\, s \text{ such that } \tau = \tau' + s(\tau'); \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Clearly, F is strictly causal.

Now, the function of Example 3.6 has the same fixed points as that of Example 3.3, whereas that of Example 3.7 has exactly one fixed point, namely the empty signal. And having ruled out instantaneous reaction, the reason behind the lack of fixed point in Example 3.4, one might expect that every strictly causal function has a fixed point. But this is not the case.

[25] In modern physics, this would actually depend on the choice of interpretation of quantum mechanics, especially with regard to paradoxes such as Bell's theorem (e.g., see [40]) and Wheeler's delayed choice (e.g., see [23]). Steering clear of the far-from-settled debate here, we believe that, regardless of personal stand, the reader will acknowledge the overwhelming plethora of physical systems that fall under this casual description.

[26] We write $[0, \infty)$ for the set of all non-negative real numbers.

[27] We write $(0, \infty)$ for the set of all positive real numbers.

Example 3.8. Suppose that $\mathrm{T} = \mathbb{Z}$, and $\preceq$ is the standard order on $\mathbb{Z}$.

Suppose that $\mathrm{V} = \mathbb{N}$.

Let F be a function on S such that for every $s \in \mathrm{S}$ and every $\tau \in \mathrm{T}$,

$$F(s)(\tau) \simeq \begin{cases} s(\tau - 1) + 1 & \text{if } \tau - 1 \in \mathsf{dom}\, s; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, F is strictly causal. However, F does not have a fixed point; any fixed point of F would be an order-embedding from the integers into the natural numbers, which is of course impossible.

Example 3.8 alone is enough to suggest that the classical notion of strictly causality is by itself too general to support a useful theory of timed systems. But lack of fixed point is not the only source of concern.

Example 3.9. Suppose that $\mathrm{T} = (0, \infty)$, and $\preceq$ is the standard order on $(0, \infty)$.

Suppose that $\mathrm{V} = \mathbb{R}$.

Let a be a real number greater than 1.

Let v be a value in V.

Let F be a function on S such that for every $s \in \mathrm{S}$ and every $\tau \in \mathrm{T}$,

$$F(s)(\tau) \simeq \begin{cases} s(\tau / a) & \text{if } \tau / a \in \mathsf{dom}\, s; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Clearly, F is strictly causal.

Let b and c be two distinct positive real numbers.

Let $s_b = \{\langle b \cdot a^i, v\rangle \mid i \in \mathbb{Z}\}$.

Let $s_c = \{\langle c \cdot a^i, v\rangle \mid i \in \mathbb{Z}\}$.

Then s_b and s_c are two distinct fixed points of F.

Imagine arranging a component that realizes the function Example 3.9 in a feedback configuration, as in the simple block-diagram of Figure 1. How ought it to behave? To tell what the output of the component ought to be at any particular time instance, we need to look at what it was at some earlier time instance. Iterating this argument, we find ourselves entangled in an infinite descending causal chain, where nothing can be traced back to anything, an infinite regress. At the same time, we have no reason to reject any particular option, in hope that we might determine the behavior by some law of exclusion. All in all, there can be no ground for the output of the component.

There is an interesting analogy put forward by Dummett in [13] to explain Thomas Aquinas' proof of the existence of God as First Cause, which we might use to shed some light on the situation. An infinite descending causal chain is much like an infinite proof, or to be more precise, a proof with some infinite deduction branch. Here we think of a deductive structure in the form of a rooted tree, each node standing for a statement derived by its children according to some inference rule. There is no reason whatever to accept any of the statements along the infinite branch as true, and hence, no reason to accept the conclusion of the proof as true.

The skeptical reader might well argue that this is but an ostensible issue, and that, much in the spirit of Kahn's principle for networks of asynchronous processes (see [24]), the component will simply settle at the empty signal; if there is no reason to output something, it will output nothing. This, however, would entail a bias toward absence of event, a view that components rather remain idle if they can. And the soundness of such a view would ultimately rest on the operational semantics of our systems, about which we remain agnostic. For example, such a view would be consistent with the approach to absence of event taken in the semantics of statecharts presented in [48], but inconsistent with that taken in the constructive semantics of pure Esterel (see [7]). All the same, our next example seems to leave little room for such skepticism.

Example 3.10. Suppose that $\mathrm{T} = \mathbb{Z}$, and $\preceq$ is the standard order on $\mathbb{Z}$.

Let v be a value in V.

Let F be a function on S such that for every $s \in \mathrm{S}$ and every $\tau \in \mathrm{T}$,

$$F(s)(\tau) \simeq \begin{cases} \text{undefined} & \text{if } \tau - 1 \in \mathsf{dom}\, s; \\ v & \text{otherwise.} \end{cases}$$

Clearly, F is strictly causal.

Let $s_{\text{odd}} = \{\langle \tau, v \rangle \mid \tau \text{ is an odd integer}\}$.

Let $s_{\text{even}} = \{\langle \tau, v \rangle \mid \tau \text{ is an even integer}\}$.

Then s_{odd} and s_{even} are two distinct fixed points of F.

Unlike Example 3.9, there are only two fixed points here, s_{odd} and s_{even}. The corresponding feedback system is just as indeterminate though, due once more to the occurrence of an infinite descending causal chain, and the perfect symmetry among presence and absence of event seems detrimental to any attempt to ground any preference to either of s_{odd} and s_{even}.

Finally, we note that strictly causal functions are closed under function composition with causal functions.

4 Contracting and strictly contracting functions

In view of the examples of the previous section, the classical notion of strictly causality is by itself too general to support a useful theory of timed systems. Indeed, Example 3.8 alone should be enough to convince one of the absurdity of the definition of strict causality at that level of generality. Here, we compare causal and strictly causal functions to the functions that are contracting and strictly contracting with respect to the generalized distance function, and argue that strictly contracting functions are actually the functions that one ought to be interested in.

4.1 Contracting functions

There is another, intuitively equivalent way to articulate the property of causality: a component is causal just as long as any two possible output signals differ no earlier than the input signals that produced them (see [27, p. 36], [8, p. 11], and [29, p. 383]). And this can be very elegantly expressed using the generalized distance function of Section 2.2.

We say that F is *contracting* if and only if for any $s_1, s_2 \in \mathsf{dom}\, F$,

$$\mathrm{d}(F(s_1), F(s_2)) \supseteq \mathrm{d}(s_1, s_2).$$

In other words, a function is contracting just as long as the generalized distance between any two signals in the range of the function is smaller than or equal to that between the signals in the domain of the function that map to them. Notice that, because $\preceq$ is not necessarily a total order, this is different, in general, from the generalized distance between any two signals in the domain of the function being no bigger than that between the signals in the range of the function that those map to, which is why we have opted for the term "contracting" over the term "non-expanding".

In [33, def. 5], causal functions were defined to be the contracting functions. Here, we prove that indeed they are.

Theorem 4.1. *F is causal if and only if F is contracting.*

Proof. Suppose that F is causal.

Then there is a partial function f such that for any $s \in \mathsf{dom}\, F$ and every $\tau \in \mathrm{T}$,

$$F(s)(\tau) \simeq f(s \restriction \{\tau' \mid \tau' \preceq \tau\}, \tau).$$

Assume $s_1, s_2 \in \mathsf{dom}\, F$.

Suppose, toward contradiction, that

$$\mathrm{d}(F(s_1), F(s_2)) \not\supseteq \mathrm{d}(s_1, s_2).$$

Then there is τ such that $\tau \in \mathrm{d}(s_1, s_2)$, but $\tau \notin \mathrm{d}(F(s_1), F(s_2))$. Thus, there is $\tau' \preceq \tau$ such that $F(s_1)(\tau') \not\simeq F(s_2)(\tau')$. But since $\tau' \preceq \tau$, $\tau' \in \mathrm{d}(s_1, s_2)$, and thus,

$$s_1 \restriction \{\tau'' \mid \tau'' \preceq \tau'\} = s_2 \restriction \{\tau'' \mid \tau'' \preceq \tau'\}.$$

Hence,

$$\begin{aligned} F(s_1)(\tau') &\simeq f(s_1 \restriction \{\tau'' \mid \tau'' \preceq \tau'\}, \tau') \\ &\simeq f(s_2 \restriction \{\tau'' \mid \tau'' \preceq \tau'\}, \tau') \\ &\simeq F(s_2)(\tau'), \end{aligned}$$

obtaining a contradiction.

Therefore,

$$\mathrm{d}(F(s_1), F(s_2)) \supseteq \mathrm{d}(s_1, s_2).$$

Thus, by generalization, F is contracting.

Conversely, suppose that F is contracting.

Let f be a partial function such that for any $s \in \mathsf{dom}\, F$ and every $\tau \in \mathrm{T}$,

$$f(s \restriction \{\tau' \mid \tau' \preceq \tau\}, \tau) \simeq (\textstyle\prod \{F(s') \mid s' \in \mathsf{dom}\, F \text{ and } \mathrm{d}(s, s') \supseteq \{\tau' \mid \tau' \preceq \tau\}\})(\tau).$$

Assume $s \in \mathsf{dom}\, F$ and $\tau \in \mathrm{T}$.

Let $X = \{s' \mid s' \in \mathsf{dom}\, F \text{ and } \mathrm{d}(s, s') \supseteq \{\tau' \mid \tau' \preceq \tau\}\}$.

Then, by the generalized ultrametric inequality, for every $s_1, s_2 \in X$,

$$\mathrm{d}(s_1, s_2) \supseteq \{\tau' \mid \tau' \preceq \tau\}.$$

And since F is contracting, for every $s_1, s_2 \in X$,

$$\mathrm{d}(F(s_1), F(s_2)) \supseteq \{\tau' \mid \tau' \preceq \tau\}.$$

Thus,

$$\bigcap \{\mathrm{d}(F(s_1), F(s_2)) \mid s_1, s_2 \in X\} \supseteq \{\tau' \mid \tau' \preceq \tau\}.$$

However, by Proposition 2.13,

$$\bigsqcap \{F(s') \mid s' \in X\} = F(s) \restriction \bigcap \{\mathrm{d}(F(s_1), F(s_2)) \mid s_1, s_2 \in X\}.$$

and hence,

$$\begin{aligned} F(s)(\tau) &\simeq (\bigsqcap \{F(s') \mid s' \in X\})(\tau) \\ &\simeq f(s \restriction \{\tau' \mid \tau' \preceq \tau\}, \tau). \end{aligned}$$

Thus, by generalization, F is causal. □

4.2 Strictly contracting functions

Following the same line of reasoning, one might expect that a component is strictly causal just as long as any two possible output signals differ later, if at all, than the signals that produced them (see [27, p. 36]).

We say that F is *strictly contracting* if and only if for any $s_1, s_2 \in \mathsf{dom}\, F$ such that $s_1 \neq s_2$,

$$\mathrm{d}(F(s_1), F(s_2)) \supset \mathrm{d}(s_1, s_2).$$

The following is immediate:

Proposition 4.2. *If F is strictly contracting, then F is contracting.*

In [44, p. 484], Naundorf defined strictly causal functions as the functions that we here call strictly contracting, and in [33, def. 6], that definition was rephrased using the generalized distance function to explicitly identify strictly causal functions with the strictly contracting functions. But the relationship between the proposed definition and the classical notion of strict causality was never formally examined. The next proposition implies that, in fact, the two are not the same.

Proposition 4.3. *If F is strictly contracting, then F has at most one fixed point.*

Proof. Suppose that F is strictly contracting.

Suppose, toward contradiction, that s_1 and s_2 are two distinct fixed points of F. Then

$$\mathrm{d}(F(s_1), F(s_2)) = \mathrm{d}(s_1, s_2),$$

obtaining a contradiction.

Thus, F has at most one fixed point. □

By Proposition 4.3, the function of Example 3.9, as well as that of Example 3.10, is not strictly contracting. By the next theorem, neither is the function of Example 3.8.

Assume $X \subseteq \mathrm{S}$.

Theorem 4.4. *If $\langle X, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d}\rangle$ is non-empty and spherically complete, then every strictly contracting function on X has exactly one fixed point.*

Theorem 4.4 follows immediately from the fixed-point theorem of Priess-Crampe and Ribenboim for strictly contracting functions on spherically complete generalized ultrametric spaces (see [49, thm. 1]), which is sometimes, and perhaps a little too liberally, referred to as a generalization of the *Banach Fixed-Point Theorem*. The following, which follows immediately from another theorem of Priess-Crampe and Ribenboim (e.g., see *Banach's Fixed Point Theorem* in [57]), justifies the use of the stronger property of spherical completeness in place of the standard property of Cauchy-completeness used in the latter:

Theorem 4.5. *If $\langle \mathrm{T}, \preceq \rangle$ is totally ordered, then $\langle X, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d} \rangle$ is non-empty and spherically complete if and only if every strictly contracting function on X has a fixed point.*

It will later follow from Theorem 5.5 and Example 5.8 that the hypothesis of $\langle \mathrm{T}, \preceq \rangle$ being totally ordered in Theorem 4.5 cannot be discarded.

Our next example shows that even the strictly causal functions that do have exactly one fixed point need not be strictly contracting.

Example 4.6. Suppose that $\mathrm{T} = \{-\infty\} \cup \mathbb{Z}$, and $\preceq$ is the standard order on $\{-\infty\} \cup \mathbb{Z}$.

Let v be a value in V.

Let F be a function on S such that for every $s \in \mathrm{S}$ and every $\tau \in \mathrm{T}$,

$$F(s)(\tau) \simeq \begin{cases} \text{undefined} & \text{if } \tau = -\infty; \\ \text{undefined} & \text{if } \tau \neq -\infty \text{ and } -\infty \notin \mathsf{dom}\, s; \\ \text{undefined} & \text{if } \tau \neq -\infty,\ -\infty \in \mathsf{dom}\, s, \text{ and } \tau - 1 \in \mathsf{dom}\, s; \\ \mathrm{v} & \text{otherwise.} \end{cases}$$

Clearly, F is strictly causal, and has a unique fixed point, namely the empty signal.

Let $s_{-\infty,\mathrm{odd}} = \{\langle \tau, v \rangle \mid \tau = -\infty \text{ or } \tau \text{ is an odd integer}\}$.

Let $s_{-\infty,\mathrm{even}} = \{\langle \tau, v \rangle \mid \tau = -\infty \text{ or } \tau \text{ is an even integer}\}$.

Then

$$\mathrm{d}(F(s_{-\infty,\mathrm{odd}}), F(s_{-\infty,\mathrm{even}})) = \mathrm{d}(s_{-\infty,\mathrm{odd}}, s_{-\infty,\mathrm{even}}),$$

and thus, F is not strictly contracting.

A less contrived example of a traditionally strictly causal system that has a unique behavior, but nevertheless cannot be modelled using a strictly contracting function, is a continuous-time dynamical system specified in terms of an ordinary differential equation of the form

$$\dot{s}(t) = f(s(t), t),$$

with t a non-negative real number, and $s(0) = v$ for some value v. This is really more of a declarative specification that we typically conceptualize as a component in feedback realizing the function

$$F(s)(t) = v + \int_0^t f(s(t'), t')dt'.$$

We can then identify the source of the problem to be the integrator, which although strictly causal in the traditional sense, cannot be modelled by a strictly contracting function. And this would seem to cast dynamical systems of this kind outside the range of the fixed-point theory of Section 5. Nevertheless, when computing a numerical solution to the differential equation, one is effectively transforming the component realizing F into a discrete-event component that progresses in discrete steps, as dictated by the particular solver in use, and does in fact realize a strictly contracting function (see also [30]).

Parenthetically, we note that a discrete-time dynamical system specified in terms of a recurrence relation of the form

$$s(n+1) = f(s(n), n),$$

with n a natural number, and $s(0) = v$ as before, always defines a strictly contracting function.

What is then the use, if any, of strictly contracting functions in a fixed-point theory for strictly causal functions? The next couple of theorems are key in answering this question.

Theorem 4.7. *If $\langle \mathrm{T}, \preceq \rangle$ is totally ordered, and $\langle \mathsf{dom}\, F, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d} \rangle$ is non-empty and spherically complete, then F is strictly contracting if and only if for every causal function F' from $\mathsf{ran}\, F$ to $\mathsf{dom}\, F$, $F' \circ F$ has a fixed point.*

Proof. Suppose that $\langle \mathrm{T}, \preceq \rangle$ is totally ordered, and $\langle \mathsf{dom}\, F, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d} \rangle$ is spherically complete.

If F is strictly contracting, then, by Theorem 4.1, for every causal function F' from $\mathsf{ran}\, F$ to $\mathsf{dom}\, F$, $F' \circ F$ is strictly contracting, and thus, by Theorem 4.4, has a fixed point.

Conversely, suppose that F is not strictly contracting.

Then there are $s_1, s_2 \in \mathsf{dom}\, F$ such that $s_1 \neq s_2$, but

$$\mathrm{d}(F(s_1), F(s_2)) \not\supset \mathrm{d}(s_1, s_2).$$

And since $\langle \mathrm{T}, \preceq \rangle$ is totally ordered,

$$\mathrm{d}(s_1, s_2) \supseteq \mathrm{d}(F(s_1), F(s_2)) \tag{10}$$

Let F' be a function from $\mathsf{ran}\, F$ to $\mathsf{dom}\, F$ such that for every $s \in \mathsf{ran}\, F$,

$$F'(s) = \begin{cases} s_1 & \text{if } \mathrm{d}(F(s_2), s) \supset \mathrm{d}(F(s_1), F(s_2)); \\ s_2 & \text{otherwise.} \end{cases}$$

Assume $s'_1, s'_2 \in \mathsf{ran}\, F$.

Since $\langle \mathrm{T}, \preceq \rangle$ is totally ordered, either

$$\mathrm{d}(F(s_1), F(s_2)) \supseteq \mathrm{d}(s'_1, s'_2),$$

or

$$\mathrm{d}(s'_1, s'_2) \supset \mathrm{d}(F(s_1), F(s_2)).$$

If

$$\mathrm{d}(F(s_1), F(s_2)) \supseteq \mathrm{d}(s'_1, s'_2),$$

then, by (10),

$$\begin{aligned} \mathrm{d}(F'(s'_1), F'(s'_2)) &\supseteq \mathrm{d}(s_1, s_2) \\ &\supseteq \mathrm{d}(s'_1, s'_2). \end{aligned}$$

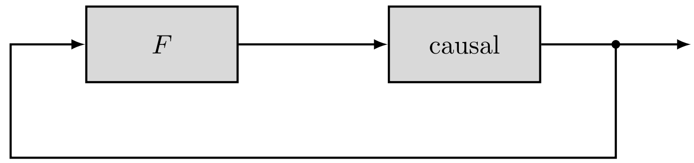

Figure 2. A functional component realizes a strictly contracting function F if and only if the cascade of the component and any arbitrary causal component has a unique, well defined behaviour when arranged in a feedback configuration.

