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AN INTUITIVE DERIVATION OF A REALIZATION

by

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AN INTUITIVE DERIVATION OF A REALIZATION

PROCEDURE BASED ON SINGULAR VALUE DECOMPOSITION

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ABSTRACT

This memorandum presents an intuitive derivation of the minimal realization of $G(s) \in \mathbb{R}(s)^{n_o \times n_i}$ based on singular value decomposition. The original work is due to P. Van Dooren, et al.

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I. Introduction

The problem of constructing a minimal realization of a given matrix of rational functions has been studied in literature, but the numerical aspects of the suggested procedures have rarely been considered. Van Dooren has proposed a numerically stable algorithm for constructing a minimal realization [1]. This paper, based on Van Dooren's result, gives an intuitive insight in such a realization, and gives a simple proof of minimality.

It is well known that if a strictly proper rational matrix is decomposed into the sum of the principal parts of its Laurent expansion at each of its poles, say

$$H(s) = \sum_{i=1}^{p} H_i(s)$$

then the direct sum of minimal realizations of each of the H_i's is a minimal realization of H. Thus in section II, we derive a minimal realization of a matrix of strictly proper rational functions with a pole of order 2. In section III, we prove that the proposed realization is minimal. In section IV, we generalize the method proposed in section II; by induction, we obtain a realization of a matrix of strictly proper rational functions with a single pole of arbitrary order.

To illustrate the spirit of the method, we consider the simple problem of the minimal realization of a matrix with a first order pole.

Let $G(s) = N_1^{(1)}/(s-\lambda)$ where $N_1^{(1)} \in \mathbb{C}^{n_o \times n_i}$.

Perform a singular value decomposition on $N_1^{(1)}$:

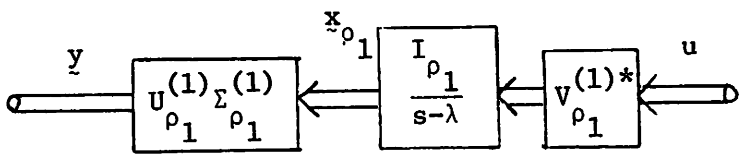
$$N_1^{(1)} = U^{(1)} \Sigma^{(1)} V^{(1)*}$$

where $U^{(1)} \in \mathbb{C}^{n_o \times n_o}$, $V^{(1)} \in \mathbb{C}^{n_i \times n_i}$ are unitary and $\Sigma^{(1)} = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_{\rho_1}, 0, 0, \ldots, 0) \in \mathbb{R}^{n_o \times n_i}$. Let $\rho_1 := \text{rank } N_1^{(1)}$.

Let $U^{(1)}_{\rho_1}$ and $V^{(1)}_{\rho_1}$ denote the first ρ_1 columns of $U^{(1)}$ and $V^{(1)}$, resp.

Let $\Sigma_{\rho_1} = \mathrm{diag}(\sigma_1,\sigma_2,\ldots,\sigma_{\rho_1})$; then

$$N_1^{(1)} = U^{(1)}_{\rho_1}\Sigma^{(1)}_{\rho_1}V^{(1)}_{\rho_1}$$



$$\begin{array}{cc} & \leftarrow \rho_1 \rightarrow \\ A := & \boxed{\lambda I_{\rho_1}} \end{array} \updownarrow \rho_1 \qquad\qquad \boxed{V^{(1)*}_{\rho_1}} =: B$$

$$C := \boxed{U^{(1)}_{\rho_1}\Sigma^{(1)}_{\rho_1}}$$

Since B and C have both rank ρ_1,

$$\mathrm{rank}[sI-A \,\vdots\, B] = \rho_1, \quad \forall s \in \mathbb{C}$$

$$\mathrm{rank}\begin{bmatrix} sI-A \\ \hdashline C \end{bmatrix} = \rho_1, \quad \forall s \in \mathbb{C}$$

Hence the realization is completely controllable and observable, hence minimal. $\square$

II. Minimal realization of a pole of order 2.

We consider a matrix of rational functions $G(s) \in \mathbb{R}^{n_o \times n_i}$, where $G(s)$ is strictly proper. Let $G(s)$ has a single pole λ of order 2, hence we write $G(s)$ as

$$G(s) = \frac{N_2^{(2)}}{(s-\lambda)^2} + \frac{N_1^{(2)}}{s-\lambda} \tag{1}$$

where $N_2^{(2)} \in \mathbb{C}^{n_o \times n_i}$, $N_1^{(2)} \in \mathbb{C}^{n_o \times n_i}$.

It is clear that to realize the second order term $\frac{N_2^{(2)}}{(s-\lambda)^2}$ requires at least $2\cdot\text{rank } N_2^{(2)}$ integrators. Then making maximum use of these $2\cdot\text{rank } N_2^{(2)}$ integrators with some additional integrators, we will realize the term $\frac{N_1^{(2)}}{s-\lambda}$.