Otherwise,

$$\mathrm{d}(s'_1, s'_2) \supset \mathrm{d}(F(s_1), F(s_2)). \tag{11}$$

Suppose, toward contradiction, that $F'(s'_1) \neq F(s'_2)$. Without loss of generality, assume that $F(s'_1) = s_1$. Then

$$\mathrm{d}(F(s_2), s'_1) \supset \mathrm{d}(F(s_1), F(s_2)). \tag{12}$$

Since $\langle \mathrm{T}, \preceq \rangle$ is totally ordered, by (11), (12), and Proposition 2.7,

$$\mathrm{d}(F(s_2), s'_2) \supset \mathrm{d}(F(s_1), F(s_2)).$$

Thus, $F(s'_2) = s_1$, obtaining a contradiction.

Therefore, $F(s'_1) = F(s'_2)$, and thus,

$$\mathrm{d}(F(s'_1), F(s'_2)) \supseteq \mathrm{d}(s'_1, s'_2).$$

Thus, by generalization, F' is contracting, and hence, by Theorem 4.1, causal. But clearly,

$$(F' \circ F)(s_1) = s_2$$

and

$$(F' \circ F)(s_2) = s_1,$$

and thus, $F' \circ F$ does not have a fixed point. □

An informal but informative way of reading Theorem 4.7 is the following: a functional component realizes a strictly contracting function if and only if the cascade of the component and any arbitrary causal filter has a unique, well defined behaviour when arranged in a feedback configuration (see Figure 2); that is, the components that realize strictly contracting functions are those functional components that maintain the consistency of the feedback loop no matter how we chose to filter the signal transmitted over the feedback wire, as long as we do so in a causal way.

Under the hypothesis of $\langle \mathrm{T}, \preceq \rangle$ being totally ordered, Theorem 4.7 completely characterizes strictly contracting functions in terms of the classical notion of causality, identifying the class of all such functions as the largest class of functions that have a fixed point not by some fortuitous coincidence, but as a direct consequence of their causality properties. And under the same hypothesis, all such functions are strictly causal.

Theorem 4.8. *If $\langle \mathrm{T}, \preceq \rangle$ is totally ordered, then if F is strictly contracting, then F is strictly causal.*

Proof. Suppose that $\langle \mathrm{T}, \preceq \rangle$ is totally ordered.

Suppose that F is strictly contracting.

Let f be a partial function such that for any $s \in \mathsf{dom}\, F$ and every $\tau \in \mathrm{T}$,

$$f(s \restriction \{\tau' \mid \tau' \prec \tau\}, \tau) \simeq (\textstyle\bigsqcap \{F(s') \mid s' \in \mathsf{dom}\, F \text{ and } \mathrm{d}(s, s') \supseteq \{\tau' \mid \tau' \prec \tau\}\})(\tau).$$

Assume $s \in \mathsf{dom}\, F$ and $\tau \in \mathrm{T}$.

Let $X = \{s' \mid s' \in \mathsf{dom}\, F \text{ and } \mathrm{d}(s, s') \supseteq \{\tau' \mid \tau' \prec \tau\}\}$.

Then, by the generalized ultrametric inequality, for every $s_1, s_2 \in X$,

$$\mathrm{d}(s_1, s_2) \supseteq \{\tau' \mid \tau' \prec \tau\}.$$

And since $\langle \mathrm{T}, \preceq \rangle$ is totally ordered, and F is strictly contracting, for every $s_1, s_2 \in X$,

$$\mathrm{d}(F(s_1), F(s_2)) \supseteq \{\tau' \mid \tau' \preceq \tau\}.$$

Thus,

$$\textstyle\bigcap \{\mathrm{d}(F(s_1), F(s_2)) \mid s_1, s_2 \in X\} \supseteq \{\tau' \mid \tau' \preceq \tau\}.$$

However, by Proposition 2.13,

$$\textstyle\bigsqcap \{F(s') \mid s' \in X\} = F(s) \restriction \bigcap \{\mathrm{d}(F(s_1), F(s_2)) \mid s_1, s_2 \in X\}.$$

and hence,

$$\begin{aligned} F(s)(\tau) &\simeq (\textstyle\bigsqcap \{F(s') \mid s' \in X\})(\tau) \\ &\simeq f(s \restriction \{\tau' \mid \tau' \prec \tau\}, \tau). \end{aligned}$$

Thus, by generalization, F is strictly causal. □

The following shows that the hypothesis of $\langle \mathrm{T}, \preceq \rangle$ being totally ordered in Theorem 4.8 cannot be discarded:

Example 4.9. Suppose that $\mathrm{T} = \{0, 1\}$, and $\preceq$ is the discrete order on $\{0, 1\}$.

Let v_1, v_1', v_2, v_2', and v be five distinct values in V.

Let F be a partial function on S defined by the following mapping:

$$\begin{aligned} &\{\ \} \mapsto \{\langle 0, v\rangle, \langle 1, v\rangle\}; \\ &\{\langle 0, v\rangle\} \mapsto \{\langle 0, v\rangle, \langle 1, v\rangle\}; \\ &\{\langle 0, v_1\rangle, \langle 1, v_1\rangle\} \mapsto \{\langle 0, v\rangle, \langle 1, v_1'\rangle\}; \\ &\{\langle 0, v_2\rangle, \langle 1, v_2\rangle\} \mapsto \{\langle 0, v\rangle, \langle 1, v_2'\rangle\}; \\ &\{\langle 0, v\rangle, \langle 1, v_1'\rangle\} \mapsto \{\langle 0, v\rangle, \langle 1, v\rangle\}; \\ &\{\langle 0, v\rangle, \langle 1, v_2'\rangle\} \mapsto \{\langle 0, v\rangle, \langle 1, v\rangle\}; \\ &\{\langle 0, v\rangle, \langle 1, v\rangle\} \mapsto \{\langle 0, v\rangle, \langle 1, v\rangle\}. \end{aligned}$$

Then $\langle \mathsf{dom}\, F, \sqsubseteq \rangle$ is a finite, and thus, trivially, complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$, and F is strictly contracting, but not strictly causal.

The implication of Theorem 4.7 and 4.8, we believe, is that, under the hypothesis of $\langle \mathrm{T}, \preceq \rangle$ being totally ordered, the class of strictly contracting functions is the largest class of strictly causal functions that one can reasonably hope to attain a uniform fixed-point theory for.

Finally, if we further require that $\prec$ be well founded[28] on the domain of any signal in the domain of a function, which would effectively preclude the occurrence of infinite descending causal chains in feedback configurations, then the difference between a strictly causal function and a strictly contracting one vanishes.

Theorem 4.10. *If for any $s \in \mathsf{dom}\, F$, $\prec$ is well founded on $\mathsf{dom}\, s$, then if F is strictly causal, then F is strictly contracting.*

Proof. Suppose that for any $s \in \mathsf{dom}\, F$, $\prec$ is well founded on $\mathsf{dom}\, s$.

Suppose that F is strictly causal.

Then, by Proposition 3.5 and Theorem 4.1, F is contracting.

Assume $s_1, s_2 \in \mathsf{dom}\, F$ such that $s_1 \neq s_2$.

Since F is contracting,

$$\mathrm{d}(F(s_1), F(s_2)) \supseteq \mathrm{d}(s_1, s_2).$$

Suppose, toward contradiction, that $\prec$ is not well founded on $\{\tau \mid s_1(\tau) \not\simeq s_2(\tau)\}$. Then, by the Axiom of Dependent Choice, there is an infinite sequence $\langle \tau_n \mid n \in \omega \rangle$ over $\{\tau \mid s_1(\tau) \not\simeq s_2(\tau)\}$ such that for every $n \in \omega$, $\tau_{n+1} \prec \tau_n$.

If

$$\{\tau_n \mid n \in \omega\} \cap \mathsf{dom}\, s_1 = \emptyset,$$

then $\langle \tau_n \mid n \in \omega \rangle$ is an infinite sequence over $\mathsf{dom}\, s_2$, and thus, $\prec$ is not well founded on $\mathsf{dom}\, s_2$, obtaining a contradiction.

Otherwise,

$$\{\tau_n \mid n \in \omega\} \cap \mathsf{dom}\, s_1 \neq \emptyset.$$

Then, since $\prec$ is well founded on $\mathsf{dom}\, s_1$, there is $m \in \omega$ such that τ_m is $\prec$-minimal in $\{\tau_n \mid n \in \omega\} \cap \mathsf{dom}\, s_1$. Thus, $\langle \tau_n \mid m < n \rangle$ is an infinite sequence over $\mathsf{dom}\, s_2$, and hence, $\prec$ is not well founded on $\mathsf{dom}\, s_2$, obtaining a contradiction.

Therefore, $\prec$ is well founded on $\{\tau \mid s_1(\tau) \not\simeq s_2(\tau)\}$.

Let τ' be a tag that is $\prec$-minimal in $\{\tau \mid s_1(\tau) \not\simeq s_2(\tau)\}$.

Then

$$\mathrm{d}(s_1, s_2) \supseteq \{\tau \mid \tau \prec \tau'\},$$

or equivalently,

$$s_1 \restriction \{\tau \mid \tau \prec \tau'\} = s_2 \restriction \{\tau \mid \tau \prec \tau'\}.$$

Since F is contracting,

$$\mathrm{d}(F(s_1), F(s_2)) \supseteq \{\tau \mid \tau \prec \tau'\},$$

[28] For every set A, and every binary relation R on A, R is *well founded* on A if and only if for every non-empty $S \subseteq A$, there is $a \in S$ such that a is R-minimal in S.

and since F is strictly causal, $F(s_1)(\tau') \simeq F(s_2)(\tau')$. Thus, $\tau' \in \mathrm{d}(F(s_1), F(s_2))$, and hence,

$$\mathrm{d}(F(s_1), F(s_2)) \supset \mathrm{d}(s_1, s_2).$$

Thus, by generalization, F is strictly contracting. □

The following is immediate from Theorem 4.10 and 4.8:

Corollary 4.11. *If $\langle \mathrm{T}, \preceq \rangle$ is totally ordered, and for any $s \in \mathsf{dom}\, F$, $\langle \mathsf{dom}\, s, \prec \rangle$ is well ordered*[29]*, then F is strictly causal if and only if F is strictly contracting.*

For example, the function of Example 3.7 is not strictly contracting, as witnessed by the signals $\{\langle \frac{1}{2n+1}, \frac{2}{4n^2-1} \rangle \mid n \in \mathbb{N}\}$ and $\{\langle \frac{1}{2n}, \frac{1}{2n(n-1)} \rangle \mid n \in \mathbb{N} \text{ and } n \geq 2\}$, but its restriction to, say, the set of all discrete-event[30] signals is.

Corollary 4.11, immediately applicable in the case of discrete-event systems, is most pleasing considering our emphasis on timed computation. It implies that for all kinds of computational timed systems, where components are expected to operate on discretely generated signals, including all programming languages and model-based design tools mentioned in the beginning of the introduction, the strictly contracting functions are exactly the strictly causal ones.

5 Fixed-point theory

We henceforth concentrate on the strictly contracting functions, and begin to develop the rudiments of a constructive fixed-point theory for such functions.

5.1 Existence

We start by proving another fixed-point existence result for strictly contracting functions, which is similar to Theorem 4.4, but has a different premise. The proof is more like Naundorf's proof in [44], but, as also possible in the case of the existence part of Theorem 4.4 (see [49, p. 229]), our main theorem applies to a more general type of function.

Assume a partial endofunction[31] F on S.

We say that F is *strictly contracting on orbits* if and only if for any $s \in \mathsf{dom}\, F$ such that $s \neq F(s)$,

$$\mathrm{d}(F(s), F(F(s))) \supset \mathrm{d}(s, F(s)).$$

In other words, F is strictly contracting on orbits just as long as the generalized distance between every two successive signals in the orbit[32] of any $s \in \mathsf{dom}\, F$ under F gets smaller and smaller along the orbit.

The following is immediate:

Proposition 5.1. *If F is strictly contracting, then F is strictly contracting on orbits.*

[29] An ordered set $\langle P, \leqslant \rangle$ is *well ordered* if and only if for every non-empty $S \subseteq P$, there is $p \in S$ such that p is least in $\langle S, \leqslant \rangle$.
[30] A signal s is *discrete-event* if and only if there is an order-embedding from $\langle \mathsf{dom}\, s, \preceq \rangle$ to $\langle \mathbb{N}, \leq \rangle$ (see [27]).
[31] A function f is an endofunction if and only if $\mathsf{ran}\, f \subseteq \mathsf{dom}\, f$.
[32] For every set A, every function f on A, and any $a \in A$, the *orbit* of a under f is the sequence $\langle f^n(a) \mid n \in \omega \rangle$.

Theorem 5.2. *If $\langle X, \sqsubseteq \rangle$ is a non-empty, directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$, then every contracting function on X that is strictly contracting on orbits has a fixed point.*

Before we embark on the proof of the theorem, we prove two important lemmas that will be useful throughout this section.

For every partial endofunction F on S, and any $s \in \mathsf{dom}\, F$, we say that s is a *post-fixed point* of F if and only if $s \sqsubseteq F(s)$.

Lemma 5.3. *If $\langle X, \sqsubseteq \rangle$ is a subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$, then for every contracting function F on X, and any $s \in X$, the following are true:*

1. *$F(s) \sqcap F(F(s))$ is a post-fixed point of F;*
2. *if s is a post-fixed point of F, then $s \sqsubseteq F(s) \sqcap F(F(s))$.*

Proof. Suppose that $\langle X, \sqsubseteq \rangle$ is a subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$.

Assume a contracting function F on X, and $s \in X$.

Since F is contracting, by Proposition 2.15.2,

$$\begin{aligned} \mathrm{d}(F(F(s) \sqcap F(F(s))), F(F(s))) &\supseteq \mathrm{d}(F(s) \sqcap F(F(s)), F(s)) \\ &= \mathrm{d}(F(s) \sqcap F(F(s)), F(s) \sqcap F(s)) \\ &\supseteq \mathrm{d}(F(s), F(F(s))), \end{aligned}$$

and thus, by Proposition 2.15.1,

$$\begin{aligned} F(s) \sqcap F(F(S)) &\sqsubseteq F(F(s) \sqcap F(F(S))) \sqcap F(F(S)) \\ &\sqsubseteq F(F(s) \sqcap F(F(S))). \end{aligned}$$

Thus, 1 is true.

Suppose that $s \sqsubseteq F(s)$.

Since F is contracting,

$$\mathrm{d}(F(s), F(F(s))) \supseteq \mathrm{d}(s, F(s)),$$

and thus, by Proposition 2.15.1,

$$s \sqcap F(s) \sqsubseteq F(s) \sqcap F(F(s)).$$

And since $s \sqsubseteq F(s)$, $s \sqcap F(s) = s$, and thus,

$$s \sqsubseteq F(s) \sqcap F(F(s)).$$

Thus, 2 is true. ∎

Lemma 5.4. *If $\langle X, \sqsubseteq \rangle$ is a subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$, then for every contracting function F on X, and any set P of post-fixed points of F, if P has a least upper bound in $\langle X, \sqsubseteq \rangle$, then $\bigsqcup_X P$ is a post-fixed point of F.*

Proof. Suppose that $\langle X, \sqsubseteq \rangle$ is a subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$.

Assume a contracting function F on X, and a set P of post-fixed points of F that has a least upper bound in $\langle X, \sqsubseteq \rangle$.

Assume $s \in P$.

Since F is contracting,

$$\mathrm{d}(F(s), F(\textstyle\bigsqcup_X P)) \supseteq \mathrm{d}(s, \textstyle\bigsqcup_X P). \tag{13}$$

By Proposition 2.15.2 and (13),

$$\mathrm{d}((\textstyle\bigsqcup_X P) \sqcap F(s), (\textstyle\bigsqcup_X P) \sqcap F(\textstyle\bigsqcup_X P)) \supseteq \mathrm{d}(s, \textstyle\bigsqcup_X P). \tag{14}$$

Also, since s is a post-fixed point of F, by Proposition 2.15.2,

$$\begin{aligned} \mathrm{d}(s, (\textstyle\bigsqcup_X P) \sqcap F(s)) &= \mathrm{d}(F(s) \sqcap s, F(s) \sqcap \textstyle\bigsqcup_X P) \\ &\supseteq \mathrm{d}(s, \textstyle\bigsqcup_X P). \end{aligned} \tag{15}$$

By (14), (15), and the generalized ultrametric inequality,

$$\mathrm{d}(s, (\textstyle\bigsqcup_X P) \sqcap F(\textstyle\bigsqcup_X P)) \supseteq \mathrm{d}(s, \textstyle\bigsqcup_X P).$$

Then, by the generalized ultrametric inequality,

$$\mathrm{d}(\textstyle\bigsqcup_X P, (\textstyle\bigsqcup_X P) \sqcap F(\textstyle\bigsqcup_X P)) \supseteq \mathrm{d}(s, \textstyle\bigsqcup_X P),$$

and thus, by Proposition 2.15.1,

$$\begin{aligned} s \sqcap \textstyle\bigsqcup_X P &\sqsubseteq (\textstyle\bigsqcup_X P) \sqcap (\textstyle\bigsqcup_X P) \sqcap F(\textstyle\bigsqcup_X P) \\ &= (\textstyle\bigsqcup_X P) \sqcap F(\textstyle\bigsqcup_X P). \end{aligned}$$

However, since $s \in P$, $s \sqsubseteq \bigsqcup_X P$, and thus, $s \sqcap \bigsqcup_X P = s$. Thus,

$$\begin{aligned} s &\sqsubseteq (\textstyle\bigsqcup_X P) \sqcap F(\textstyle\bigsqcup_X P) \\ &\sqsubseteq F(\textstyle\bigsqcup_X P). \end{aligned}$$

Thus, by generalization, $F(\bigsqcup_X P)$ is an upper bound of P in $\langle X, \sqsubseteq \rangle$. And since $\bigsqcup_X P$ is the least upper bound of P in $\langle X, \sqsubseteq \rangle$, $\bigsqcup_X P \sqsubseteq F(\bigsqcup_X P)$. Thus, $\bigsqcup_X P$ is a post-fixed point of P. □

Proof of Theorem 5.2. Suppose that $\langle X, \sqsubseteq \rangle$ is a non-empty, directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$.