To determine the rank of $N_2^{(2)} \in \mathbb{C}^{n_o \times n_i}$, we perform a singular value decomposition (abbreviated by SVD) on $N_2^{(2)}$ and obtain

$$N_2^{(2)} = U^{(2)}\Sigma^{(2)}V^{(2)*} \tag{2}$$

where $U^{(2)} \in \mathbb{C}^{n_o \times n_o}$ is unitary; $V^{(2)} \in \mathbb{C}^{n_i \times n_i}$ is unitary;

$$\Sigma^{(2)} \in \mathbb{R}^{n_o \times n_i};\ \Sigma^{(2)} := \left[\begin{array}{ccc|c} \sigma_1^{(2)} & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_{\rho_2}^{(2)} & \\ \hline & 0 & & 0 \end{array}\right] \quad \text{with}$$

$$\sigma_1^{(2)} \geq \sigma_2^{(2)} \ldots \geq \sigma_{\rho_2}^{(2)} > 0.$$

Hence

$$\text{rank } N_2^{(2)} = \rho_2.$$

Partitioning both $U^{(2)}$, $V^{(2)}$ as follows, we obtain

$$U^{(2)} = \begin{array}{c} \leftarrow \rho_2 \rightarrow \quad \leftarrow n_o-\rho_2 \rightarrow \\ \left[\begin{array}{c|c} U_{\rho_2}^{(2)} & U_{n_o-\rho_2}^{(2)} \end{array}\right] \end{array} \updownarrow n_o \qquad V^{(2)} = \begin{array}{c} \leftarrow \rho_2 \rightarrow \quad \leftarrow n_i-\rho_2 \rightarrow \\ \left[\begin{array}{c|c} V_{\rho_2}^{(2)} & V_{n_i-\rho_2}^{(2)} \end{array}\right] \end{array} \updownarrow n_i \tag{3}$$

Let's define

$$\Sigma_{\rho_2}^{(2)} := \begin{bmatrix} \sigma_1^{(2)} & & 0 \\ & \ddots & \\ 0 & & \sigma_{\rho_2}^{(2)} \end{bmatrix} \tag{4}$$

With the notations defined in (3) and (4), Eq. (2) is rewritten as

$$N_2^{(2)} = U_{\rho_2}^{(2)} \Sigma_{\rho_2}^{(2)} V_{\rho_2}^{(2)*} \tag{5}$$

<u>Remark</u>: If $N_1^{(2)} = 0$, using (5), a minimal realization of $G(s)$ is immediate:

$$A := \left[\begin{array}{c|c} \lambda I_{\rho_2} & I_{\rho_2} \\ \hline 0 & \lambda I_{\rho_2} \end{array}\right] \qquad \begin{array}{c} \leftarrow n_i \rightarrow \\ \left[\begin{array}{c} 0 \\ \hline V_{\rho_2}^{(2)*} \end{array}\right] \end{array} =: B$$

$$C := \begin{array}{cc} \left[\begin{array}{c|c} U_{\rho_2}^{(2)} \Sigma_{\rho_2}^{(2)} & 0 \end{array}\right] & \updownarrow n_o \\ \leftarrow \rho_2 \rightarrow & \end{array}$$

By inspection, $\forall s \in \mathbb{C}$, $\text{rank}[sI-A \,\vdots\, B] = 2\rho_2$ and $\text{rank} \left[\begin{array}{c} sI-A \\ \hline C \end{array}\right] = 2\rho_2$, hence the realization is minimal. ◻

Let $\underset{\sim}{u} \in \mathbb{R}^{n_i}$ denote the input. Let

$$\begin{array}{c} \updownarrow \rho_2 \\ \updownarrow n_i-\rho_2 \end{array} \left[\begin{array}{c} \underset{\sim}{v}_{\rho_2} \\ \hline \underset{\sim}{v}_{n_i-\rho_2} \end{array}\right] := V^{(2)*} \underset{\sim}{u} \tag{6}$$

To construct a realization intuitively, consider

$$G(s)\underset{\sim}{u} = \left[\frac{N_2^{(2)}}{(s-\lambda)^2} + \frac{N_1^{(2)}}{s-\lambda}\right] V^{(2)} \cdot V^{(2)*} \underset{\sim}{u}$$

$$= \left[\frac{N_2^{(2)} V^{(2)}}{(s-\lambda)^2} + \frac{N_1^{(2)} V^{(2)}}{s-\lambda}\right] \left[\begin{array}{c} \underset{\sim}{v}_{\rho_2} \\ \hline v_{n_i-\rho_2} \end{array}\right]$$

Using the partition of $V^{(2)}$ from Eq. (3), we obtain

$$G(s)\underset{\sim}{u} = \frac{N_2^{(2)}V_{\rho_2}}{s-\lambda} \cdot \frac{I_{\rho_2}}{s-\lambda}\underset{\sim}{v}_{\rho_2} + \frac{N_1^{(2)}V^{(2)}_{\underline{n_i-\rho_2}}}{s-\lambda}\underset{\sim}{v}_{n_i-\rho_2} + N_1^{(2)}V^{(2)}_{\rho_2}\frac{I_{\rho_2}}{s-\lambda}\underset{\sim}{v}_{\rho_2} \tag{7}$$

Let us use ρ_2 integrators to realize

$$\underset{\sim}{x}_{\rho_2} := \frac{I_{\rho_2}}{s-\lambda}\underset{\sim}{v}_{\rho_2}$$

then the realization of the third term of (7) is immediate:

$$\frac{N_1^{(2)}V^{(2)}_{\rho_2}}{s-\lambda}\underset{\sim}{v}_{\rho_2} = N_1^{(2)}V^{(2)}_{\rho_2} \cdot \underset{\sim}{x}_{\rho_2} .$$

In terms of $\underset{\sim}{x}_{\rho_2}$ and $\underset{\sim}{v}_{\underline{n_i-\rho_2}}$, the first two terms of (7) become

$$\underset{\sim}{z} := \frac{N_2^{(2)}V^{(2)}_{\rho_2}}{s-\lambda}\underset{\sim}{x}_{\rho_2} + \frac{N_1^{(2)}V^{(2)}_{\underline{n_i-\rho_2}}}{s-\lambda}\underset{\sim}{v}_{n_i-\rho_2}$$

$$= \frac{1}{s-\lambda}\begin{bmatrix} N_1^{(1)} \end{bmatrix}\begin{bmatrix} \underset{\sim}{x}_{\rho_2} \\ \hline \underset{\sim}{v}_{\underline{n_i-\rho_2}} \end{bmatrix} \tag{8}$$

where

$$N_1^{(1)} := \left[\, N_2^{(2)}V^{(2)}_{\rho_2} \;\vdots\; N_1^{(2)}V^{(2)}_{\underline{n_i-\rho_2}} \,\right] \tag{9}$$

The minimum no. of integrators required for realizing (8) is $\rho_1 := \text{rank } N_1^{(1)}$. To determine ρ_1, we perform a singular value decomposition on $N_1^{(1)}$ and obtain

$$N_1^{(1)} = U^{(1)}\Sigma^{(1)}V^{(1)} \tag{10}$$

where $U^{(1)} \in \mathbb{C}^{n_o x n_o}$ is unitary, $V^{(1)} \in \mathbb{C}^{n_i x n_i}$ is unitary; $\Sigma^{(1)} \in \mathbb{R}^{n_o x n_i}$

$$\Sigma^{(1)} := \left[\begin{array}{ccc|c} \sigma_1^{(1)} & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_{\rho_1}^{(1)} & \\ \hline & 0 & & 0 \end{array}\right] \tag{10a}$$

with $\sigma_1^{(1)} \geq \sigma_2^{(1)} \; \cdots \geq \sigma_{\rho_1}^{(1)} > 0.$

Partitioning both $U^{(1)}$, $V^{(1)}$ as follows, we obtain

$$U^{(1)} = \begin{array}{cc} \left[\begin{array}{c|c} U_{\rho_1}^{(1)} & U_{n_o-\rho_1}^{(1)} \end{array}\right] & \updownarrow n_o \\ \leftarrow \rho_1 \rightarrow \quad \leftarrow n_o-\rho_1 \rightarrow & \end{array} \qquad V^{(1)} = \begin{array}{cc} \left[\begin{array}{c|c} V_{\rho_1}^{(1)} & V_{n_i-\rho_1}^{(1)} \end{array}\right] & \updownarrow n_i \\ \leftarrow \rho_1 \rightarrow \quad \leftarrow n_i-\rho_i \rightarrow & \end{array} \tag{11}$$

We further partition $V_{\rho_1}^{(1)}$ as follows:

$$V_{\rho_1}^{(1)} = \begin{array}{cc} & \leftarrow \rho_1 \rightarrow \\ \updownarrow \rho_2 & \left[\begin{array}{c} \hat{V}_{\rho_1}^{(1)} \\ \hline \check{V}_{\rho_1}^{(1)} \end{array}\right] \\ \updownarrow n_i-\rho_2 & \end{array} \tag{12}$$

We define

$$\Sigma_{\rho_1}^{(1)} := \begin{bmatrix} \sigma_1^{(1)} & & & 0 \\ & \sigma_2^{(1)} & & \\ & & \ddots & \\ 0 & & & \sigma_{\rho_1}^{(1)} \end{bmatrix}, \tag{13a}$$

and

$$\rho_1 := \text{rank } N_1^{(1)} \tag{13b}$$

With the notations defined in (11), (12), (13), Eq. (10) is written as

$$N_1^{(1)} = U_{\rho_1}^{(1)}\Sigma_{\rho_1}^{(1)}V_{\rho_1}^{(1)*} = \left(U_{\rho_1}^{(1)}\Sigma_{\rho_1}^{(1)}V_{\rho_1}^{(1)*}\right)\left(V_{\rho_1}^{(1)}I_{\rho_1}V_{\rho_1}^{(1)*}\right)$$

$$= N_1^{(1)}V_{\rho_1}^{(1)}I_{\rho_1}V_{\rho_1}^{(1)*} \tag{14}$$

and Eq. (8) becomes

$$\underset{\sim}{z} := N_1^{(1)}V_{\rho_1}^{(1)} \cdot \frac{I_{\rho_1}}{s-\lambda} \cdot \left[\hat{V}_{\rho_1}^{(1)*} \,\vdots\, \check{V}_{\rho_1}^{(1)*}\right]\begin{bmatrix} \underset{\sim}{x}_{\rho_2} \\ \hline \underset{\sim}{v}_{\underline{n_i-\rho_2}} \end{bmatrix} \tag{15}$$

This shows that $\underset{\sim}{z}$ can be realized by ρ_1 integrators.