Assume a contracting function F on X that is strictly contracting on orbits.

Let $P = \{s \mid s \sqsubseteq F(s)\}$.

Let s be a signal in X.

By Lemma 5.3.1,

$$F(s) \sqcap F(F(s)) \sqsubseteq F(F(s) \sqcap F(F(s))),$$

and thus, $P \neq \emptyset$. Then, by Kuratowski's Lemma (see [12, sec. 10.2]), every chain in $\langle P, \sqsubseteq \rangle$ is contained in a $\subset$-maximal chain in $\langle P, \sqsubseteq \rangle$.

Let C be a $\subset$-maximal chain in $\langle P, \sqsubseteq \rangle$.

Since $\langle X, \sqsubseteq \rangle$ is directed-complete, C has a least upper bound in $\langle X, \sqsubseteq \rangle$.

We claim that $\bigsqcup_X C$ is a fixed point of F.

Suppose, toward contradiction, that $\bigsqcup_X C$ is not a fixed point of F.

Let $x = F(\bigsqcup_X C) \sqcap F(F(\bigsqcup_X C))$.

By Lemma 5.4, $\bigsqcup_X C \sqsubseteq F(\bigsqcup_X C)$, and thus, by Lemma 5.3.2, $\bigsqcup_X C \sqsubseteq x$.

Suppose, toward contradiction, that $\bigsqcup_X C = x$. Since F is strictly contracting on orbits, and $\bigsqcup_X C$ is not a fixed point of F,

$$\mathrm{d}(F(\textstyle\bigsqcup_X C), F(F(\bigsqcup_X C))) \supset \mathrm{d}(\bigsqcup_X C, F(\bigsqcup_X C)). \tag{16}$$

However, since $x = F(\bigsqcup_X C) \sqcap F(F(\bigsqcup_X C))$ and $\bigsqcup_X C = x$, by Proposition 2.15.2,

$$\begin{aligned}\mathrm{d}(\textstyle\bigsqcup_X C, F(\bigsqcup_X C)) &= \mathrm{d}(F(\textstyle\bigsqcup_X C), \bigsqcup_X C)\\ &= \mathrm{d}(F(\textstyle\bigsqcup_X C), F(\bigsqcup_X C) \sqcap F(F(\bigsqcup_X C)))\\ &= \mathrm{d}(F(\textstyle\bigsqcup_X C) \sqcap F(\bigsqcup_X C), F(\bigsqcup_X C) \sqcap F(F(\bigsqcup_X C)))\\ &\supseteq \mathrm{d}(F(\textstyle\bigsqcup_X C), F(F(\bigsqcup_X C))),\end{aligned}$$

contrary to (16).

Therefore, $\bigsqcup_X C \sqsubset x$. Thus, $x \notin C$. And by Lemma 5.3.1, $x \sqsubseteq F(x)$, and thus, $x \in P$. Thus, $C \cup \{x\}$ is a chain in $\langle P, \sqsubseteq \rangle$, and $C \subset C \cup \{x\}$, contrary to C being a $\subset$-maximal chain in $\langle P, \sqsubseteq \rangle$.

Therefore, $\bigsqcup_X C$ is a fixed point of F. □

There are two things to notice here. First, the proof of Theorem 5.2 is inherently non-constructive, overtly appealing to the Axiom of Choice through the use of Kuratowski's Lemma. And second, there need not be only one fixed point; indeed, the identity function on S is trivially causal and strictly contracting on orbits, yet every signal is a fixed point of it.

The following is immediate from Proposition 4.2, 4.3, and 5.1, and Theorem 5.2:

Theorem 5.5. *If $\langle X, \sqsubseteq \rangle$ is a non-empty, directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$, then every strictly contracting function on X has exactly one fixed point.*

For every strictly contracting partial endofunction F on S such that $\langle \mathsf{dom}\, F, \sqsubseteq \rangle$ is a non-empty, directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$, we write $\mathsf{fix}\, F$ for the unique fixed point of F.

The following is immediate from Theorem 4.5 and 5.5:

Corollary 5.6. *If $\langle \mathrm{T}, \preceq \rangle$ is totally ordered, then if $\langle X, \sqsubseteq \rangle$ is a directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$, then $\langle X, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d} \rangle$ is spherically complete.*

Example 2.19 showed that, even under the hypothesis of $\langle \mathrm{T}, \preceq \rangle$ being totally ordered, a directed-complete semilattice of signals that is not a subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$ need not be spherically complete. The following shows that a subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$ that is not directed-complete need not be spherically complete either:

Example 5.7. Suppose that $\mathrm{T} = \mathbb{N}$, and $\preceq$ is the standard order on $\mathbb{N}$.

Suppose that V is a singleton set.

Let $X = \{\{0\} \times \mathrm{V}, \{0, 1\} \times \mathrm{V}, \{0, 1, 2\} \times \mathrm{V}, \ldots\}$.

$\langle X, \sqsubseteq \rangle$ is totally ordered, and thus, trivially, a subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$.

Let $C = \{\{s \mid s \in X \text{ and } \mathrm{d}(s, \{m \mid m \leq n\} \times \mathrm{V}) \supseteq \{m \mid m \leq n\}\} \mid n \in \mathbb{N}\}$.

The ordered set $\langle C, \subseteq \rangle$ is a non-empty chain of balls in $\langle X, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d} \rangle$, but $\bigcap C = \emptyset$. Thus, $\langle X, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d} \rangle$ is not spherically complete.

Therefore, Corollary 5.6 is, in a sense, tight.

The following shows that the hypothesis of $\langle \mathrm{T}, \preceq \rangle$ being totally ordered in Corollary 5.6 cannot be discarded:

Example 5.8. Suppose that $\mathrm{T} = \{0, 1\} \times \mathbb{N}$, and $\preceq$ is an order relation on $\{0, 1\} \times \mathbb{N}$ such that for every $\langle i_1, n_1 \rangle, \langle i_2, n_2 \rangle \in \{0, 1\} \times \mathbb{N}$,

$$\langle i, n_1 \rangle \preceq \langle i_2, n_2 \rangle \iff i_1 = i_2 \text{ and } n_1 \leq n_2.$$

Let v be a value in V.

Let $X = \{\{\langle \langle 0, m \rangle, v \rangle \mid m \leq n\} \mid n \in \mathbb{N}\} \cup \{\mathrm{T} \times \{v\}\}$.

It is not hard to verify that $\langle X, \sqsubseteq \rangle$ is a directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$.

Let $C = \{\{s \mid s \in X \text{ and } \mathrm{d}(s, \{\langle \langle 0, m \rangle, v \rangle \mid m \leq n\}) \supseteq \{\langle 0, m \rangle \mid m \leq n\} \cup (\{1\} \times \mathbb{N})\} \mid n \in \mathbb{N}\}$.

The ordered set $\langle C, \subseteq \rangle$ is a non-empty chain of balls in $\langle X, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d} \rangle$, but $\bigcap C = \emptyset$. Thus, $\langle X, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d} \rangle$ is not spherically complete.

This has two notable consequences. First, by Theorem 5.5, the hypothesis of $\langle \mathrm{T}, \preceq \rangle$ being totally ordered in Theorem 4.5 cannot be discarded. And second, Theorem 4.4 and 5.5 are incomparable with respect to deduction; that is, one cannot deduce Theorem 5.5 from Theorem 4.4, nor Theorem 4.4 from Theorem 5.5.

5.2 Construction

Although theoretically pleasing, mere existence of fixed points is practically moot. Theorem 5.2 and 5.5, just like Theorem 4.4, offer little if no means of deductive reasoning about the fixed points ascertained to exist. And unless we construct these fixed points, we can have little insight into how they relate to the operational behaviour of actual systems.

But how are we to construct these fixed points? Theorem A.2 and A.4 seem to render standard fixed-point theories of ordered sets and metric spaces more or less irrelevant. At the same time, it may well be that the relevant fixed-point theorem of Priess-Crampe and Ribenboim is independent of the theory of generalized ultrametric spaces in the classical Zermelo-Fraenkel set theory without choice, thus lacking a constructive proof altogether.[33]

The answer lies in the non-constructive proof of Theorem 5.2. Indeed, the proof contains all the ingredients of a transfinite recursion facilitating the construction of a chain that may effectively substitute for the maximal one only asserted to exist therein by an appeal to Kuratowski's Lemma. We may start with any arbitrary post-fixed point of the function F, and iterate through the function $\lambda x : X \,.\, F(x) \sqcap F(F(x))$ to form an ascending chain of such points. Every so often, we may take the supremum of all signals theretofore constructed, and resume the process therefrom, until no further progress can be made. Of course, the phrase "every so often" is to be interpreted rather liberally here, and certain groundwork is required before we can formalize its transfinite intent.

[33] A purportedly constructive proof for the fixed-point theorem of Priess-Crampe and Ribenboim under the hypothesis of a totally ordered set of distances was presented in [20, thm. 1.3.9]. However, the proof covertly appeals to the Axiom of Choice through a potentially transfinite sequence of choices.

We henceforth assume some familiarity with transfinite set theory, and in particular, ordinal numbers. The unversed reader may refer to any introductory textbook on set theory for details (e.g., see [15]).

Assume a subsemilattice $\langle X, \sqsubseteq \rangle$ of $\langle \mathrm{S}, \sqsubseteq \rangle$, and a function F on X.

We write $\mathsf{1m2}\, F$ for a function on X such that for any $s \in X$,

$$(\mathsf{1m2}\, F)(s) = F(s) \sqcap F(F(s)).$$

In other words, $\mathsf{1m2}\, F$ is the function $\lambda x : X \,.\, F(x) \sqcap F(F(x))$.

Assume a directed-complete subsemilattice $\langle X, \sqsubseteq \rangle$ of $\langle \mathrm{S}, \sqsubseteq \rangle$, a contracting function F on X, and a post-fixed point s of F.

We let

$$(\mathsf{1m2}\, F)^0(s) = s,$$

for every ordinal α,

$$(\mathsf{1m2}\, F)^{\alpha+1}(s) = (\mathsf{1m2}\, F)((\mathsf{1m2}\, F)^{\alpha}(s)),$$

and for every limit ordinal λ,

$$(\mathsf{1m2}\, F)^{\lambda}(s) = \bigsqcup_X \{(\mathsf{1m2}\, F)^{\alpha}(s) \mid \alpha \in \lambda\}.$$

The following implies that for every ordinal α, $(\mathsf{1m2}\, F)^{\alpha}(s)$ is well defined:

Lemma 5.9. *If $\langle X, \sqsubseteq \rangle$ is a directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$, then for every contracting function F on X, any post-fixed point s of F, and every ordinal α,*

1. $(\mathsf{1m2}\, F)^{\alpha}(s) \sqsubseteq F((\mathsf{1m2}\, F)^{\alpha}(s))$;
2. *for any $\beta \in \alpha$,* $(\mathsf{1m2}\, F)^{\beta}(s) \sqsubseteq (\mathsf{1m2}\, F)^{\alpha}(s)$.

Proof. Suppose that $\langle X, \sqsubseteq \rangle$ is a directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$,

Assume a contracting function F on X, a post-fixed point s of F, and an ordinal α.

We use transfinite induction on the ordinal α to jointly prove that 1 and 2 are true.

If $\alpha = 0$, then $(\mathsf{1m2}\, F)^{\alpha}(s) = s$. Thus, 1 is trivially true, whereas 2 is vacuously true.

Suppose that there is an ordinal β such that $\alpha = \beta + 1$.

Then

$$\begin{aligned}(\mathsf{1m2}\, F)^{\alpha}(s) &= (\mathsf{1m2}\, F)((\mathsf{1m2}\, F)^{\beta}(s)) \\ &= F((\mathsf{1m2}\, F)^{\beta}(s)) \sqcap F(F((\mathsf{1m2}\, F)^{\beta}(s))).\end{aligned} \tag{17}$$

Thus, by Lemma 5.3.1, 1 is true.

For every $\gamma \in \alpha$, either $\gamma = \beta$, or $\gamma \in \beta$, and thus, by the induction hypothesis,

$$(\mathsf{1m2}\, F)^{\gamma}(s) \sqsubseteq (\mathsf{1m2}\, F)^{\beta}(s). \tag{18}$$

Also, by the induction hypothesis,

$$(\mathsf{1m2}\, F)^{\beta}(s) \sqsubseteq F((\mathsf{1m2}\, F)^{\beta}(s)).$$

Thus, by Lemma 5.3.2 and (17),

$$\begin{aligned}(\mathsf{1m2}\,F)^{\beta}(s) &\sqsubseteq F((\mathsf{1m2}\,F)^{\beta}(s)) \sqcap F(F((\mathsf{1m2}\,F)^{\beta}(s))) \\ &= (\mathsf{1m2}\,F)^{\alpha}(s). \end{aligned} \tag{19}$$

And by (18) and (19), $(\mathsf{1m2}\,F)^{\gamma}(s) \sqsubseteq (\mathsf{1m2}\,F)^{\alpha}(s)$. Thus, 2 is true.

Otherwise, α is a limit ordinal. By the induction hypothesis, $\langle\{(\mathsf{1m2}\,F)^{\beta}(s) \mid \beta \in \alpha\}, \sqsubseteq\rangle$ is totally ordered, and thus, $\{(\mathsf{1m2}\,F)^{\beta}(s) \mid \beta \in \alpha\}$ is directed in $\langle X, \sqsubseteq\rangle$. And since $\langle X, \sqsubseteq\rangle$ is directed-complete, $\{(\mathsf{1m2}\,F)^{\beta}(s) \mid \beta \in \alpha\}$ has a least upper bound in $\langle X, \sqsubseteq\rangle$, and

$$(\mathsf{1m2}\,F)^{\alpha}(s) = \bigsqcup_X \{(\mathsf{1m2}\,F)^{\beta}(s) \mid \beta \in \alpha\}.$$

Thus, 2 is trivially true.

By the induction hypothesis, for every $\beta \in \alpha$, $(\mathsf{1m2}\,F)^{\beta}(s) \sqsubseteq F((\mathsf{1m2}\,F)^{\beta}(s))$. Thus, by Lemma 5.4, 1 is true. □

By Lemma 5.9.2, and a simple cardinality argument, there is an ordinal α such that for every ordinal β such that $\alpha \in \beta$, $(\mathsf{1m2}\,F)^{\beta}(s) = (\mathsf{1m2}\,F)^{\alpha}(s)$. In fact, for every directed-complete subsemilattice $\langle X, \sqsubseteq\rangle$ of $\langle \mathrm{S}, \sqsubseteq\rangle$, there is a least ordinal α such that for every contracting function F on X, any post-fixed point s of F, and every ordinal β such that $\alpha \in \beta$, $(\mathsf{1m2}\,F)^{\beta}(s) = (\mathsf{1m2}\,F)^{\alpha}(s)$.

We write $\mathsf{oh}\,\langle X, \sqsubseteq\rangle$ for the least ordinal α such that there is no function φ from α to X such that for every $\beta, \gamma \in \alpha$, if $\beta \in \gamma$, then $\varphi(\beta) \sqsubset \varphi(\gamma)$.

In other words, $\mathsf{oh}\,\langle X, \sqsubseteq\rangle$ is the least ordinal that cannot be orderly embedded in $\langle X, \sqsubseteq\rangle$, which we may think of as the *ordinal height* of $\langle X, \sqsubseteq\rangle$. Notice that the Hartogs number of X is an ordinal that cannot be orderly embedded in $\langle X, \sqsubseteq\rangle$, and thus, $\mathsf{oh}\,\langle X, \sqsubseteq\rangle$ is well defined, and in particular, smaller than or equal to the Hartogs number of X.

Lemma 5.10. *If $\langle X, \sqsubseteq\rangle$ is a directed-complete subsemilattice $\langle X, \sqsubseteq\rangle$ of $\langle \mathrm{S}, \sqsubseteq\rangle$, then for every contracting function F on X, any post-fixed point s of F, and every ordinal α, if $(\mathsf{1m2}\,F)^{\alpha}(s)$ is not a fixed point of $\mathsf{1m2}\,F$, then $\alpha + 2 \in \mathsf{oh}\,\langle X, \sqsubseteq\rangle$.*

Proof. Suppose that $\langle X, \sqsubseteq\rangle$ is a directed-complete subsemilattice $\langle X, \sqsubseteq\rangle$ of $\langle \mathrm{S}, \sqsubseteq\rangle$.

Assume a contracting function F, a post-fixed point s of F, and an ordinal α.

Suppose that $(\mathsf{1m2}\,F)^{\alpha}(s)$ is not a fixed point of $\mathsf{1m2}\,F$.

We claim that for any $\beta, \gamma \in \alpha + 2$, if $\beta \neq \gamma$, then

$$(\mathsf{1m2}\,F)^{\beta}(s) \neq (\mathsf{1m2}\,F)^{\gamma}(s).$$

Suppose, toward contradiction, that there are $\beta, \gamma \in \alpha + 2$ such that $\beta \neq \gamma$, but

$$(\mathsf{1m2}\,F)^{\beta}(s) = (\mathsf{1m2}\,F)^{\gamma}(s).$$

Without loss of generality, assume that $\beta \in \gamma$. Since F is contracting, by Lemma 5.9.2,

$$\begin{aligned}(\mathsf{1m2}\,F)^{\beta}(s) &\sqsubseteq (\mathsf{1m2}\,F)^{\beta+1}(s) \\ &\sqsubseteq (\mathsf{1m2}\,F)^{\gamma}(s),\end{aligned}$$

and thus,

$$(\mathsf{1m2}\,F)^{\beta}(s) = (\mathsf{1m2}\,F)^{\beta+1}(s).$$

And since $\beta \in \gamma \in \alpha + 2$, either $\beta \in \alpha$, or $\beta = \alpha$. Thus, by an easy transfinite induction,

$$(\mathsf{1m2}\,F)^{\beta}(s) = (\mathsf{1m2}\,F)^{\alpha}(s),$$

contrary to the assumption that $(\mathsf{1m2}\,F)^{\alpha}(s)$ is not a fixed point of $\mathsf{1m2}\,F$.