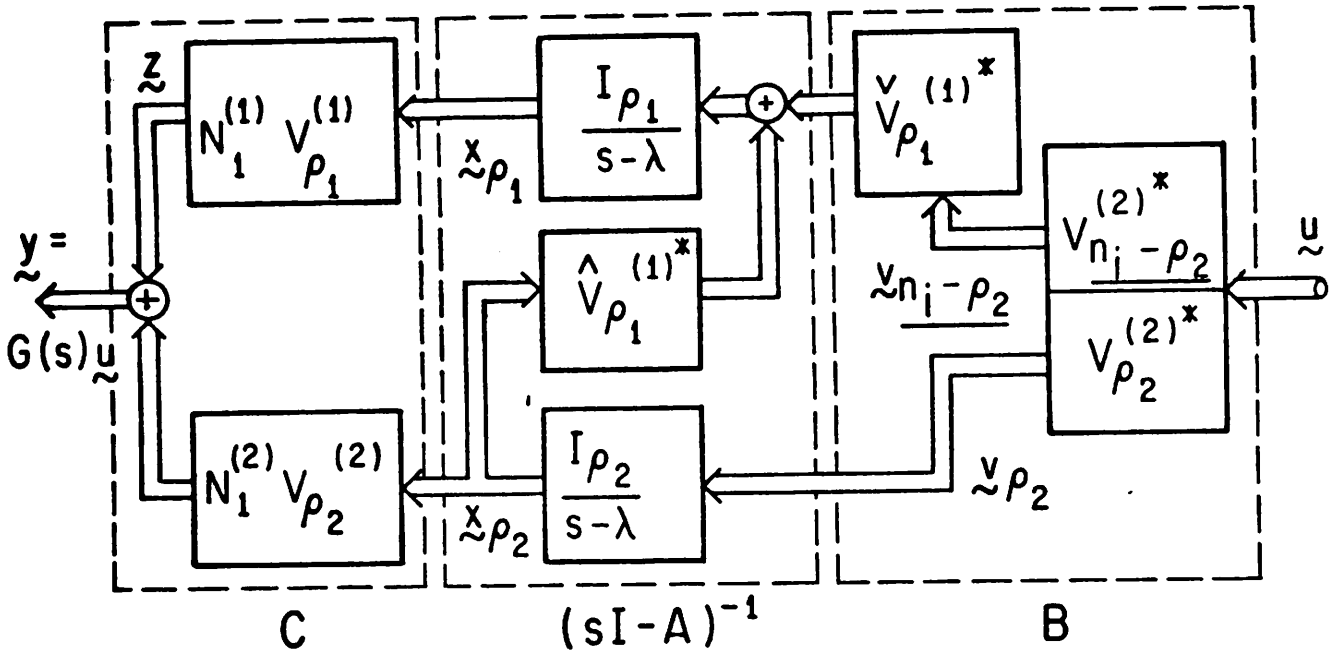
We define

$$\underset{\sim}{x}_{\rho_1} := \frac{I_{\rho_1}}{s-\lambda}\left[\hat{V}_{\rho_1}^{(1)*} \,\vdots\, \check{V}_{\rho_1}^{(1)*}\right]\begin{bmatrix} \underset{\sim}{x}_{\rho_2} \\ \hline \underset{\sim}{v}_{\underline{n_i-\rho_2}} \end{bmatrix} \tag{16}$$

<u>Remark</u>: Since $N_1^{(1)} := \left[N_2^{(2)}V_{\rho_2}^{(2)} \,\vdots\, N_1^{(2)}V_{\underline{n_i-\rho_2}}^{(2)}\right]$ from (9),

$\rho_2 := \text{rank } N_2^{(2)} = \text{rank } N_2^{(2)}V_{\rho_2}^{(2)} \leq \text{rank } N_1^{(1)} =: \rho_1$. ¤

Based on the above analysis, G(s) is realized by the following block diagram.



Hence, a realization $\{A,B,C\}$ is given as follows:

$$A := \begin{array}{c} \begin{array}{cc} \longleftarrow \rho_1 \longrightarrow & \longleftarrow \rho_2 \longrightarrow \end{array} \\ \left[\begin{array}{c|c} \lambda I_{\rho_1} & \hat{V}^{(1)*}_{\rho_1} \\ \hline \bigcirc & \lambda I_{\rho_2} \end{array}\right] \end{array} \begin{array}{c} \updownarrow \rho_1 \\ \updownarrow \rho_2 \end{array} \qquad \begin{array}{c} \updownarrow \rho_1 \\ \updownarrow \rho_2 \end{array} \begin{array}{c} \longleftarrow n_i \longrightarrow \\ \left[\begin{array}{c} \check{V}^{(1)*}_{\rho_1} \underline{V^{(2)*}_{n_i-\rho_2}} \\ \hline V^{(2)*}_{\rho_2} \end{array}\right] \end{array} =: B \tag{17}$$

$$C := \left[\begin{array}{c|c} N_1^{(1)} V^{(1)}_{\rho_1} & N_1^{(2)} V^{(2)}_{\rho_2} \end{array}\right] \updownarrow n_o \ ;$$

with $\underset{\sim}{x} := \left[\begin{array}{c} \underset{\sim}{x}_{\rho_1} \\ \hline \underset{\sim}{x}_{\rho_2} \end{array}\right]$

<u>Remark</u>: The $\rho_1 \times n_i$ matrix $\check{V}^{(1)*}_{\rho_1} \cdot \underline{V^{(2)*}_{n_i-\rho_2}}$ is the product of a $\rho_1 \times (n_i-\rho_2)$ by a $(n_i-\rho_2) \times n_i$ matrix. ⌑

The procedure of constructing a realization $\{A,B,C\}$ of $G(s) = \dfrac{N_2^{(2)}}{(s-\lambda)^2} + \dfrac{N_1^{(2)}}{s-\lambda}$ is summarized by the following algorithm:

Algorithm

Step 1 Perform the SVD of $N_2^{(2)}$

$$N_2^{(2)} = U^{(2)}\Sigma^{(2)}V^{(2)*} = U_{\rho_2}^{(2)}\Sigma_{\rho_2}^{(2)}V_{\rho_2}^{(2)}$$

where ρ_2 := rank $N_2^{(2)}$ and $V^{(2)}$ is partitioned as

$$V^{(2)} = \begin{array}{cc} \begin{array}{cc} \leftarrow\rho_2\rightarrow & \leftarrow n_i-\rho_2\rightarrow \end{array} & \\ \left[\begin{array}{c|c} V_{\rho_2}^{(2)} & \underline{V_{n_i-\rho_2}^{(2)}} \end{array}\right] & \updownarrow n_i \end{array}$$

Step 2 Define

$$N^{(1)} := \begin{array}{cc} \left[\begin{array}{c|c} N_2^{(2)}V_{\rho_2}^{(2)} & N_1^{(2)}\underline{V_{n_i-\rho_2}^{(2)}} \end{array}\right] & \updownarrow n_o \\ \begin{array}{cc} \leftarrow \rho_2 \longrightarrow & \longleftarrow n_i-\rho_2 \longrightarrow \end{array} & \end{array}$$

and perform the SVD of $N_1^{(1)}$

$$N_1^{(1)} = U^{(1)}\Sigma^{(1)}V^{(1)*} = U_{\rho_1}^{(1)}\Sigma_{\rho_1}V_{\rho_1}^{(1)*}$$

where ρ_1 := rank $N_1^{(1)}$, and $V^{(1)}$ is partitioned as

$$V^{(1)} = \begin{array}{cc} \begin{array}{cc} \leftarrow\rho_1\rightarrow & \leftarrow n_i-\rho_1\rightarrow \end{array} & \\ \left[\begin{array}{c|c} V_{\rho_1}^{(1)} & \underline{V_{n_i-\rho_1}^{(1)}} \end{array}\right] & \updownarrow n_i \end{array}$$

We further partition $V_{\rho_1}^{(1)}$ as

$$V_{\rho_1}^{(1)} = \begin{array}{cc} & \leftarrow\rho_1\rightarrow \\ \updownarrow \rho_2 & \left[\begin{array}{c} \hat{V}_{\rho_1}^{(1)} \\ \hline \check{V}_{\rho_1}^{(1)} \end{array}\right] \\ \updownarrow n_i-\rho_2 & \end{array}$$

Step 3 A realization {A,B,C} of G(s) is

$$
A := \left[\begin{array}{c:c} \lambda I_{\rho_1} & \hat{V}^{(1)*}_{\rho_1} \\ \hdashline \bigcirc & \lambda I_{\rho_2} \end{array}\right]; \quad \begin{array}{c} \xleftarrow{\quad n_i \quad} \\ \left[\begin{array}{cc} V^{(1)*}_{\rho_1} & \underline{V^{(2)*}_{n_i-\rho_2}} \\ \hline \multicolumn{2}{c}{V^{(2)*}_{\rho_2}} \end{array}\right] \end{array} =: B \qquad (17)
$$

$$
C := \left[\begin{array}{c:c} N^{(1)}_1 V^{(1)}_{\rho_1} & N^{(2)}_1 V^{(2)}_{\rho_2} \end{array}\right] \updownarrow n_o
$$

$\square$

III. Proof of Minimality

We show that the realization $\{A,B,C\}$ given by Eq. (17) is minimal.

Theorem Consider $G(s) \in \mathbb{R}(s)^{n_o x n_i}$ given by (1). Then $\{A,B,C\}$ given by (17) is a minimal realization of $G(s)$.

Proof From the analysis of section II, it is clear that $\{A,B,C\}$ is a realization of $G(s)$. Hence the remaining task is to show minimality, or equivalently, to show that $\{A,B,C\}$ is completely controllable <u>and</u> completely observable.

To show complete controllability, we show that $[sI-A \vdots B]$ is full rank $\forall s \in \mathbb{C}$. Now by (17), $\forall s \neq \lambda$, $[sI-A \vdots B]$ is full rank. Now for $s = \lambda$, we have

$$
\text{rank}[\lambda I-A \vdots B] = \text{rank} \begin{array}{c} \\ \updownarrow \rho_1 \\ \updownarrow \rho_2 \end{array} \begin{array}{c} \leftrightarrow\rho_1 \ \leftrightarrow\rho_2 \quad \xleftarrow{\quad n_i \quad} \\ \left[\begin{array}{c:c:cc} \bigcirc & \hat{V}^{(1)*}_{\rho_1} & \check{V}^{(1)*}_{\rho_1} & \underline{V^{(2)*}_{n_i-\rho_2}} \\ \hdashline \bigcirc & \bigcirc & \multicolumn{2}{c}{V^{(2)*}_{\rho_2}} \end{array}\right] \end{array}
$$

$$
= \text{rank} \begin{array}{c} \leftarrow \rho_1 \rightarrow \quad \xleftarrow{\quad n_i \quad} \\ \left[\begin{array}{c:cc} \hat{V}^{(1)*}_{\rho_1} & \check{V}^{(1)*}_{\rho_1} & \underline{V^{(2)*}_{n_i-\rho_2}} \\ \hdashline \bigcirc & \multicolumn{2}{c}{V^{(2)*}_{\rho_2}} \end{array}\right] \end{array} \begin{array}{c} \updownarrow \rho_1 \\ \updownarrow \rho_2 \end{array}
$$

$$\operatorname{rank}[\lambda I-A \,\vdots\, B] = \operatorname{rank}\left[\begin{array}{c:c} \hat{V}^{(1)*}_{\rho_1} & \check{V}^{(1)*}_{\rho_1} \underline{V^{(2)*}_{n_i-\rho_2}} \\ \hdashline 0 & V^{(2)*}_{\rho_2} \end{array}\right]\left[\begin{array}{c:c} I_{\rho_1} & 0 \\ \hdashline 0 & V^{(2)} \end{array}\right] \quad \text{since } V^{(2)} \in \mathbb{C}^{n_i \times n_i} \text{ is of rank } n_i$$

$$= \operatorname{rank}\left[\begin{array}{c:c:c} \hat{V}^{(1)*}_{\rho_1} & 0 & \check{V}^{(1)*}_{\rho_1} \\ \hdashline 0 & I_{\rho_2} & 0 \end{array}\right]$$

$$= \rho_1 + \rho_2 \qquad \text{because } V^{(1)}_{\rho_1} \text{ is full rank and (12).}$$