Therefore, for any $\beta, \gamma \in \alpha + 2$,

$$(\mathsf{1m2}\,F)^{\beta}(s) = (\mathsf{1m2}\,F)^{\gamma}(s)$$

if and only if $\beta = \gamma$. Thus, since F is contracting, by Lemma 5.9.2, there is a function φ from $\alpha + 2$ to X such that for every $\beta, \gamma \in \alpha + 2$, if $\beta \in \gamma$, then $\varphi(\beta) \sqsubset \varphi(\gamma)$. Thus, by definition of $\mathsf{oh}\,\langle X, \sqsubseteq\rangle$, $\alpha + 2 \in \mathsf{oh}\,\langle X, \sqsubseteq\rangle$. ☐

By Lemma 5.10, $(\mathsf{1m2}\,F)^{\mathsf{oh}\,\langle X,\sqsubseteq\rangle}(s)$ is a fixed point of $\mathsf{1m2}\,F$. Nevertheless, $(\mathsf{1m2}\,F)^{\mathsf{oh}\,\langle X,\sqsubseteq\rangle}(s)$ need not be a fixed point of F as intended. For example, if F is the function of Example 3.4, then for every ordinal α, $(\mathsf{1m2}\,F)^{\alpha}(\emptyset) = \emptyset$, even though $\emptyset$ is not a fixed point of F. This rather trivial example demonstrates how the recursion process might start stuttering at points that are not fixed under the function in question. If the function is strictly contracting on orbits, however, progress at such points is guaranteed.

Lemma 5.11. *If $\langle X, \sqsubseteq\rangle$ is a subsemilattice of $\langle \mathrm{S}, \sqsubseteq\rangle$, then for every function F on X that is strictly contracting on orbits, s is a fixed point of F if and only if s is a fixed point of $\mathsf{1m2}\,F$.*

Proof. Suppose that $\langle X, \sqsubseteq\rangle$ is a subsemilattice of $\langle \mathrm{S}, \sqsubseteq\rangle$.

Assume a function F on X that is strictly contracting on orbits.

If s is a fixed point of F, then

$$\begin{aligned} s &= F(s) \\ &= F(F(s)), \end{aligned}$$

and thus,

$$\begin{aligned} s &= F(s) \sqcap F(F(s)) \\ &= (\mathsf{1m2}\,F)(s). \end{aligned}$$

Conversely, suppose that s is a fixed point of $\mathsf{1m2}\,F$.

Then, by Proposition 2.15.2,

$$\begin{aligned} \mathrm{d}(s, F(s)) &= \mathrm{d}((\mathsf{1m2}\,F)(s), F(s)) \\ &= \mathrm{d}(F(s) \sqcap F(F(s)), F(s)) \\ &= \mathrm{d}(F(s) \sqcap F(F(s)), F(s) \sqcap F(s)) \\ &\supseteq \mathrm{d}(F(s), F(F(s))). \end{aligned} \tag{20}$$

Suppose, toward contradiction, that s is not a fixed point of F. Then, since F is strictly contracting on orbits,

$$\mathrm{d}(F(s), F(F(s))) \supset \mathrm{d}(s, F(s)),$$

contrary to (20).

Therefore, s is a fixed point of F. ☐

We may at last put all the different pieces together to obtain a constructive version of Theorem 5.2.

Theorem 5.12. *If $\langle X, \sqsubseteq \rangle$ is a directed-complete subsemilattice $\langle X, \sqsubseteq \rangle$ of $\langle \mathrm{S}, \sqsubseteq \rangle$, then for every contracting function F on X that is strictly contracting on orbits, and any post-fixed point s of F, $(\mathsf{1m2}\, F)^{\mathsf{oh}\,\langle X, \sqsubseteq \rangle}(s)$ is a fixed point of F.*

Proof. Suppose that $\langle X, \sqsubseteq \rangle$ is a directed-complete subsemilattice $\langle X, \sqsubseteq \rangle$ of $\langle \mathrm{S}, \sqsubseteq \rangle$.

Assume a contracting function F that is strictly contracting on orbits, and a post-fixed point s of F.

Suppose, toward contradiction, that $(\mathsf{1m2}\, F)^{\mathsf{oh}\,\langle X, \sqsubseteq \rangle}(s)$ is not a fixed point of $\mathsf{1m2}\, F$. Then, by Lemma 5.10, $\mathsf{oh}\,\langle X, \sqsubseteq \rangle + 2 \in \mathsf{oh}\,\langle X, \sqsubseteq \rangle$, a contradiction.

Therefore, $(\mathsf{1m2}\, F)^{\mathsf{oh}\,\langle X, \sqsubseteq \rangle}(s)$ is a fixed point of $\mathsf{1m2}\, F$. And since F is strictly contracting on orbits, by Lemma 5.11, $(\mathsf{1m2}\, F)^{\mathsf{oh}\,\langle X, \sqsubseteq \rangle}(s)$ is a fixed point of F. □

To be pedantic, Theorem 5.12 does not directly prove that F has a fixed point; unless there is a post-fixed point of F, the theorem is true vacuously. But if X is non-empty, then, by Lemma 5.3.1, for every $s \in X$, $(\mathsf{1m2}\, F)(s)$ is a post-fixed point of F.

The following is immediate from Proposition 4.2 and 5.1, Lemma 5.3.1, and Theorem 5.12:

Theorem 5.13. *If $\langle X, \sqsubseteq \rangle$ is a non-empty, directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$, then for every strictly contracting function F on X, and every $s \in X$,*

$$\mathsf{fix}\, F = (\mathsf{1m2}\, F)^{\mathsf{oh}\,\langle X, \sqsubseteq \rangle}((\mathsf{1m2}\, F)(s)).$$

This construction of fixed points as "limits of stationary transfinite iteration sequences" is very similar to the construction of extremal fixed points of monotone operators in [11] and references therein, where the function iterated is not $\mathsf{1m2}\, F$, but F itself. Notice, however, that if F preserves the prefix relation, then for any post-fixed point of F, $(\mathsf{1m2}\, F)(s) = F(s)$.

The astute reader will at this point anticipate the following:

Theorem 5.14. *If $\langle X, \sqsubseteq \rangle$ is a non-empty, directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$, then for every strictly contracting function F on X,*

$$\mathsf{fix}\, F = \bigsqcup\nolimits_X \{s \mid s \in X \text{ and } s \sqsubseteq F(s)\}.$$

Proof. Suppose that $\langle X, \sqsubseteq \rangle$ is a directed-complete subsemilattice $\langle X, \sqsubseteq \rangle$ of $\langle \mathrm{S}, \sqsubseteq \rangle$.

Assume a strictly contracting function F on X.

Assume a post-fixed point s of F.

By Lemma 5.9.2, $s \sqsubseteq (\mathsf{1m2}\, F)^{\mathsf{oh}\,\langle X, \sqsubseteq \rangle}(s)$, and thus, since F is strictly contracting, by Proposition 4.2 and 5.1, Lemma 5.9.2, and Theorem 5.12, $s \sqsubseteq \mathsf{fix}\, F$.

Thus, by generalization, $\mathsf{fix}\, F$ is an upper bound of $\{s \mid s \in X \text{ and } s \sqsubseteq F(s)\}$ in $\langle X, \sqsubseteq \rangle$. And since $\mathsf{fix}\, F$ is a post-fixed point of F, for every upper bound u of $\{s \mid s \in X \text{ and } s \sqsubseteq F(s)\}$ in $\langle X, \sqsubseteq \rangle$, $\mathsf{fix}\, F \sqsubseteq u$. Thus,

$$\mathsf{fix}\, F = \bigsqcup\nolimits_X \{s \mid s \in X \text{ and } s \sqsubseteq F(s)\}.$$

□

In retrospect, we find that Theorem 5.14 may be derived directly from first principles. In particular, and under the premise of the corollary, it is easy to establish without any use of Theorem 5.12 that for every $s \in X$, $s \sqsubseteq \mathsf{fix}\, F$ if and only if $s \sqsubseteq F(s)$, as the reader may wish to verify.

The construction of Theorem 5.14 is identical in form to Tarski's well known construction of greatest fixed points of order-preserving functions on complete lattices (see [59, thm. 1]). The question naturally arises whether the dual construction might also be of use here. In particular, we might be tempted to speculate that $\mathsf{fix}\, F = \bigsqcap_X \{s \mid s \in X \text{ and } F(s) \sqsubseteq s\}$. The following rejects this:

Example 5.15. Suppose that $\mathrm{T} = \{0, 1\}$, and $\preceq$ is the standard numerical order on $\{0, 1\}$.

Suppose that $\mathrm{V} = \{v\}$.

Let F be a function on S defined by the following mapping:

$$\begin{aligned} \emptyset &\mapsto \{\langle 1, v\rangle\}; \\ \{\langle 0, v\rangle\} &\mapsto \emptyset; \\ \{\langle 1, v\rangle\} &\mapsto \{\langle 1, v\rangle\}; \\ \{\langle 0, v\rangle, \langle 1, v\rangle\} &\mapsto \emptyset. \end{aligned}$$

It is easy to verify that F is strictly contracting. However,

$$\mathsf{fix}\, F \not\sqsubseteq \{\langle 0, v\rangle\},$$

whereas

$$F(\{\langle 0, v\rangle\}) \sqsubseteq \{\langle 0, v\rangle\}.$$

Example 5.15 is also sufficient to dispose of any lingering suspicion that $\mathsf{1m2}\, F$ might be order-preserving under the above premises.

5.3 Induction

Having used transfinite recursion to construct fixed points, we may use transfinite induction to prove properties of them. And in the case of strictly contracting endofunctions, which have exactly one fixed point, we may use Theorem 5.13 to establish a special proof rule.

Assume $P \subseteq \mathrm{S}$.

We say that P is *strictly inductive* if and only if every non-empty chain in $\langle P, \sqsubseteq\rangle$ has a least upper bound in $\langle P, \sqsubseteq\rangle$.

Note that P is strictly inductive if and only if $\langle P, \sqsubseteq\rangle$ is directed-complete (see [36, cor. 2]).

Theorem 5.16. *If $\langle X, \sqsubseteq\rangle$ is a non-empty, directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq\rangle$, then for every strictly contracting function F on X, and every non-empty, strictly inductive $P \subseteq X$, if for every $s \in P$, $(\mathsf{1m2}\, F)(s) \in P$, then $\mathsf{fix}\, F \in P$.*

Proof. Suppose that $\langle X, \sqsubseteq\rangle$ is a non-empty, directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq\rangle$.

Assume a strictly contracting function F on X, and non-empty, strictly inductive $P \subseteq X$.

Suppose that for every $s \in P$, $(\mathsf{1m2}\, F)(s) \in P$.

Let s be a signal in P.

By Lemma 5.3.1, $(\mathsf{1m2}\,F)(s)$ is a post-fixed point of F.

We use transfinite induction to prove that for every ordinal α, $(\mathsf{1m2}\,F)^{\alpha}((\mathsf{1m2}\,F)(s)) \in P$.

If $\alpha = 0$, then

$$(\mathsf{1m2}\,F)^{\alpha}((\mathsf{1m2}\,F)(s)) = (\mathsf{1m2}\,F)(s),$$

and thus, since P is closed under $\mathsf{1m2}\,F$, $(\mathsf{1m2}\,F)^{\alpha}((\mathsf{1m2}\,F)(s)) \in P$.

If there is an ordinal β such that $\alpha = \beta + 1$, then

$$(\mathsf{1m2}\,F)^{\alpha}((\mathsf{1m2}\,F)(s)) = (\mathsf{1m2}\,F)((\mathsf{1m2}\,F)^{\beta}((\mathsf{1m2}\,F)(s))).$$

By the induction hypothesis, $(\mathsf{1m2}\,F)^{\beta}((\mathsf{1m2}\,F)(s)) \in P$, and thus, since P is closed under $\mathsf{1m2}\,F$, $(\mathsf{1m2}\,F)^{\alpha}((\mathsf{1m2}\,F)(s)) \in P$.

Otherwise, α is a limit ordinal, and thus,

$$(\mathsf{1m2}\,F)^{\alpha}((\mathsf{1m2}\,F)(s)) = \bigsqcup_X \{(\mathsf{1m2}\,F)^{\beta}((\mathsf{1m2}\,F)(s)) \mid \beta \in \alpha\}.$$

By the induction hypothesis,

$$\{(\mathsf{1m2}\,F)^{\beta}((\mathsf{1m2}\,F)(s)) \mid \beta \in \alpha\} \subseteq P,$$

and by Lemma 5.9.2, $\langle\{(\mathsf{1m2}\,F)^{\beta}((\mathsf{1m2}\,F)(s)) \mid \beta \in \alpha\}, \sqsubseteq\rangle$ is totally ordered. Thus, since P is strictly inductive, $(\mathsf{1m2}\,F)^{\alpha}((\mathsf{1m2}\,F)(s)) \in P$.

Therefore, by transfinite induction, for every ordinal α, $(\mathsf{1m2}\,F)^{\alpha}((\mathsf{1m2}\,F)(s)) \in P$.

By Theorem 5.13,

$$\mathsf{fix}\,F = (\mathsf{1m2}\,F)^{\mathsf{oh}\,\langle X, \sqsubseteq\rangle}((\mathsf{1m2}\,F)(s)),$$

and thus, $\mathsf{fix}\,F \in P$. □

Theorem 5.16 is an induction principle that one may use to prove properties of fixed points of strictly contracting endofunctions. We think of properties extensionally here; that is, a property is a set of signals. And the properties that are admissible for use with this principle are those that are non-empty and strictly inductive. According to the principle, then, for every strictly contracting function F on any non-empty, directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq\rangle$, every non-empty, strictly inductive property that is preserved by $\mathsf{1m2}\,F$ is true of $\mathsf{fix}\,F$.

It is interesting to compare this principle with the fixed-point induction principle for order-preserving functions on complete partial orders (see [58]), which we will here refer to as *Scott-de Bakker induction*, and the fixed-point induction principle for contraction mappings on complete metric spaces (see [52]), which we will here refer to as *Reed-Roscoe induction* (see also [55], [54], [25]).

For a comparison between our principle and Scott-de Bakker induction, let F be a function of the most general kind of function to which both our principle and Scott-de Bakker induction apply, namely an order-preserving, strictly contracting function on a pointed, directed-complete subsemilattice $\langle X, \sqsubseteq\rangle$ of $\langle \mathrm{S}, \sqsubseteq\rangle$. Now assume a property $P \subseteq X$. If P is admissible for use with Scott-de Bakker induction, that is, closed under suprema in $\langle X, \sqsubseteq\rangle$ of arbitrary chains in $\langle P, \sqsubseteq\rangle$, then $\{s \mid s \in P \text{ and } s \sqsubseteq F(s)\}$ is non-empty and strictly inductive. And if P is closed under F, then $\{s \mid s \in P \text{ and } s \sqsubseteq F(s)\}$ is trivially closed under $\mathsf{1m2}\,F$. Therefore, given any reasonable property-specification logic, our principle is at least as strong a proof rule as Scott-de Bakker induction. At the same time, the often inconvenient requirement that a property P that is admissible for use with Scott-de Bakker induction contain the least upper bound in

$\langle X, \sqsubseteq \rangle$ of the empty chain, namely the least element in $\langle X, \sqsubseteq \rangle$, and the insistence that the least upper bound of every non-empty chain in $\langle P, \sqsubseteq \rangle$ be the same as in $\langle X, \sqsubseteq \rangle$ make it less likely that every property true of $\mathsf{fix}\, F$ that can be proved using our principle can also be proved using Scott-de Bakker induction. For this reason, we are inclined to say that, given any reasonable property-specification logic, our principle is a strictly stronger proof rule than Scott-de Bakker induction, in the case, of course, where both apply.

The relationship between our principle and Reed-Roscoe induction is less clear. $\langle \mathrm{S}, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d} \rangle$ being a generalized ultrametric space rather than a metric one, it might even seem that there can be no common ground for a meaningful comparison between the two. Nevertheless, it is possible to generalize Reed-Roscoe induction in a way that extends its applicability to the present case, while preserving its essence. According to the generalized principle, then, for every strictly contracting function F on any Cauchy complete, non-empty, directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$ such that every orbit under F is a Cauchy sequence, every non-empty property closed under limits of Cauchy sequences that is preserved by F is true of $\mathsf{fix}\, F$. One similarity between this principle and our own, and an interesting difference from Scott-de Bakker induction, is the lack of an explicit basis for the induction; as long as the property in question is non-empty, there is some basis available. In terms of closure and preservation of admissible properties, however, the two principles look rather divergent from one another. For example, the property of a signal having only a finite number of events in any finite interval of time is Cauchy complete, but not strictly inductive. On the other hand, by Lemma 5.3.1 and Theorem 5.14, our principle is better fit for proving properties that are closed under prefixes, such as, for example, the property of a signal having at most one event in any time interval of a certain fixed size. And for this reason, we suspect that, although complimentary to the generalized Read-Roscoe induction principle in theory, our principle might turn out to be more useful in practice, what can of course only be evaluated empirically.

Finally, we note that another simple proof rule may be associated with the construction of Theorem 5.14, which is nothing more than rephrasing the theorem to assert that any post-fixed point of F is a prefix of $\mathsf{fix}\, F$. In the context of order-preserving functions on complete lattices, this is known as the *coinduction* proof method (see [41], [56]). Its dual, known as *Park's principle of fixpoint induction* (see [47]), is not valid in our setting, as demonstrated in Example 5.15.

5.4 Convergence

From a computational point of view, Theorem 5.12 and 5.13 are not entirely satisfying. The transfinite iteration sequence of post-fixed points constructed may grow arbitrarily long en route to the fixed point. For every non-empty, directed-complete subsemilattice $\langle X, \sqsubseteq \rangle$ of $\langle \mathrm{S}, \sqsubseteq \rangle$, the length of the sequence will of course be bounded by $\mathsf{oh}\, \langle X, \sqsubseteq \rangle$. But for every ordinal α, it is easy to come up with a strictly contracting endofunction F and a post-fixed point s of F such that for any $\beta \in \alpha$, $(\mathsf{1m2}\, F)^{\beta}(s) \sqsubset (\mathsf{1m2}\, F)^{\beta+1}(s)$. And this is one obstacle to using these theorems as effective procedures for "computing" the sought fixed point. Of course, the latter may very well be an infinite object, and we understand "computing" that object as a process of successive approximations converging to it. But the notions of approximation and convergence are formalized topologically, and depend on the topology chosen. And both the generalized distance function and the prefix relation induce topologies on signals, each lending its own perspective on the problem.

We begin with the notion of approximation associated with the generalized distance function, and start probing into the convergence properties of Theorem 5.12 and 5.13 with the following proposition:

Proposition 5.17. *If $\langle X, \sqsubseteq \rangle$ is a directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$, then for every function F on X, any post-fixed point s of F, and every ordinal α and β,*

1. *if F is contracting, then*

$$\mathrm{d}((\mathsf{1m2}\, F)^{\alpha+1}(s), (\mathsf{1m2}\, F)^{\beta+1}(s)) \supseteq \mathrm{d}((\mathsf{1m2}\, F)^{\alpha}(s), (\mathsf{1m2}\, F)^{\beta}(s));$$

2. *if F is contracting and strictly contracting on orbits, and* $(\mathsf{1m2}\,F)^{\alpha}(s) \neq (\mathsf{1m2}\,F)^{\beta}(s)$*, then*

$$\mathrm{d}((\mathsf{1m2}\,F)^{\alpha+1}(s), (\mathsf{1m2}\,F)^{\beta+1}(s)) \supset \mathrm{d}((\mathsf{1m2}\,F)^{\alpha}(s), (\mathsf{1m2}\,F)^{\beta}(s)).$$

Proof. Suppose that $\langle X, \sqsubseteq\rangle$ is a directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq\rangle$.

Assume a function F on X, and a post-fixed point s of F.

For every ordinal α, let $s_\alpha = (\mathsf{1m2}\,F)^{\alpha}(s)$.

Assume ordinals α and β.

Suppose that F is contracting.

Since F is contracting,

$$\begin{aligned}\mathrm{d}(F(F(s_\alpha)), F(F(s_\beta))) &\supseteq \mathrm{d}(F(s_\alpha), F(s_\beta))\\ &\supseteq \mathrm{d}(s_\alpha, s_\beta).\end{aligned}$$

Then, by Proposition 2.15.2,

$$\mathrm{d}(F(s_\alpha) \sqcap F(F(s_\alpha)), F(s_\alpha) \sqcap F(F(s_\beta))) \supseteq \mathrm{d}(s_\alpha, s_\beta)$$

and

$$\mathrm{d}(F(F(s_\beta)) \sqcap F(s_\alpha), F(F(s_\beta)) \sqcap F(s_\beta)) \supseteq \mathrm{d}(s_\alpha, s_\beta).$$

Thus, by the generalized ultrametric inequality,

$$\mathrm{d}(F(s_\alpha) \sqcap F(F(s_\alpha)), F(s_\beta) \sqcap F(F(s_\beta))) \supseteq \mathrm{d}(s_\alpha, s_\beta),$$

and hence, by definition of $\mathsf{1m2}\,F$, s_α, and s_β,

$$\mathrm{d}((\mathsf{1m2}\,F)^{\alpha+1}(s), (\mathsf{1m2}\,F)^{\beta+1}(s)) \supseteq \mathrm{d}((\mathsf{1m2}\,F)^{\alpha}(s), (\mathsf{1m2}\,F)^{\beta}(s)).$$

Thus, by generalization, 1 is true.

Suppose that F is contracting and strictly contracting on orbits, and $s_\alpha \neq s_\beta$.

Since F is contracting, by 1,

$$\mathrm{d}(s_{\alpha+1}, s_{\beta+1}) \supseteq \mathrm{d}(s_\alpha, s_\beta).$$

Suppose, toward contradiction, that

$$\mathrm{d}(s_{\alpha+1}, s_{\beta+1}) = \mathrm{d}(s_\alpha, s_\beta). \tag{21}$$

Without loss of generality, assume that $\alpha \in \beta$. Then, by Lemma 5.9.2,

$$s_\alpha \sqsubseteq s_{\alpha+1} \sqsubseteq s_\beta \sqsubseteq s_\beta. \tag{22}$$

By Proposition 2.15.2,

$$\mathrm{d}(s_\beta \sqcap s_{\alpha+1}, s_\beta \sqcap s_{\beta+1}) \supseteq \mathrm{d}(s_{\alpha+1}, s_{\beta+1}).$$

However, by (22), $s_\beta \sqcap s_{\alpha+1} = s_{\alpha+1}$ and $s_\beta \sqcap s_{\beta+1} = s_\beta$, and thus,

$$\mathrm{d}(s_{\alpha+1}, s_\beta) \supseteq \mathrm{d}(s_{\alpha+1}, s_{\beta+1}). \tag{23}$$

Then, by (21), (23), and the generalized ultrametric inequality,

$$\mathrm{d}(s_\alpha, s_{\alpha+1}) \supseteq \mathrm{d}(s_{\alpha+1}, s_{\beta+1}), \tag{24}$$

and thus, by Proposition 2.15.1 and (22),

$$s_{\alpha+1} \sqcap s_{\beta+1} \sqsubseteq s_\alpha \sqcap s_{\alpha+1}.$$

However, by (22), $s_{\alpha+1} \sqcap s_{\beta+1} = s_{\alpha+1}$ $s_\alpha \sqcap s_{\alpha+1} = s_\alpha$, and $s_\alpha \sqsubseteq s_{\alpha+1}$, and thus, $s_\alpha = s_{\alpha+1}$. Thus, s_α is a fixed point of $\mathsf{1m2}\,F$, and by an easy transfinite induction, $s_\alpha = s_\beta$, contrary to the assumption that $s_\alpha \neq s_\beta$.

Therefore,

$$\mathrm{d}(s_{\alpha+1}, s_{\beta+1}) \supset \mathrm{d}(s_\alpha, s_\beta).$$

Thus, by generalization, 2 is true. □

Assume a function F on a non-empty, directed-complete subsemilattice $\langle X, \sqsubseteq\rangle$ of $\langle \mathrm{S}, \sqsubseteq\rangle$, and consider the first ω terms in the transfinite iteration sequence of $\mathsf{1m2}\,F$ starting from a post-fixed point s of F. If F is contracting and strictly contracting on orbits, and for every $n \in \omega$, $(\mathsf{1m2}\,F)^n(s) \neq (\mathsf{1m2}\,F)^{n+1}(s)$, then, by Proposition 5.17.2, the sequence $\langle \mathrm{d}((\mathsf{1m2}\,F)^0(s), (\mathsf{1m2}\,F)^1(s)), \mathrm{d}((\mathsf{1m2}\,F)^1(s), (\mathsf{1m2}\,F)^2(s)), \ldots\rangle$ is a strictly descending chain in $\langle \mathscr{L}\langle \mathrm{T}, \preceq\rangle, \supseteq\rangle$. Nevertheless, the sequence $\langle (\mathsf{1m2}\,F)^0(s), (\mathsf{1m2}\,F)^1(s), \ldots\rangle$ need not be Cauchy (see Theorem 5.18 and 5.23), and thus, need not converge in the topology induced by d on X.

This kind of convergence failure is a manifestation of Zeno's paradox. This is particularly true whenever $\langle \mathrm{T}, \preceq\rangle$ is totally ordered, in which case, $\langle X, \mathscr{L}\langle \mathrm{T}, \preceq\rangle, \supseteq, \mathrm{T}, \mathrm{d}\rangle$ is Cauchy-complete, indeed spherically complete (see Corollary 5.6), and therefore, $\langle (\mathsf{1m2}\,F)^0(s), (\mathsf{1m2}\,F)^1(s), \ldots\rangle$ fails to converge in the related topology exactly when $\bigcup\{\mathrm{d}((\mathsf{1m2}\,F)^n(s), (\mathsf{1m2}\,F)^{n+1}(s)) \mid n \in \omega\}$ is a strict subset of T, or equivalently, there is $\tau \in \mathrm{T}$ such that for every $n \in \omega$, there is $\tau' \prec \tau$ such that $(\mathsf{1m2}\,F)^n(s)(\tau') \not\simeq (\mathsf{1m2}\,F)^{n+1}(s)(\tau')$. By Lemma 5.9.2 and Theorem 5.12, then, each term of the sequence $\langle (\mathsf{1m2}\,F)^0(s), (\mathsf{1m2}\,F)^1(s), \ldots\rangle$ will contribute at least one new event to the fixed point, each with some tag $\tau' \prec \tau$. For a timed system, this means that there will be an infinite number of events accruing before some particular instance of time, a variant of Zeno's paradox.

This is clearly an issue whenever $(\mathsf{1m2}\,F)^\omega(s)$ is not a fixed point of F, and it may still be an issue even when $(\mathsf{1m2}\,F)^\omega(s)$ is a fixed point of F. But whenever the sequence $\langle (\mathsf{1m2}\,F)^0(s), (\mathsf{1m2}\,F)^1(s), \ldots\rangle$ is Cauchy, and converges in the topology induced by d on X, $(\mathsf{1m2}\,F)^\omega(s)$ is a fixed point of F.

Theorem 5.18. *If $\langle \mathrm{T}, \preceq\rangle$ is totally ordered, and $\langle X, \sqsubseteq\rangle$ is a directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq\rangle$, then for every contracting function F on X that is strictly contracting on orbits, and any post-fixed point s of F, if $\langle (\mathsf{1m2}\,F)^n(s) \mid n \in \omega\rangle$ is Cauchy in $\langle X, \mathscr{L}\langle \mathrm{T}, \preceq\rangle, \supseteq, \mathrm{T}, \mathrm{d}\rangle$, then $(\mathsf{1m2}\,F)^\omega(s)$ is a fixed point of F.*

Proof. Suppose that $\langle \mathrm{T}, \preceq\rangle$ is totally ordered, and $\langle X, \sqsubseteq\rangle$ is a directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq\rangle$.

Assume a contracting function F on X that is strictly contracting on orbits, and a post-fixed point s of F.

For every ordinal α, let $s_\alpha = (\mathsf{1m2}\,F)^\alpha(s)$.

Suppose that $\langle s_n \mid n \in \omega\rangle$ is Cauchy in $\langle X, \mathscr{L}\langle \mathrm{T}, \preceq\rangle, \supseteq, \mathrm{T}, \mathrm{d}\rangle$.

Suppose, toward contradiction, that s_ω is not a fixed point of F. Then, since F is strictly contracting on orbits, by Lemma 5.11, s_ω is not a fixed point of $\mathsf{1m2}\,F$, and thus,

$$\mathrm{T} \supset \mathrm{d}(s_\omega, s_{\omega+1}). \tag{25}$$

Assume $n_1, n_2 \in \omega$.

Without loss of generality, assume that $n_1 < n_2$. Then, by Proposition 2.15.2 and Lemma 5.9.2,

$$\begin{aligned} \mathrm{d}(s_{n_1}, s_\omega) &\supseteq \mathrm{d}(s_{n_2} \sqcap s_{n_1}, s_{n_2} \sqcap s_\omega) \\ &= \mathrm{d}(s_{n_1}, s_{n_2}) \end{aligned}$$

and

$$\begin{aligned} \mathrm{d}(s_{n_1}, s_{\omega+1}) &\supseteq \mathrm{d}(s_{n_2} \sqcap s_{n_1}, s_{n_2} \sqcap s_{\omega+1}) \\ &= \mathrm{d}(s_{n_1}, s_{n_2}) \end{aligned}$$

and thus, by the generalized ultrametric inequality,

$$\mathrm{d}(s_\omega, s_{\omega+1}) \supseteq \mathrm{d}(s_{n_1}, s_{n_2}).$$

Thus, by generalization and (25), $\langle s_n \mid n \in \omega \rangle$ is not Cauchy in $\langle X, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d} \rangle$, contrary to our assumption.

Therefore, s_ω is a fixed point of F. □

The following is immediate from Proposition 5.1 and Theorem 5.18:

Theorem 5.19. *If $\langle \mathrm{T}, \preceq \rangle$ is totally ordered, and $\langle X, \sqsubseteq \rangle$ is a non-empty, directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$, then for every strictly contracting function F on X, and every $s \in X$, if $\langle (\mathsf{1m2}\, F)^n(s) \mid n \in \omega \rangle$ is Cauchy in $\langle X, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d} \rangle$, then*

$$\mathsf{fix}\, F = (\mathsf{1m2}\, F)^\omega((\mathsf{1m2}\, F)(s)).$$

The following shows that the hypothesis of $\langle \mathrm{T}, \preceq \rangle$ being totally ordered in Theorem 5.18 and 5.19 cannot be discarded:

Example 5.20. Suppose that $\mathrm{T} = \{0, 1\} \times \mathbb{N}$, and $\preceq$ is an order relation on $\{0, 1\} \times \mathbb{N}$ such that for every $\langle i_1, n_1 \rangle, \langle i_2, n_2 \rangle \in \{0, 1\} \times \mathbb{N}$,

$$\langle i, n_1 \rangle \preceq \langle i_2, n_2 \rangle \iff i_1 = i_2 \text{ and } n_1 \leq n_2.$$

Let v be a value in V.

For every $\langle i, n \rangle \in \mathrm{T}$, let $s_{\langle i,n \rangle} = \{\langle \langle j, m \rangle, v \rangle \mid j < i \text{ and } m \in \mathbb{N}, \text{ or } j = i \text{ and } m < n\}$.

Let $X = \{s_{\langle i,n \rangle} \mid \langle i, n \rangle \in \mathrm{T}\} \cup \{\mathrm{T} \times \{v\}\}$.

It is not hard to verify that $\langle X, \sqsubseteq \rangle$ is a directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$.

Let F be a function on X such that for every $\langle i, n \rangle \in \mathrm{T}$,

$$F(s_{\langle i,n \rangle}) = s_{\langle i,n+1 \rangle},$$

and

$$F(\{\mathrm{T} \times \{v\}\}) = \{\mathrm{T} \times \{v\}\}.$$

It is easy to verify that F is strictly contracting. But although $\langle (\mathsf{1m2}\, F)^n(s_{\langle 0,0 \rangle})) \mid n \in \omega \rangle$ is Cauchy in $\langle X, \mathscr{L}\langle \mathrm{T}, \preceq \rangle, \supseteq, \mathrm{T}, \mathrm{d} \rangle$, $(\mathsf{1m2}\, F)^\omega(s)$ is not a fixed point of F.

In Banach's fixed-point theorem, and any reasonable generalization of it (e.g., see [39], [10]), it is of course not the convergence of the orbit of s under $\mathsf{1m2}\,F$, but ultimately, the convergence of the orbit of s under F that is exploited. It is therefore interesting to see what the orbit of s under $\mathsf{1m2}\,F$ does when the orbit of s under F converges. The following shows that even when every orbit under F is Cauchy, and converges in the topology induced by d on X, the orbit of s under $\mathsf{1m2}\,F$ need not be Cauchy, and $(\mathsf{1m2}\,F)^{\omega}(s)$ need not be a fixed point of F:

Example 5.21. Suppose that $\mathrm{T} = \omega + 2$, and $\preceq$ is the standard order on $\omega + 2$.

Let v_1 and v_2 be two distinct values in V.

Let $X = \{\alpha \times \{v_1\} \mid \alpha \in \omega + 2\} \cup \{\alpha \times \{v_1\} \cup ((\omega + 2) - \alpha) \times \{v_2\} \mid \alpha \in \omega + 2\}$.

It is easy to verify that $\langle X, \sqsubseteq\rangle$ is a directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq\rangle$.

Let F be a function on X such that for every $s \in X$ and every $\tau \in \mathrm{T}$,

$$F(s)(\tau) \simeq \begin{cases} v_1 & \text{if } \tau \in \mathsf{dom}\, s + 1; \\ v_2 & \text{otherwise.} \end{cases}$$

It is easy to verify that F is strictly contracting. Furthermore, for every $s \in X$ and every $n > 1$,

$$F^n(s) = (\omega + 2) \times \{v_1\}.$$

However, for every $\alpha \in \omega + 2$,

$$(\mathsf{1m2}\,F)^{\alpha}(\emptyset) = \alpha \times \{v_1\},$$

and thus, $(\mathsf{1m2}\,F)^{\omega}(\emptyset)$ is not a fixed point of F.

Now, thus far, we have been concerned with the effects of convergence relative to the topology induced by d. But ultimately, what we are really interested in is convergence relative to a topology induced by $\sqsubseteq$. A handy topology here is of course the Scott topology, but the details are not important. What is important is that, for our purposes, topological convergence can be construed in terms of suprema of ω-chains. And to require that the ω-chain $\langle(\mathsf{1m2}\,F)^0(s), (\mathsf{1m2}\,F)^1(s), \ldots\rangle$ converge to the sought fixed point is to require that $(\mathsf{1m2}\,F)^{\omega}(s)$ be that fixed point.

Ideally, we would like to impose some kind of continuity condition on F, to require that its value at the "infinite" arguments be completely determined by its value at the "finite" ones. This idea is inspired by the Scott-continuity paradigm, and its merit is not limited to convergent fixed-point constructions. But whether this is feasible or even meaningful in our context is a research topic on its own. Here, we only consider a simple condition on the domain of F.

Proposition 5.22. *If $\langle X, \sqsubseteq\rangle$ is a directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq\rangle$, and $\mathsf{oh}\,\langle X, \sqsubseteq\rangle \in \omega + 3$, then for every contracting function F on X that is strictly contracting on orbits, and any post-fixed point s of F, $(\mathsf{1m2}\,F)^{\omega}(s)$ is a fixed point of F.*

Proof. We prove the contrapositive.

Assume a contracting function F on X, and a post-fixed point s of F.

Suppose that $(\mathsf{1m2}\,F)^{\omega}(s)$ is not a fixed point of F. Then, since F is strictly contracting on orbits, by Lemma 5.11, $(\mathsf{1m2}\,F)^{\omega}(s)$ is not a fixed point of $\mathsf{1m2}\,F$. Thus, by Lemma 5.10, $\omega + 2 \in \mathsf{oh}\,\langle X, \sqsubseteq\rangle$, and hence, $\mathsf{oh}\,\langle X, \sqsubseteq\rangle \notin \omega + 3$. □

In other words, $\langle (\mathsf{1m2}\,F)^0(s), (\mathsf{1m2}\,F)^1(s), \ldots \rangle$ will converge to a fixed point of F whenever there is no ascending chain of length $\omega + 2$ in $\langle X, \sqsubseteq \rangle$. Trivial as it may be, this fact is of note because of its direct bearing on discrete-event systems: if F models a component that operates on discrete-even signals, then $(\mathsf{1m2}\,F)^\omega(s)$ is a fixed point of F.

If we further require that $\langle \mathrm{T}, \preceq \rangle$ be totally ordered, then we may also reverse the implication of Proposition 5.22, and moreover, do so in a rather strong way.