To show complete observability, we show that $\begin{bmatrix} sI-A \\ \hdashline C \end{bmatrix}$ is full rank $\forall s \in \mathbb{C}$. Again by (17), $\forall s \neq \lambda$, $\begin{bmatrix} sI-A \\ \hdashline C \end{bmatrix}$ is full rank. Now for $s = \lambda$, we have

$$\operatorname{rank}\begin{bmatrix} \lambda I-A \\ \hdashline C \end{bmatrix} = \operatorname{rank} \begin{array}{cc} \begin{array}{cc} \longleftarrow \rho_1 \longrightarrow & \longleftarrow \rho_2 \longrightarrow \end{array} & \\ \left[\begin{array}{c:c} 0 & \hat{V}^{(1)*}_{\rho_1} \\ \hdashline 0 & 0 \\ \hdashline N_1^{(1)} V^{(1)}_{\rho_1} & N_1^{(2)} V^{(2)}_{\rho_2} \end{array}\right] & \begin{array}{c} \updownarrow \rho_1 \\ \updownarrow \rho_2 \\ \updownarrow n_o \end{array} \end{array}$$

$$= \operatorname{rank} \begin{array}{cc} \begin{array}{cc} \longleftarrow \rho_1 \longrightarrow & \longleftarrow \rho_2 \longrightarrow \end{array} & \\ \left[\begin{array}{c:c} 0 & \hat{V}^{(1)*}_{\rho_1} \\ \hdashline N_1^{(1)} V^{(1)}_{\rho_1} & N_1^{(2)} V^{(2)}_{\rho_2} \end{array}\right] & \begin{array}{c} \updownarrow \rho_1 \\ \updownarrow n_o \end{array} \end{array}$$

Now $N_1^{(1)} V^{(1)}_{\rho_1} = U^{(1)}_{\rho_1} \Sigma^{(1)}_{\rho_1}$ is of rank ρ_1 because $\Sigma^{(1)}_{\rho_1}$ is square and of rank ρ_1, (see (13a)), and $U^{(1)}$ being unitary has its first ρ_1 columns, namely $U^{(1)}_{\rho_1}$, forming an independent family. Consider now $\hat{V}^{(1)*}_{\rho_1} \in \mathbb{C}^{\rho_1 \times \rho_2}$:

$$\operatorname{rank} \hat{V}^{(1)*}_{\rho_1} = \operatorname{rank}\left\{V^{(1)*}_{\rho_1}\begin{bmatrix} I_{\rho_2} \\ \hline 0 \end{bmatrix}\right\} \quad \text{by (12)}$$

$$= \operatorname{rank}\left\{\Sigma^{(1)}_{\rho_1} V^{(1)*}_{\rho_1}\begin{bmatrix} I_{\rho_2} \\ \hline 0 \end{bmatrix}\right\}$$

since $\Sigma^{(1)}_{\rho_1} \in \mathbb{C}^{\rho_1 x \rho_1}$ and is of rank ρ_1

$$= \operatorname{rank}\left\{\Sigma^{(1)} V^{(1)*}\begin{bmatrix} I_{\rho_2} \\ \hline 0 \end{bmatrix}\right\}$$

by (10a), (11) and (13a)

$$= \operatorname{rank}\left\{U^{(1)}\Sigma^{(1)} V^{(1)*}\begin{bmatrix} I_{\rho_2} \\ \hline 0 \end{bmatrix}\right\}$$

since $U^{(1)} \in \mathbb{C}^{n_o x n_o}$ and is of rank n_o.

$$= \operatorname{rank}[N_2^{(2)} V^{(2)}_{\rho_2}]$$

by (9) and (10)

$$= \operatorname{rank}[U^{(2)}_{\rho_2} \Sigma^{(2)}_{\rho_2}]$$

by (5)

$$= \rho_2$$

because $U^{(2)}$ is unitary, hence rank $U^{(2)}_{\rho_2} = \rho_2$.

Hence

$$\operatorname{rank}\left[\begin{array}{c|c} 0 & \hat{V}^{(1)*}_{\rho_1} \\ \hline N_1^{(1)} V^{(1)}_{\rho_1} & N_1^{(2)} V^{(2)}_{\rho_2} \end{array}\right] = \rho_1 + \rho_2$$

and the pair (C,A) of (17) is observable. ¤

IV. <u>An induction step for the realization of a pole of order $\ell > 2$</u>.

We have constructed a minimal realization of a matrix of rational functions with a single pole of order 2. We now consider a matrix of rational functions with a single pole of order $\ell > 2$:

$$G(s) = \frac{N_\ell^{(\ell)}}{(s-\lambda)^\ell} + \frac{N_{\ell-1}^{(\ell)}}{(s-\lambda)^{\ell-1}} + \ldots + \frac{N_1^{(\ell)}}{s-\lambda} \tag{18}$$

where $N_i^{(\ell)} \in \mathbb{C}^{n_o \times n_i} \quad \forall i \in \{1,2,\ldots,\ell\}$.