Theorem 5.23. *If $\langle \mathrm{T}, \prec \rangle$ is totally ordered, and $\langle X, \sqsubseteq \rangle$ is a directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$, then the following are equivalent:*

1. $\mathsf{oh}\,\langle X, \sqsubseteq \rangle \in \omega + 3$*;*
2. *for every contracting function F on X that is strictly contracting on orbits, and any post-fixed point s of F, $(\mathsf{1m2}\,F)^\omega(s)$ is a fixed point of F.*

Proof. Suppose that $\langle \mathrm{T}, \prec \rangle$ is totally ordered.

Suppose that $\langle X, \sqsubseteq \rangle$ is a directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$.

If 1 is true, then by 5.22, 2 is true.

Conversely, suppose that 1 is not true. Then there is a function φ from $\omega + 2$ to X such that for every $\alpha, \beta \in \omega + 2$, if $\alpha \in \beta$, then $\varphi(\alpha) \sqsubset \varphi(\beta)$.

Let F be a function on X such that for any $s \in X$,

$$F(s) = \begin{cases} \varphi(\min\{\alpha \mid \alpha \in \omega + 1 \text{ and } \varphi(\alpha) \not\sqsubseteq s\}) & \text{if } \{\alpha \mid \alpha \in \omega + 1 \text{ and } \varphi(\alpha) \not\sqsubseteq s\} \neq \emptyset; \\ \varphi(\omega + 1) & \text{otherwise.} \end{cases}$$

Assume $s_1, s_2 \in X$ such that $s_1 \neq s_2$.

Suppose, toward contradiction, that

$$\mathrm{d}(F(s_1), F(s_2)) \not\supset \mathrm{d}(s_1, s_2).$$

Then, since $\langle \mathrm{T}, \prec \rangle$ is totally ordered,

$$\mathrm{d}(s_1, s_2) \supseteq \mathrm{d}(F(s_1), F(s_2)). \tag{26}$$

Let α be the unique ordinal in $\omega + 2$ such that $\varphi(\alpha) = F(s_1)$.

Let β be the unique ordinal in $\omega + 2$ such that $\varphi(\beta) = F(s_2)$.

Then, by (26),

$$\mathrm{d}(s_1, s_2) \supseteq \mathrm{d}(\varphi(\alpha), \varphi(\beta)),$$

and thus, by Proposition 2.15.2,

$$\mathrm{d}(\varphi(\alpha) \sqcap s_1, \varphi(\alpha) \sqcap s_2) \supseteq \mathrm{d}(\varphi(\alpha), \varphi(\beta)). \tag{27}$$

Without loss of generality, assume that $\alpha \in \beta$. Then $\alpha \in \omega + 1$.

Suppose, toward contradiction, that $\varphi(\alpha) \not\sqsubseteq s_2$. Then, by definition of F, $\alpha \notin \beta$, contrary to our assumption.

Therefore, $\varphi(\alpha) \not\sqsubseteq s_2$, and thus,

$$\varphi(\alpha) \sqcap s_2 = \varphi(\alpha). \tag{28}$$

By (27) and (28),

$$\mathrm{d}(\varphi(\alpha) \sqcap s_1, \varphi(\alpha)) \supseteq \mathrm{d}(\varphi(\alpha), \varphi(\beta)),$$

and thus, by Proposition 2.15.1,

$$\begin{aligned} \varphi(\alpha) \sqcap \varphi(\beta) &\sqsubseteq (\varphi(\alpha) \sqcap s_1) \sqcap \varphi(\alpha) \\ &= \varphi(\alpha) \sqcap s_1. \end{aligned}$$

However, since $\alpha \in \beta$, $\varphi(\alpha) \sqsubset \varphi(\beta)$, and thus, $\varphi(\alpha) \sqcap \varphi(\beta) = \varphi(\alpha)$. Hence, $\varphi(\alpha) = \varphi(\alpha) \sqcap s_1$, and thus, $\varphi(\alpha) \sqsubseteq s_1$, contrary to the definition of F.

Therefore,

$$\mathrm{d}(F(s_1), F(s_2)) \not\supset \mathrm{d}(s_1, s_2).$$

Thus, by generalization, F is strictly contracting.

Trivially, $\varphi(0)$ is a post-fixed point of F, and by an easy transfinite induction, for every $\alpha \in \omega + 2$,

$$(\mathsf{1m2}\, F)^{\alpha}(\varphi(0)) = \varphi(\alpha).$$

Suppose, toward contradiction, that $(\mathsf{1m2}\, F)^{\omega}(\varphi(0))$ is a fixed point of F. Then, since F is strictly contracting, by Proposition 5.1 and Lemma 5.11, $(\mathsf{1m2}\, F)^{\omega}(\varphi(0)) = (\mathsf{1m2}\, F)^{\omega+1}(\varphi(0))$, contrary to the fact that $\varphi(\omega) \sqsubset \varphi(\omega + 1)$.

Therefore, $(\mathsf{1m2}\, F)^{\omega}(\varphi(0))$ is not a fixed point of f, and thus, 2 is not true. □

The following is immediate from the proof of Theorem 5.23:

Theorem 5.24. *If $\langle \mathrm{T}, \prec \rangle$ is totally ordered, and $\langle X, \sqsubseteq \rangle$ is a non-empty, directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$, then the following are equivalent:*

1. $\mathsf{oh}\, \langle X, \sqsubseteq \rangle \in \omega + 3$;
2. *for every strictly contracting function F on X, and every $s \in X$,*

$$\mathsf{fix}\, F = (\mathsf{1m2}\, F)^{\omega}((\mathsf{1m2}\, F)(s)).$$

The following shows that the hypothesis of $\langle \mathrm{T}, \preceq \rangle$ being totally ordered in Theorem 5.23 and 5.24 cannot be discarded:

Example 5.25. Suppose that $\mathrm{T} = \omega + 2$, and $\preceq$ is the discrete order on $\omega + 2$.

Suppose that V is a singleton set.

Let $X = \{\alpha \times \mathrm{V} \mid \alpha \in \omega + 2\}$.

It is not hard to verify that the ordered set $\langle X, \sqsubseteq \rangle$ is a directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$, and that $\mathsf{oh}\, \langle X, \sqsubseteq \rangle = \omega + 3$. However, for every contracting function F on X that is strictly contracting on orbits, and every $s \in X$, $F(s)$ is a fixed point of F, as the reader is invited to verify.

As a hint, notice that for every $\alpha, \beta, \gamma, \delta \in \omega + 2$, if $\alpha \in \beta$, then

$$\mathrm{d}(\gamma \times \mathrm{V}, \delta \times \mathrm{V}) \supseteq \mathrm{d}(\alpha \times \mathrm{V}, \beta \times \mathrm{V})$$

if and only if

$$\{\gamma, \delta\} \subseteq (\beta + 1) - \alpha.$$

6 Discussion

From an academic standpoint, the fixed-point theory of Section 5 is quite satisfying. It is, we believe, elegant, reasonably abstract, and remarkably general. Indeed, it is hard to see how one could relax the premise that the domain of the function be a directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$ in any reasonable way without compromising the possibility of a constructive argument. But how meaningful is it in practice to assume that a component operates on a directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$? And how accurately does our fixed-point construction model the operation of a component in feedback? In the absence of a formal operational semantics, these questions are admittedly vague, and must be evaluated on informal grounds. Hence the discussion.

Let us start from the concept of signal. Definition 2.2 is inclusive enough to accurately capture the input or output behaviour of any sort of system that one might sensibly think of as timed. Any sort of variation in time, be it continuous, discrete, or even hybrid, will readily fit in it. The very notion of time is extremely versatile: real time, discrete time, superdense time, they surely do not exhaust the overwhelming array of options.

On the grounds of such generality, we feel comfortable articulating the following thesis: every determinate single-input, single-output timed system can be modelled as a partial function on signals. This is effectively a definition, and thus, not really susceptible to formal arguments. It is, however, a plausible formalization of our conception of what a timed system is, and the reader should find no trouble subscribing to it.

Notice that a modelling function need not be defined at every signal. This should come as no surprise. For example, a discrete-event component is expected to operate on a possibly infinite sequence of time-stamped values, processing them in the order determined by their time stamps. Now, unless it has no internal state, it makes no sense feeding that component with a set of time-stamped values that cannot be arranged into a sequence according to their time stamps. It is therefore unreasonable to demand that a function modelling that component be defined at any signal that is not a discrete-event one.

On the other hand, the domain of a modelling function is not entirely arbitrary either. To return to the foregoing example, we are happy to find that, for any choice of tag set, discrete-event signals form a directed-complete subsemilattice of $\langle \mathrm{S}, \sqsubseteq \rangle$. In fact, they form a directed-complete lower set of $\langle \mathrm{S}, \sqsubseteq \rangle$. And this, we claim, is not incidental to the discrete-event case, but true of any reasonably specified domain of operation. Arbitrary signals, well-ordered signals, even bare streams of values, all attest to our claim as natural, common examples.

An interesting exception is the case of total signals, that is, signals that are defined over the entire tag set. Completeness with respect to directed suprema is still valid in that case, albeit trivially so, every directed set of total signals being a singleton one. But closure under prefixes fails dramatically. And yet there is nothing unreasonable in ruling out absence of event. For absence of event is absurd when it comes to components that demand an input value at every time instance, such as, for example, physical components operating on continuous variations.

Fortunately, we can easily reduce the fixed-point problem of a strictly contracting function on the set of all total signals to that of one on that of all signals. For example, let v be a fixed value in V, and H a function on S such that for every $s \in \mathrm{S}$ and $\tau \in \mathrm{T}$,

$$H(s)(\tau) = \begin{cases} s(\tau) & \text{if } \tau \in \mathsf{dom}\, s; \\ v & \text{otherwise.} \end{cases}$$

H maps every signal to a corresponding total signal by filling any idle tag slots with the value v. Now assume a function F on the set of all total signals. If F is contracting and strictly contracting on orbits then, $F \circ H$ is also contracting and strictly contracting on orbits. Moreover, s is a fixed point of F if and only if s is a fixed point of $F \circ H$. Hence, we can use $F \circ H$, which does satisfy our premise, to construct and reason about the fixed points of F, which does not. Notice that if F is strictly contracting, then the choice of v is completely irrelevant. This is not true in the more general case, however, where that choice can actually bias the fixed-point construction process.

Of course, one might be interested not in the set of all total signals, but in some particular subset of it, such as, for example, that of all continuous signals. In that case, one can accordingly adjust the foregoing reduction to make H not a function from the set of all signals to that subset, but one from the closure of that subset under prefixes to it.

There is a different, yet instructive way to look at such reductions: we may use the function H to induce an order on the set of total signals in question that arranges that set into a directed-complete semilattice satisfying Proposition 2.15 when $\sqcap$ is interpreted as the meet operation of that semilattice. We invite the reader to sort out the details, and muse over the implications of this approach.

We go on to discuss how our fixed-point construction relates to the actual operation of a component in feedback. To keep our discussion tractable, we assume that the systems under consideration are either physically or logically timed, such that the output of each component is built up incrementally, in step with progress of time in the system.

Consider then a directed-complete subsemilattice $\langle X, \sqsubseteq \rangle$ of $\langle \mathrm{S}, \sqsubseteq \rangle$, and a component that realizes a strictly contracting function F on X, and is configured in feedback as in Figure 1. Initially, there are no events at the input of the component, which thus sets out to produce the events that make up the signal $F(\emptyset)$. But as these events begin to appear at the output of the component, they instantly modify the input that the component operates on, causing the component to evaluate anew what events to produce. Because the component realizes a strictly contracting function, however, the effect of this retroaction is discerned only after the events making up the largest common prefix of $F(\emptyset)$ and $F(F(\emptyset))$ have been produced, at which point the component is seen to diverge from its original path, and proceed with the events that make up the signal $F(F(\emptyset) \sqcap F(F(\emptyset)))$, or equivalently, $F((\mathsf{1m2}\, F)(\emptyset))$. Iterating this kind of reasoning, we notice that the transfinite sequence $\langle (\mathsf{1m2}\, F)^{\alpha}(\emptyset) \mid \alpha \in \mathsf{oh}\, \langle X, \sqsubseteq \rangle \rangle$ is really a trace of the operation of the system. Therefore, we expect that the events produced throughout the operation of the system be precisely those that make up the unique fixed point of F.

However informal, the preceding argument is still quite tenable, especially when the system in question is linearly timed. But turning the argument into a rigorous proof would call for a formal operational semantics, and is thus outside the scope of this work.

Finally, it might seem natural to extrapolate this line of reasoning to the situation where the realized function F is only contracting and strictly contracting on orbits, in an attempt to argue that, again, the events produced throughout the operation of the system are those that make up $(\mathsf{1m2}\, F)^{\mathsf{oh}\, \langle X, \sqsubseteq \rangle}(\emptyset)$. At closer inspection though, we find that the argument outlined above is no longer sound under the revised conditions. Specifically, there is no longer the guarantee that for every $\alpha \in \mathsf{oh}\, \langle X, \sqsubseteq \rangle$, and after having produced the events of $(\mathsf{1m2}\, F)^{\alpha}(\emptyset)$, the system will ever go on to reach $(\mathsf{1m2}\, F)^{\alpha+1}(\emptyset)$. This is perhaps best understood through an example.

Example 6.1. Suppose that $\mathrm{T} = \{0, 1\}$, and $\preceq$ is the standard order on $\{0, 1\}$.

Let $\mathrm{V} = \{v\}$.

Let F be a function on S defined by the following mapping:

$$\emptyset \mapsto \{\langle 0, v \rangle, \langle 1, v \rangle\};$$
$$\{\langle 0, v \rangle\} \mapsto \{\langle 0, v \rangle\};$$
$$\{\langle 1, v \rangle\} \mapsto \{\langle 0, v \rangle, \langle 1, v \rangle\};$$
$$\{\langle 0, v \rangle, \langle 1, v \rangle\} \mapsto \{\langle 0, v \rangle, \langle 1, v \rangle\}.$$

It is easy to verify that F is contracting and strictly contracting on orbits, and

$$(\mathsf{1m2}\,F)^{\mathsf{oh}\,\langle \mathrm{S},\sqsubseteq\rangle}(\emptyset) = \{\langle 0, v\rangle, \langle 1, v\rangle\}.$$

However,

$$\begin{aligned}(\mathsf{1m2}\,F)^{0}(\emptyset) &\sqsubseteq \{\langle 0, v\rangle\}\\ &\sqsubseteq (\mathsf{1m2}\,F)^{1}(\emptyset),\end{aligned}$$

and $\{\langle 0, v\rangle\}$ is another fixed point of F.

If the component of the system realizes the function F of Example 6.1, then it will initially set out to produce the events that make up $\{\langle 0, v\rangle, \langle 1, v\rangle\}$. However, just after having produced its first event, the component will be found operating on $\{\langle 0, v\rangle\}$, a fixed point of F. Thus, the system will never go on to produce the event $\langle 1, v\rangle$, what would imply that the component is actually able to distinguish between an extraneously produced event and an identical one produced by itself in response to absence of an event at that same time, and thus, cannot really realize a function on S.

However paradoxical, the idea of a component reacting instantaneously to its own stimuli is commonplace in reactive systems, and its potential bearing on logically timed systems should be thoroughly thought out before declared nonsensical.

7 Related work

Fixed points have been used extensively in the construction of mathematical models in computer science. In most cases, ordered sets and monotone functions have been the more natural choice. But in the case of timed computation, metric spaces and contraction mappings have proved a better fit, and Tarski's fixed-point theorem and its variants have given place to Banach's contraction principle. To our knowledge, the first to use this kind of modelling framework were Reed and Roscoe in their work on a real-time extension of CSP (see [52], [53]). Yates later used more or less the same methods to develop what was probably the first extensional model of timed computation: a real-time extension of Kahn's process networks (see [60]). Müller and Scholz introduced another such extension in [43], working with metric spaces of dense signals rather than timed streams. And a uniform framework encompassing both kinds of models was presented in [28], [27], [30].

Common to all [52], [53], [60], [43], [28], [27], and [30] is the requirement of a positive lower bound on the reaction time of each component in a system. This constraint is used to guarantee that the functions modelling these components are actually contraction mappings with respect to the defined metrics. The motivation is of course the ability to use Banach's fixed-point theorem in the interpretation of feedback, but a notable consequence is the absence of non-trivial Zeno phenomena, what has always been considered a precondition for realism in the real-time systems community. Even in the verification literature, where it has not really been necessary to bound the reaction time of a component, divergence of time has been demanded almost by default (e.g., see [5], [4], [18], [6], [35]). And yet in modelling and simulation, where time is represented as an ordinary program variable, Zeno behaviours are not only realizable, but occasionally desirable as well. Simulating the dynamics of a bouncing ball, for example, will naturally give rise to a Zeno behaviour, and the mathematical model used to study or even define the semantics of the simulation environment should allow for that behaviour. This is impossible with the kind of metric spaces found in [52], [53], [60], [43], [28], [27], and [30] (see [33, sec. 4.1]). Even worse, it is possible to come up with simulation models that do not exhibit any kind of Zeno behaviour, and are used to specify embedded and distributed real-time systems (see [65] and [14]), but consist of components that cannot be handled within the kind of framework used in the above references.

It is worthwhile noting that the requirement of time divergence is absent from the real-time process calculi that emerged around the same time (e.g., see [42], [62], [45], [17]). The reason behind this is that such a requirement would call for a treatment similar to that of fairness or the finite delay property in the corresponding untimed calculi, which has always been problematic with traditional interleaving theories based on labelled transition systems (see [38], [37]).

Another limiting factor in the applicability of the existing approaches based on metric spaces is the choice of tag set. The latter is typically some unbounded subset of the real numbers, excluding other interesting choices, such as, for example, that of superdense time (see [33, sec. 4.3]).

Naundorf was the first to address these issues, abolishing the bounded reaction time constraint, and allowing for arbitrary tag sets (see [44]). He defined strictly causal functions as the functions that we here call strictly contracting, and used an ad hoc, non-constructive argument to prove the existence of a unique fixed point for every such function. Unlike that in [28], [27], and [30], Naundorf's definition of strict causality was at least sound under the hypothesis of a totally ordered tag set (see Theorem 4.8), but nevertheless incomplete (e.g., see Example 3.9). It was rephrased in [33] using the generalized distance function to explicitly identify strictly causal functions with the strictly contracting ones. This provided access to the fixed-point theory of generalized ultrametric spaces, which, however, proved less useful than one might have hoped. The main fixed-point theorem of Priess-Crampe and Ribenboim for strictly contracting endofunctions offered little more than another non-constructive proof of Naundorf's theorem, improving only marginally on the latter by allowing the domain of the function to be any arbitrary spherically complete set of signals, and the few constructive fixed-point theorems that we know to be of any relevance (e.g., see *Proof of Theorem 9 for ordinal distances* in [22], [26, thm. 43]) were of limited applicability.

An interesting generalization of Naundorf's theorem was proved in [19]. The overall approach is vaguely reminiscent of our effort to understand the relationship between the generalized distance function and the prefix relation on signals, and abstract from the internal structure of the latter. But the intent is to eliminate any reference to generalized distances, and the proof is again non-constructive.

A constructive fixed-point theorem for a restricted class of strictly causal functions on signals over a superdense time domain was proved in [10]. Although a bit more generally applicable, the theorem was explicitly applied to so-called "eventually delta-causal" functions, which model components subject to a simple generalization of the bounded reaction-time constraint to the case of superdense time.

There have also been a few attempts to use complete partial orders and least fixed points in the study of timed systems. In [61], Yates and Gao reduced the fixed-point problem related to a system of so-called "Δ-causal" components to that of a suitably constructed Scott-continuous function, transferring the Kahn principle to networks of real-time processes, but once more, under the usual bounded reaction-time constraint. A more direct application of the principle in the context of timed systems was put forward in [31] and [32]. A special value was used to make absence of event explicit, and signals were constrained to be defined on lower sets of the tag set, making progress of time part of the semantics of a system. Strictly causal functions were defined to be the monotone functions that extend the domain of definition of signals, and strictly causal functions that were also Scott-continuous were proved to have unique fixed points in which time diverges. This meant relaxing the bounded reaction-time constraint to allow for certain components whose reaction time is locally rather than globally bounded. But the proposed definition of strict causality was still incomplete, unable to accommodate components with more arbitrarily varying reaction-times, such as the one modelled by the function of Example 3.7 restricted to the set of all discrete-event signals, and ultimately, systems with non-trivial Zeno behaviours. Finally, a more naive approach was proposed in [9], where components were modelled as Scott-continuous functions with respect to the prefix relation on signals, creating, of course, all kinds of causality problems, which, however, seem to have gone largely unnoticed.

Lastly, we mention Broy's work in [8], where he proves, without the use of "more sophisticated theoretical concepts such as least fixpoints, complete partially ordered sets or metric spaces", that every so-called "time-guarded" function on timed streams has a unique fixed point, but again, under the usual bounded reaction-time constraint.

8 Conclusion

An interesting way to communicate the contribution of this work is through a comparison, or perhaps contrast, with Kahn's seminal work on networks of asynchronous processes (see [24]), much as was done in [60]. Just as in [24], our objective is ultimately a mathematical semantics facilitating the definition, construction, and analysis of complex systems. But the systems we have in mind are radically different from those considered in [24], setting up a very different problem indeed.

Kahn was interested in sequential processes that compute in parallel and communicate asynchronously through finite-delay, first in, first out, unbounded queues. It was his brilliant insight to model each process as a Scott-continuous function from the complete partial order of histories over its input queues to that of histories over its output queues, and the behavior of a network of such processes as the least solution to a system of mutually recursive equations, one for each queue. To better motivate his presentation, he sketched a toy, ALGOL-like programming language for such networks, which was meant as a concrete illustration of the proposed computational paradigm.

Our interest is in components that are also autonomous, at least conceptually, but communicate through timed signals rather than untimed streams. And unlike Kahn's processes, these components are very much aware of time. As a consequence, they can behave in ways that cannot be modelled using any sort of order-preserving function. Thus, we can never hope to find a mathematical semantics for systems made up of such components within standard domain theory, as in [24]. Rather, we have to build a new theory, starting with a fixed-point theory for the kind of functions used to model them. Kahn took an established, well understood mathematical model, and matched it with a computational paradigm. We already have the paradigm; what we need is the model.

Looking back, our fixed-point theory is surprisingly similar to that of order-preserving functions. In fact, every feature of our theory is characterized in a purely order-theoretic fashion. The reason for this is that, even though strictly contracting functions need not be order-preserving, systems made up of components that realize such functions still build up their behaviour in a monotone way, never invalidating what they have already output, which is possibly the only similarity between these systems and those considered by Kahn in [24].

The reader will likely protest here that, unlike [24], we have only dealt with feedback configurations, and specifically, only those involving a single component with a single input and a single output. This is not entirely true though. Assuming a non-empty set C of channels mediating the communication between the individual components within a system, and for each channel c, a non-empty set V_c of values that may be communicated over c, we can model a component with more than one input and output again as a partial function on signals, but this time, on signals whose values range over the non-empty, single-valued subsets of $\bigcup \{\{c\} \times \mathrm{V}_c \mid c \in \mathrm{C}\}$. And since every system can be thought of as a single component receiving and transmitting over every channel, every system is effectively amenable to our theory.

Recursion, in the sense of [24], poses a greater challenge. In principle, it is possible to extend the results of Section 5 to the case of strictly contracting operators on strictly contracting functions, suitably defined, to handle recursive schemata of the kind considered in [24], suitably guarded. But as we stand, if we were to do so, we would have to relapse, if only in part, into a composite view of signals, having to peek under the hood at their individual events. And the problem would become intractable if we were to more generally consider higher-order systems, where we would practically have to adjust each result and its proof to each higher-order type.

What we need is an abstract characterization of a class of structures that will support the development of the theory, and remain closed under the construction of products and function spaces of interest, enabling the treatment of arbitrary, even higher-order composition in a more standard, uniform way. Clauses 1 and 2 of Proposition 2.15 have been singled out precisely with this purpose in mind. Indeed, we have done well to derive every result of Section 5 from these two properties alone, without any reference whatever to the internal structure of signals. Because of this, our fixed-point theory is actually perfectly applicable to any directed-complete semilattice satisfying clauses 1 and 2 of Proposition 2.15 when $\sqcap$ is interpreted as the meet operation of that semilattice and d as a generalized ultrametric thereon. But whether it is necessary to restrict such structures further or not is still an open question.

We conclude with a few words on the issue of determinacy. From the outset, we have insisted that a timed system be determinate, down to every constituent component. When it comes to communication and concurrency, indeterminacy is a very powerful semantic abstraction, but one that is mainly used when the only relevant aspect of time is order. It is of course absurd to talk about indeterminacy in this sense here. It is still possible, however, for indeterminacy to enter the scene, this time with the intent of modelling the uncertainty in the precision of timing, an issue of major concern in any distributed system. Investigating the adaptation and application of our methods in that context is another interesting direction for future work.

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A Appendix

We prove that there is no non-trivial order relation that will render every strictly contracting endofunction order-preserving, and no metric function that will render every such endofunction a contraction mapping. The implication is that it is impossible to directly apply the fixed-point theory of order-preserving functions on ordered sets or that of contraction mappings on metric spaces to the fixed-point problem in hand.

A.1 Strictly contracting functions versus order-preserving functions

As first pointed out in [61] and [60], there are strictly causal functions that do not preserve the prefix relation on signals. The following illustrates this:

Example A.1. Suppose that $\mathrm{T} = [0, \infty)$, and $\preceq$ is the standard order on $[0, \infty)$.

Let v be a value in V.

Let F be a function on S such that for every $s \in \mathrm{S}$,

$$F(s) = \begin{cases} \{\langle 1, v\rangle\} & \text{if for every } \tau \in [0,1),\ \tau \notin \mathsf{dom}\, s; \\ \emptyset & \text{otherwise.} \end{cases}$$

Clearly, F is a strictly causal function, and it is easy to verify that F is in fact a strictly contracting function. However, $F(\emptyset) \not\sqsubseteq F(\{\langle 0, v\rangle\})$, whereas $\emptyset \sqsubseteq \{\langle 0, v\rangle\}$, and thus, F is not order-preserving in $\langle \mathrm{S}, \sqsubseteq \rangle$.

The function of Example A.1 models a component that operates like an alarm clock that is set to go off at time 1 unless it is reset before that time, and clearly, fails to preserve the prefix relation on signals. As a consequence, we cannot hope to use the fixed-point theory for order-preserving functions to study the behaviour of such a component in feedback, at least not if we intend to use the prefix relation as our order relation. But what if we are inclined to look for a different one?

In general terms, we may ask the following question: Is there a non-trivial order relation on signals that will render all strictly causal, or more pertinently, strictly contracting functions order-preserving? The answer is no.

Assume $X \subseteq \mathrm{S}$.

Theorem A.2. *If $\langle \mathrm{T}, \preceq \rangle$ is totally ordered, and for any $s_1, s_2 \in X$ such that $s_1 \neq s_2$, there are $s'_1, s'_2 \in X$ such that $s'_1 \neq s'_2$ and*

$$\mathrm{d}(s'_1, s'_2) \supset \mathrm{d}(s_1, s_2),$$

then for every order relation $\leqslant \subseteq X \times X$, every strictly contracting function on X is order-preserving in $\langle X, \leqslant \rangle$ if and only if $\leqslant$ is the discrete order on X.

Proof. Suppose that $\langle \mathrm{T}, \preceq \rangle$ is totally ordered.

Suppose that for any $s_1, s_2 \in X$ such that $s_1 \neq s_2$, there are $s'_1, s'_2 \in X$ such that $s'_1 \neq s'_2$ and

$$\mathrm{d}(s'_1, s'_2) \supset \mathrm{d}(s_1, s_2).$$

Assume an order relation $\leqslant \subseteq X \times X$.

Suppose that every strictly contracting function on X is order-preserving in $\langle X, \leqslant \rangle$.

Suppose, toward contradiction, that there are $s_1, s_2 \in X$ such that $s_1 \leqslant s_2$ and $s_1 \neq s_2$. Then there are $s'_1, s'_2 \in X$ such that $s'_1 \neq s'_2$ and

$$\mathrm{d}(s'_1, s'_2) \supset \mathrm{d}(s_1, s_2). \tag{29}$$

Let F_1 be a function on X such that for every $s \in X$,

$$F_1(s) = \begin{cases} s'_1 & \text{if } \mathrm{d}(s_1, s) \supset \mathrm{d}(s_1, s_2); \\ s'_2 & \text{otherwise.} \end{cases}$$

Let F_2 be a function on X such that for every $s \in X$,

$$F_2(s) = \begin{cases} s'_2 & \text{if } \mathrm{d}(s_1, s) \supset \mathrm{d}(s_1, s_2); \\ s'_1 & \text{otherwise.} \end{cases}$$

Assume $s''_1, s''_2 \in X$ such that $s''_1 \neq s''_2$.

Since $\langle \mathrm{T}, \preceq \rangle$ is totally ordered, either

$$\mathrm{d}(s_1, s_2) \supseteq \mathrm{d}(s''_1, s''_2),$$

or

$$\mathrm{d}(s''_1, s''_2) \supset \mathrm{d}(s_1, s_2).$$

If

$$\mathrm{d}(s_1, s_2) \supseteq \mathrm{d}(s''_1, s''_2),$$

then, by (29),

$$\begin{aligned} \mathrm{d}(F_1(s''_1), F_1(s''_2)) &\supseteq \mathrm{d}(s'_1, s'_2) \\ &\supset \mathrm{d}(s''_1, s''_2). \end{aligned}$$

Otherwise,

$$\mathrm{d}(s''_1, s''_2) \supset \mathrm{d}(s_1, s_2). \tag{30}$$

Suppose, toward contradiction, that $F_1(s''_1) \neq F_1(s''_2)$. Without loss of generality, assume that $F_1(s''_1) = s'_1$. Then

$$\mathrm{d}(s_1, s''_1) \supset \mathrm{d}(s_1, s_2). \tag{31}$$

Since $\langle \mathrm{T}, \preceq \rangle$ is totally ordered, by (30) and (31),

$$\mathrm{d}(s''_1, s''_2) \cap \mathrm{d}(s_1, s''_1) \supset \mathrm{d}(s_1, s_2),$$

and thus, by the generalized ultrametric inequality,

$$\mathrm{d}(s_1, s''_2) \supset \mathrm{d}(s_1, s_2).$$

Thus, $F_1(s''_2) = s'_1$, obtaining a contradiction.

Therefore, $F_1(s''_1) = F_1(s''_2)$, and since $s''_1 \neq s''_2$,

$$\mathrm{d}(F_1(s''_1), F_1(s''_2)) \supset \mathrm{d}(s''_1, s''_2).$$

Thus, by generalization, F_1 is strictly contracting. And by symmetry, F_2 is strictly contracting. Then, by hypothesis, F_1 and F_2 are order-preserving in $\langle X, \leqslant\rangle$. And since $s_1 \leqslant s_2$, $F_1(s_1) \leqslant F_1(s_2)$ and $F_2(s_1) \leqslant F_2(s_2)$, and thus, $s'_1 \leqslant s'_2$ and $s'_2 \leqslant s'_1$. Thus, $s'_1 = s'_2$, obtaining a contradiction.

Therefore, for every $s_1, s_2 \in X$, $s_1 \leqslant s_2$ if and only if $s_1 = s_2$. Thus, $\leqslant$ is the discrete order on X.

Conversely, if $\leqslant$ is the discrete order on X, then, trivially, every strictly contracting function on X is order-preserving in $\langle X, \leqslant\rangle$. □

Note that a more natural hypothesis for Theorem A.2 would be to require that $\{\mathrm{d}(s_1, s_2) \mid s_1, s_2 \in X\}$ is cofinal in $\langle \mathscr{L}\langle \mathrm{T}, \preceq\rangle, \subseteq\rangle$, but the weaker assumption of there not being a generalized distance that is $\supset$-minimal in $\{\mathrm{d}(s_1, s_2) \mid s_1, s_2 \in X \text{ and } s_1 \neq s_2\}$ is sufficient to prove the theorem.

By Theorem A.2, it is impossible, under the pertaining assumptions, to arrange signals in any non-trivial, let alone sensible, ordering that is preserved by every strictly contracting function. Whether for every particular strictly contracting function there is such an ordering preserved by that function remains an open question. But a unified framework facilitating the representation of strictly contracting functions as order-preserving functions is out of the question.

Parenthetically, we remark that functions that do preserve the prefix relation on signals need not, in general, be strictly causal either (e.g., see Example 3.3).

A.2 Strictly contracting functions versus contraction mappings

In the same spirit as before, we may ask the following question: Is there a metric function on signals that will render all strictly contracting functions contraction mappings? The existence of such a metric function would be of genuine practical interest, for one could then directly apply Banach's fixed-point theorem to solve the fixed-point problem considered in this work. But the answer is still no.

Assume an infinite sequence $\langle s_n \mid n \in \omega\rangle$ over S.

Lemma A.3. *If $\langle \mathrm{T}, \preceq\rangle$ is totally ordered, and for every $n \in \omega$,*

$$\mathrm{d}(s_{n+1}, s_{n+2}) \supset \mathrm{d}(s_n, s_{n+1}),$$

then the following are true:

1. *for every $n \in \omega$, and every $i_1, i_2 \in \omega \setminus \{0\}$,*

$$\mathrm{d}(s_n, s_{n+i_1}) = \mathrm{d}(s_n, s_{n+i_2});$$

2. *for every $n_1, n_2 \in \omega$ such that $n_1 \neq n_2$, and every $i_1, i_2 \in \omega \setminus \{0\}$,*

$$\mathrm{d}(s_{n_1+i_1}, s_{n_2+i_2}) \supset \mathrm{d}(s_{n_1}, s_{n_2}).$$

Proof. Suppose that $\langle \mathrm{T}, \preceq\rangle$ is totally ordered.

Suppose that for every $n \in \omega$,

$$\mathrm{d}(s_{n+1}, s_{n+2}) \supset \mathrm{d}(s_n, s_{n+1}),$$

Assume $n \in \omega$.

We use induction to prove that for every $i \in \omega \setminus \{0\}$,

$$\mathrm{d}(s_n, s_{n+i}) = \mathrm{d}(s_n, s_{n+1}).$$

If $i = 1$, then, trivially,

$$\mathrm{d}(s_n, s_{n+i}) = \mathrm{d}(s_n, s_{n+1}).$$

Otherwise, there is $j \in \omega \setminus \{0\}$ such that $i = j + 1$. By the induction hypothesis,

$$\mathrm{d}(s_n, s_{n+j}) = \mathrm{d}(s_n, s_{n+1}). \tag{32}$$

By hypothesis,

$$\begin{aligned} \mathrm{d}(s_{n+j}, s_{n+i}) &= \mathrm{d}(s_{n+j}, s_{n+j+1}) \\ &\supset \mathrm{d}(s_n, s_{n+1}). \end{aligned} \tag{33}$$

Suppose, toward contradiction, that

$$\mathrm{d}(s_n, s_{n+1}) \supset \mathrm{d}(s_n, s_{n+i}). \tag{34}$$

Then, since $\langle \mathrm{T}, \preceq \rangle$ is totally ordered, by (32), (33), (34), and Proposition 2.7,

$$\mathrm{d}(s_n, s_{n+i}) \supset \mathrm{d}(s_n, s_{n+i}),$$

obtaining a contradiction.

Therefore, since $\langle \mathrm{T}, \preceq \rangle$ is totally ordered,

$$\mathrm{d}(s_n, s_{n+i}) \supseteq \mathrm{d}(s_n, s_{n+1}).$$

Suppose, toward contradiction, that

$$\mathrm{d}(s_n, s_{n+i}) \supset \mathrm{d}(s_n, s_{n+1}). \tag{35}$$

Then, since $\langle \mathrm{T}, \preceq \rangle$ is totally ordered, by (33), (35), and Proposition 2.7,

$$\mathrm{d}(s_n, s_{n+j}) \supset \mathrm{d}(s_n, s_{n+1}),$$

in contradiction to (32).

Therefore,

$$\mathrm{d}(s_n, s_{n+i}) = \mathrm{d}(s_n, s_{n+1}).$$

Therefore, by induction, for every $i \in \omega \setminus \{0\}$,

$$\mathrm{d}(s_n, s_{n+i}) = \mathrm{d}(s_n, s_{n+1}).$$

Then, for every $i_1, i_2 \in \omega \setminus \{0\}$,

$$\begin{aligned} \mathrm{d}(s_n, s_{n+i_1}) &= \mathrm{d}(s_n, s_{n+1}) \\ &= \mathrm{d}(s_n, s_{n+i_2}), \end{aligned}$$

and thus, 1 is true.

Assume $n_1, n_2 \in \omega$ such that $n_1 \neq n_2$, and $i_1, i_2 \in \omega \setminus \{0\}$.