The induction assumption is that we have a method for a <u>minimal</u> realization of any matrix of rational functions with a single pole of order $\ell-1$; we denote it by $\{A^{(\ell-1)}, B^{(\ell-1)}, C^{(\ell-1)}\}$. We now construct a realization $\{A,B,C\}$ of the $G(s)$ of (18) in terms of $\{A^{(\ell-1)}, B^{(\ell-1)}, C^{(\ell-1)}\}$.

We perform a singular value decomposition on $N_\ell^{(\ell)}$ and obtain

$$N_\ell^{(\ell)} = U^{(\ell)}\Sigma^{(\ell)}V^{(\ell)*} \tag{19}$$

$$= U_{\rho_\ell}^{(\ell)}\Sigma_{\rho_\ell}^{(\ell)}V_{\rho_\ell}^{(\ell)*} \tag{20}$$

where $\rho_\ell := \operatorname{rank} N_\ell^{(\ell)}$.

As in (7), we obtain

$$G(s)\underset{\sim}{u} = \left[\frac{N_\ell^{(\ell)}}{(s-\lambda)^\ell} + \frac{N_{\ell-1}^{(\ell)}}{(s-\lambda)^{\ell-1}} + \ldots \frac{N_1^{(\ell)}}{(s-\lambda)}\right] V^{(\ell)}V^{(\ell)*}\underset{\sim}{u}$$

$$= \left[\frac{N_\ell^{(\ell)}V^{(\ell)}}{(s-\lambda)^\ell} + \frac{N_{\ell-1}^{(\ell)}V^{(\ell)}}{(s-\lambda)^{\ell-1}} + \ldots \frac{N_1^{(\ell)}V^{(\ell)}}{(s-\lambda)}\right] \begin{bmatrix} \underset{\sim}{v}_{\rho_\ell} \\ \hline \underset{\sim}{v}_{\underline{n_i-\rho_\ell}} \end{bmatrix} \tag{21}$$

$$= \frac{\left[N_\ell^{(\ell)}V_{\rho_\ell}^{(\ell)}\right]}{(s-\lambda)^{\ell-1}} \cdot \frac{I_{\rho_\ell}}{s-\lambda}\underset{\sim}{v}_{\rho_\ell} + \frac{N_{\ell-1}^{(\ell)}V_{\underline{n_i-\rho_\ell}}^{(\ell)}}{(s-\lambda)^{\ell-1}}\underset{\sim}{v}_{\underline{n_i-\rho_\ell}}$$

$$+ \frac{N_{\ell-1}^{(\ell)}V_{\rho_\ell}^{(\ell)}}{(s-\lambda)^{\ell-2}} \cdot \frac{I_{\rho_\ell}}{s-\lambda} \cdot \underset{\sim}{v}_{\rho_\ell} + \ldots + \frac{N_1^{(\ell)}V_{\underline{n_i-\rho_\ell}}^{(\ell)}}{s-\lambda}\underset{\sim}{v}_{\underline{n_i-\rho_\ell}}$$

$$+ N_1^{(\ell)}V_{\rho_\ell}^{(\ell)} \cdot \frac{I_{\rho_\ell}}{s-\lambda}\underset{\sim}{v}_{\rho_\ell} \tag{22}$$

where we defined

$$\begin{array}{c} \uparrow \\ \rho_1 \\ \downarrow \\ \uparrow \\ n_i-\rho_\ell \\ \downarrow \end{array} \left[\begin{array}{c} \underset{\sim}{v}_{\rho_\ell} \\ \hline \underset{\sim}{v}_{\underline{n_i-\rho_\ell}} \end{array}\right] := V^{(\ell)*}\underset{\sim}{u} \tag{23}$$

Note that (21) includes ℓ terms and that the use of the variables $\underset{\sim}{v}_{\rho_\ell}$ and $\underset{\sim}{v}_{\underline{n_i-\rho_\ell}}$, (defined in (23)), creates $2\ell-1$ terms in (22). (Indeed the first term of (21) leads to only one term in $\underset{\sim}{v}_{\rho_\ell}$.)

We use ρ_ℓ integrators to realize

$$\underset{\sim}{x}_{\rho_\ell} := \frac{I_{\rho_\ell}}{s-\lambda}\,\underset{\sim}{v}_{\rho_\ell}$$