Then, by 1,

$$\mathrm{d}(s_{n_1}, s_{n_2}) = \mathrm{d}(s_{\min\{n_1,n_2\}}, s_{\min\{n_1,n_2\}+1}). \tag{36}$$

Since $n_1 < n_2$ and $i_1, i_2 \in \omega \setminus \{0\}$,

$$\min\{n_1, n_2\} + 1 \leq \min\{n_1 + i_1, n_2 + i_2\}. \tag{37}$$

If $n_1 + i_1 = n_2 + i_2$, then, trivially,

$$\mathrm{d}(s_{n_1+i_1}, s_{n_2+i_2}) \supset \mathrm{d}(s_{n_1}, s_{n_2}).$$

Otherwise, $n_1 + i_1 \neq n_2 + i_2$. Then, by 1,

$$\mathrm{d}(s_{n_1+i_1}, s_{n_2+i_2}) = \mathrm{d}(s_{\min\{n_1+i_1,n_2+i_2\}}, s_{\min\{n_1+i_1,n_2+i_2\}+1}). \tag{38}$$

Thus, by (36), (37), (38), and hypothesis,

$$\mathrm{d}(s_{n_1+i_1}, s_{n_2+i_2}) \supset \mathrm{d}(s_{n_1}, s_{n_2}).$$

Thus, by generalization, 2 is true. □

Theorem A.4. *If $\langle \mathrm{T}, \preceq \rangle$ is totally ordered, and there is an infinite sequence $\langle s_n \mid n \in \omega \rangle$ over X such that for every $n \in \omega$,*

$$\mathrm{d}(s_{n+1}, s_{n+2}) \supset \mathrm{d}(s_n, s_{n+1}),$$

and $s_a, s_b \in X$ such that $s_a \neq s_b$ and for every $n \in \omega$,

$$\mathrm{d}(s_a, s_b) \supset \mathrm{d}(s_n, s_{n+1}),$$

then for every metric function[34] *d on X, there is a strictly contracting function on X that is not a contraction mapping*[35] *on $\langle X, d \rangle$.*

Proof. Suppose that $\langle \mathrm{T}, \preceq \rangle$ is totally ordered.

Suppose that there is an infinite sequence $\langle s_n \mid n \in \omega \rangle$ over X such that for every $n \in \omega$,

$$\mathrm{d}(s_{n+1}, s_{n+2}) \supset \mathrm{d}(s_n, s_{n+1}),$$

and $s_a, s_b \in X$ such that $s_a \neq s_b$ and for every $n \in \omega$,

$$\mathrm{d}(s_a, s_b) \supset \mathrm{d}(s_n, s_{n+1}).$$

[34] For every set A, a *metric function* on A is a function d from $A \times A$ to $\mathbb{R}$ such that for any $a_1, a_2, a_3 \in A$, the following are true:

1. $d(a_1, a_2) = 0$ if and only if $a_1 = a_2$;
2. $d(a_1, a_2) = d(a_2, a_1)$;
3. $d(a_1, a_2) + d(a_2, a_3) \geq d(a_1, a_3)$.

[35] For every metric space $\langle A, d \rangle$, and every function f on A, f is a *contraction mapping* on $\langle A, d \rangle$ if and only if there is $c \in [0, 1)$ such that for any $a_1, a_2 \in A$, $d(f(a_1), f(a_2)) \leq c \cdot d(a_1, a_2)$.

Assume a metric function d on X.

Suppose, toward contradiction, that every strictly contracting function on X is a contraction mapping on $\langle X, d\rangle$.

Assume $n \in \omega$.

Let F_n be a function on X such that for every $s \in X$,

$$F_n(s) = \begin{cases} s_a & \text{if } \mathrm{d}(s_n, s) \supset \mathrm{d}(s_n, s_{n+1}); \\ s_b & \text{otherwise.} \end{cases}$$

Then F_n is strictly contracting (see F_1 in proof of Theorem A.2). Thus, by hypothesis, F_n is a contraction mapping on $\langle X, d\rangle$, and hence, there is $c_n \in [0, 1)$ such that for every $s'_1, s'_2 \in X$,

$$d(F_n(s'_1), F_n(s'_2)) \leq c_n \cdot d(s'_1, s'_2).$$

Thus,

$$\begin{aligned} d(s_a, s_b) &= d(F_n(s_n), F_n(s_{n+1})) \\ &\leq c_n \cdot d(s_n, s_{n+1}). \end{aligned}$$

Thus, for every $n \in \omega$,

$$d(s_a, s_b) < d(s_n, s_{n+1}). \tag{39}$$

Suppose that there is $s_\omega \in X$ such that

$$\{n \mid n \in \omega \text{ and } \mathrm{d}(s_{n+1}, s_\omega) \not\supset \mathrm{d}(s_n, s_\omega)\} = \emptyset.$$

Let F be a function on X such that for every $s \in X$,

$$F(s) = \begin{cases} s_{\min\{n \mid n \in \omega \text{ and } \mathrm{d}(s_{n+1}, s) \not\supset \mathrm{d}(s_n, s)\}+1} & \text{if } \{n \mid n \in \omega \text{ and } \mathrm{d}(s_{n+1}, s) \not\supset \mathrm{d}(s_n, s)\} \neq \emptyset; \\ s_\omega & \text{otherwise.} \end{cases}$$

Assume $s'_1, s'_2 \in X$ such that $s'_1 \neq s'_2$.

Suppose that

$$\{n \mid n \in \omega \text{ and } \mathrm{d}(s_{n+1}, s'_1) \not\supset \mathrm{d}(s_n, s'_1)\} \neq \emptyset$$

and

$$\{n \mid n \in \omega \text{ and } \mathrm{d}(s_{n+1}, s'_2) \not\supset \mathrm{d}(s_n, s'_2)\} \neq \emptyset.$$

Let $m_1 = \min\{n \mid n \in \omega \text{ and } \mathrm{d}(s_{n+1}, s'_1) \not\supset \mathrm{d}(s_n, s'_1)\}$.

Let $m_2 = \min\{n \mid n \in \omega \text{ and } \mathrm{d}(s_{n+1}, s'_2) \not\supset \mathrm{d}(s_n, s'_2)\}$.

Then $F(s'_1) = s_{m_1+1}$ and $F(s'_2) = s_{m_2+1}$.

If $m_1 = m_2$, then, trivially,

$$\mathrm{d}(F(s'_1), F(s'_2)) \supset \mathrm{d}(s'_1, s'_2).$$

Otherwise, $m_1 \neq m_2$. Without loss of generality, assume that $m_1 < m_2$. By definition of m_1,

$$\mathrm{d}(s_{m_1+1}, s'_1) \not\supset \mathrm{d}(s_{m_1}, s'_1),$$

and since $\langle \mathrm{T}, \preceq \rangle$ is totally ordered,

$$\mathrm{d}(s_{m_1}, s'_1) \supseteq \mathrm{d}(s_{m_1+1}, s'_1). \tag{40}$$

Then, by the generalized ultrametric inequality,

$$\mathrm{d}(s_{m_1}, s_{m_1+1}) \supseteq \mathrm{d}(s_{m_1+1}, s'_1),$$

And since $m_1 < m_2$, by Lemma A.31,

$$\mathrm{d}(s_{m_1}, s_{m_2}) \supseteq \mathrm{d}(s_{m_1+1}, s'_1). \tag{41}$$

Suppose, toward contradiction, that

$$\mathrm{d}(s'_1, s'_2) \supset \mathrm{d}(s_{m_1}, s_{m_2}). \tag{42}$$

Then, by (41) and (42),

$$\mathrm{d}(s'_1, s'_2) \supset \mathrm{d}(s_{m_1+1}, s'_1). \tag{43}$$

Suppose, toward contradiction, that

$$\mathrm{d}(s_{m_1+1}, s'_2) \supset \mathrm{d}(s_{m_1+1}, s'_1). \tag{44}$$

Then, since $\langle \mathrm{T}, \preceq \rangle$ is totally ordered, by (43), (44), and Proposition 2.7,

$$\mathrm{d}(s_{m_1+1}, s'_1) \supset \mathrm{d}(s_{m_1+1}, s'_1),$$

obtaining a contradiction.

Therefore,

$$\mathrm{d}(s_{m_1+1}, s'_2) \not\supset \mathrm{d}(s_{m_1+1}, s'_1),$$

and since $\langle \mathrm{T}, \preceq \rangle$ is totally ordered,

$$\mathrm{d}(s_{m_1+1}, s'_1) \supseteq \mathrm{d}(s_{m_1+1}, s'_2). \tag{45}$$

Then, by (40), (43), (45), and the generalized ultrametric inequality,

$$\mathrm{d}(s_{m_1}, s'_2) \supseteq \mathrm{d}(s_{m_1+1}, s'_2).$$

Thus, $m_1 \in \{n \mid n \in \omega \text{ and } \mathrm{d}(s_{n+1}, s'_2) \not\supset \mathrm{d}(s_n, s'_2)\}$, and since $m_1 < m_2$,

$$\min \{n \mid n \in \omega \text{ and } \mathrm{d}(s_{n+1}, s'_2) \not\supset \mathrm{d}(s_n, s'_2)\} < m_2,$$

obtaining a contradiction.

Therefore,

$$\mathrm{d}(s'_1, s'_2) \not\supset \mathrm{d}(s_{m_1}, s_{m_2}),$$

and since $\langle \mathrm{T}, \preceq \rangle$ is totally ordered,

$$\mathrm{d}(s_{m_1}, s_{m_2}) \supseteq \mathrm{d}(s'_1, s'_2).$$

Then, by Lemma A.32,

$$d(s_{m_1+1}, s_{m_2+1}) \supset d(s'_1, s'_2),$$

and hence,

$$d(F(s'_1), F(s'_2)) \supset d(s'_1, s'_2).$$

Suppose that either

$$\{n \mid n \in \omega \text{ and } d(s_{n+1}, s'_1) \not\supset d(s_n, s'_1)\} \neq \emptyset$$

and

$$\{n \mid n \in \omega \text{ and } d(s_{n+1}, s'_2) \not\supset d(s_n, s'_2)\} = \emptyset,$$

or

$$\{n \mid n \in \omega \text{ and } d(s_{n+1}, s'_1) \not\supset d(s_n, s'_1)\} = \emptyset$$

and

$$\{n \mid n \in \omega \text{ and } d(s_{n+1}, s'_2) \not\supset d(s_n, s'_2)\} \neq \emptyset.$$

Without loss of generality, assume that

$$\{n \mid n \in \omega \text{ and } d(s_{n+1}, s'_1) \not\supset d(s_n, s'_1)\} \neq \emptyset$$

and

$$\{n \mid n \in \omega \text{ and } d(s_{n+1}, s'_2) \not\supset d(s_n, s'_2)\} = \emptyset,$$

Let $m_1 = \min \{n \mid n \in \omega \text{ and } d(s_{n+1}, s'_1) \not\supset d(s_n, s'_1)\}$.
Then $F(s'_1) = s_{m_1+1}$ and $F(s'_2) = s_\omega$. By definition of m_1,

$$d(s_{m_1+1}, s'_1) \not\supset d(s_{m_1}, s'_1),$$

and since $\langle \mathrm{T}, \preceq \rangle$ is totally ordered,

$$d(s_{m_1}, s'_1) \supseteq d(s_{m_1+1}, s'_1). \tag{46}$$

Then, by the generalized ultrametric inequality,

$$d(s_{m_1}, s_{m_1+1}) \supseteq d(s_{m_1+1}, s'_1). \tag{47}$$

By definition of s_ω,

$$d(s_{m_1+1}, s_\omega) \supset d(s_{m_1}, s_\omega), \tag{48}$$

and by the generalized ultrametric inequality,

$$d(s_{m_1}, s_{m_1+1}) \supseteq d(s_{m_1}, s_\omega).$$

Suppose, toward contradiction, that

$$d(s_{m_1}, s_{m_1+1}) \supset d(s_{m_1}, s_\omega). \tag{49}$$

Then, since $\langle \mathrm{T}, \preceq \rangle$ is totally ordered, by (48), (49), and Proposition 2.7,

$$\mathrm{d}(s_{m_1}, s_\omega) \supset \mathrm{d}(s_{m_1}, s_\omega),$$

obtaining a contradiction.

Therefore,

$$\mathrm{d}(s_{m_1}, s_{m_1+1}) = \mathrm{d}(s_{m_1}, s_\omega),$$

and hence, by (47),

$$\mathrm{d}(s_{m_1}, s_\omega) \supseteq \mathrm{d}(s_{m_1+1}, s'_1). \tag{50}$$

Suppose, toward contradiction, that

$$\mathrm{d}(s'_1, s'_2) \supset \mathrm{d}(s_{m_1}, s_\omega). \tag{51}$$

Then, by (50) and (51),

$$\mathrm{d}(s'_1, s'_2) \supset \mathrm{d}(s_{m_1+1}, s'_1). \tag{52}$$

Suppose, toward contradiction, that

$$\mathrm{d}(s_{m_1+1}, s'_2) \supset \mathrm{d}(s_{m_1+1}, s'_1). \tag{53}$$

Then, since $\langle \mathrm{T}, \preceq \rangle$ is totally ordered, by (52), (53), and Proposition 2.7,

$$\mathrm{d}(s_{m_1+1}, s'_1) \supset \mathrm{d}(s_{m_1+1}, s'_1),$$

obtaining a contradiction.

Therefore,

$$\mathrm{d}(s_{m_1+1}, s'_2) \not\supset \mathrm{d}(s_{m_1+1}, s'_1),$$

and since $\langle \mathrm{T}, \preceq \rangle$ is totally ordered,

$$\mathrm{d}(s_{m_1+1}, s'_1) \supseteq \mathrm{d}(s_{m_1+1}, s'_2). \tag{54}$$

Then, by (46), (52), (54), and the generalized ultrametric inequality,

$$\mathrm{d}(s_{m_1}, s'_2) \supseteq \mathrm{d}(s_{m_1+1}, s'_2).$$

Thus, $m_1 \in \{n \mid n \in \omega \text{ and } \mathrm{d}(s_{n+1}, s'_2) \not\supset \mathrm{d}(s_n, s'_2)\}$, obtaining a contraiction.

Therefore,

$$\mathrm{d}(s'_1, s'_2) \not\supset \mathrm{d}(s_{m_1}, s_\omega),$$

and since $\langle \mathrm{T}, \preceq \rangle$ is totally ordered,

$$\mathrm{d}(s_{m_1}, s_\omega) \supseteq \mathrm{d}(s'_1, s'_2).$$

Then, by (48),

$$\mathrm{d}(s_{m_1+1}, s_\omega) \supset \mathrm{d}(s'_1, s'_2),$$

and hence,

$$\mathrm{d}(F(s'_1), F(s'_2)) \supset \mathrm{d}(s'_1, s'_2).$$

Otherwise,

$$\{n \mid n \in \omega \text{ and } \mathrm{d}(s_{n+1}, s'_1) \not\supset \mathrm{d}(s_n, s'_1)\} = \emptyset$$

and

$$\{n \mid n \in \omega \text{ and } \mathrm{d}(s_{n+1}, s'_2) \not\supset \mathrm{d}(s_n, s'_2)\} = \emptyset,$$

Then, $F(s'_1) = s_\omega$ and $F(s'_2) = s_\omega$, and thus, trivially,

$$\mathrm{d}(F(s'_1), F(s'_2)) \supset \mathrm{d}(s'_1, s'_2).$$

Thus, by generalization, F is strictly contracting. Thus, by hypothesis, F is a contraction mapping on $\langle X, d\rangle$, and hence, there is $c \in [0, 1)$ such that for every $s'_1, s'_2 \in X$,

$$d(F(s'_1), F(s'_2)) \leq c \cdot d(s'_1, s'_2).$$

Assume $n \in \omega$.

Trivially, $n \in \{n' \mid n' \in \omega \text{ and } \mathrm{d}(s_{n'+1}, s_n) \not\supset \mathrm{d}(s_{n'}, s_n)\}$, and thus,

$$F(s_n) = s_{\min\{n' \mid n' \in \omega \text{ and } \mathrm{d}(s_{n'+1}, s_n) \not\supset \mathrm{d}(s_{n'}, s_n)\}+1}.$$

Suppose, toward contradiction, that $F(s_n) \neq s_{n+1}$. Then there is $n' \in \omega$ such that $n' < n$ and

$$\mathrm{d}(s_{n'+1}, s_n) \not\supset \mathrm{d}(s_{n'}, s_n),$$

and since $\langle \mathrm{T}, \preceq\rangle$ is totally ordered,

$$\mathrm{d}(s_{n'}, s_n) \supseteq \mathrm{d}(s_{n'+1}, s_n).$$

Then, $n' + 1 < n$, and by Lemma A.31,

$$\mathrm{d}(s_{n'+1}, s_n) = \mathrm{d}(s_{n'+1}, s_{n+1}).$$

Hence,

$$\mathrm{d}(s_{n'}, s_n) \supseteq \mathrm{d}(s_{n'+1}, s_{n+1}),$$

in contradiction to Lemma A.32.

Therefore, $F(s_n) = s_{n+1}$.

Thus, by an easy induction, for every $n \in \omega$,

$$d(F(s_n), F(s_{n+1})) \leq c^{n+1} \cdot d(s_0, s_1).$$

and hence,

$$d(s_n, s_{n+1}) \leq c^n \cdot d(s_0, s_1). \tag{55}$$

And since $c \in [0, 1)$, by (39) and (55),

$$d(s_a, s_b) = 0,$$

obtaining a contradiction.

Therefore, there is a strictly contracting function on X, namely F, that is not a contraction mapping on $\langle X, d\rangle$.

Otherwise, for every $s \in X$,

$$\{n \mid n \in \omega \text{ and } \mathrm{d}(s_{n+1}, s_\omega) \not\supset \mathrm{d}(s_n, s_\omega)\} \neq \emptyset.$$

Let F be a function on X such that for every $s \in X$,

$$F(s) = s_{\min\{n \mid n \in \omega \text{ and } \mathrm{d}(s_{n+1}, s) \not\supset \mathrm{d}(s_n, s)\}+1}.$$

Then, by the same argument, F is strictly contracting, but not a contraction mapping on $\langle X, d\rangle$.

Therefore, there is a strictly contracting function on X, namely F, that is not a contraction mapping on $\langle X, d\rangle$. □

Notice that, as before, it is still possible that for every particular strictly contracting function, there is some metric rendering that function a contraction mapping. But a unified framework facilitating the representation of strictly contracting functions as contraction mappings is impossible.