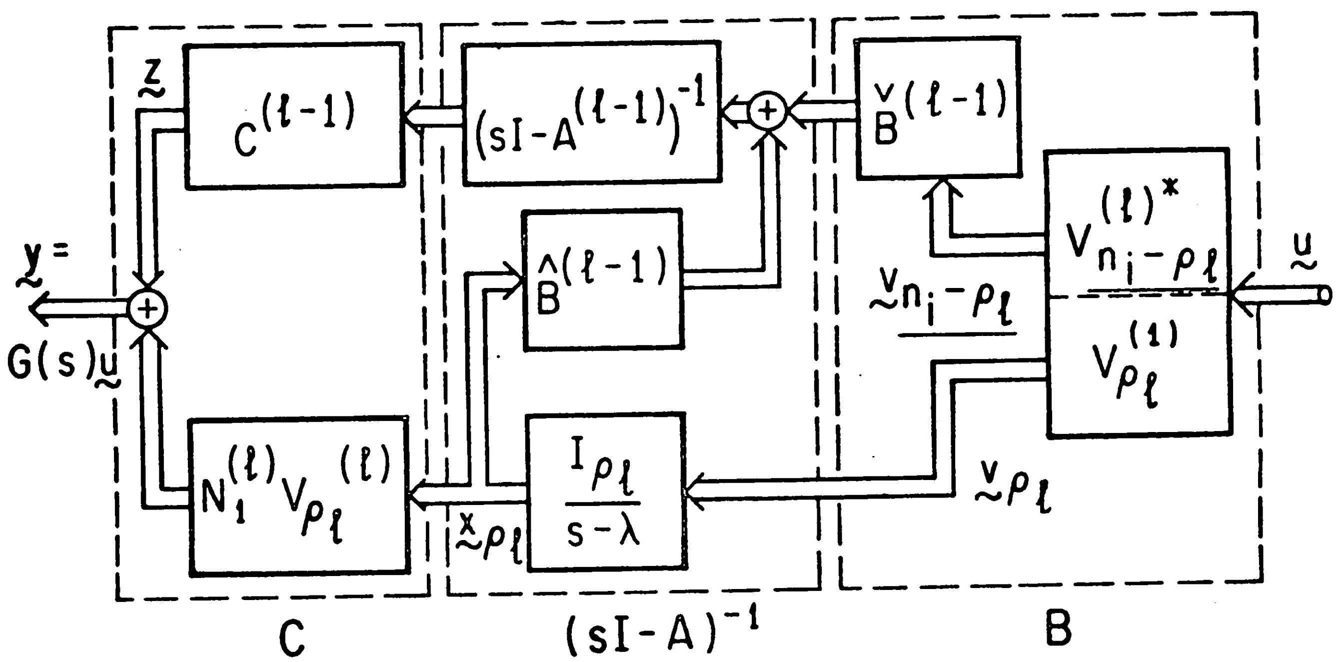
then the realization of the last term of (22) is immediate.

In terms of $\underset{\sim}{x}_{\rho_\ell}$ and $\underset{\sim}{v}_{n_i-\rho_\ell}$, the first $(\ell-1)$ terms of (22) become

$$\underset{\sim}{z} := \frac{[N_{\ell-1}^{(\ell-1)}]}{(s-\lambda)^{\ell-1}} \left[\begin{array}{c} \underset{\sim}{x}_{\rho_\ell} \\ \hline \underset{\sim}{v}_{\underline{n_i-\rho_\ell}} \end{array}\right] + \frac{[N_{\ell-1}^{(\ell-1)}]}{(s-\lambda)^{\ell-2}} \left[\begin{array}{c} \underset{\sim}{x}_{\rho_\ell} \\ \hline \underset{\sim}{v}_{\underline{n_i-\rho_\ell}} \end{array}\right] + \cdots$$

$$+ \frac{[N_1^{(\ell-1)}]}{s-\lambda} \left[\begin{array}{c} \underset{\sim}{x}_{\rho_\ell} \\ \hline v_{\underline{n_i-\rho_\ell}} \end{array}\right] \tag{24}$$

where $N_i^{(\ell-1)} := [N_{i+1}^{(\ell)}V_{\rho_\ell}^{(\ell)} \,\vdots\, N_i^{(\ell)}V_{\underline{n_i-\rho_\ell}}^{(\ell)}]$ $\forall i \in \{1,2,\ldots,\ell-1\}$. By the induction assumption, $\{A^{(\ell-1)},B^{(\ell-1)},C^{(\ell-1)}\}$ is the minimal realization of (24), we realize G(s) of (18) by the following block diagram.



where $B^{(\ell-1)}$ is partitioned as follows:

$$B^{(\ell-1)} = \left[\begin{array}{c|c} \underbrace{\hat{B}^{(\ell-1)}}_{\leftarrow \rho_\ell \rightarrow} & \check{B}^{(\ell-1)} \end{array}\right]$$

Hence a realization $\{A,B,C\}$ in terms of $\{A^{(\ell-1)}, B^{(\ell-1)}, C^{(\ell-1)}\}$ is given as follows:

$$A := \left[\begin{array}{c|c} A^{(\ell-1)} & \hat{B}^{(\ell-1)} \\ \hline O & \lambda I_{\rho_\ell} \end{array}\right] \qquad \left[\begin{array}{c} \check{B}^{(\ell-1)} \cdot V^{(\ell)*}_{n_i-\rho_\ell} \\ \hline V^{(\ell)*}_{\rho_\ell} \end{array}\right] =: B \tag{25}$$

$$C := \left[\begin{array}{c|c} C^{(\ell-1)} & N_1^{(\ell)} V_{\rho_\ell}^{(\ell)} \end{array}\right]$$

The realization of G(s) of Eq. (18) is then obtained iteratively.

For a proof of minimality, refer to [1].

V. Conclusion

Based on Van Dooren's work [1], in section II, we obtain intuitively a realization of a matrix of rational function with a single pole of order 2; we then prove the minimality. In section IV, by an induction step, we obtain a minimal realization of the matrix of rational functions with a single pole of order $\ell > 2$.

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[1] P. Van Dooren and P. Dewilde, "State-Space Realization of a General Rational Matrix: A Numerically Stable Algorithm," Proceedings of the Twentieth Midwest Symposium on Circuits and Systems, Texas Tech University, 25-27 August 1977, p. 773-781.