Copyright © 1979, by the author(s).
All rights reserved.

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. To copy otherwise, to republish, to post on servers or to redistribute to lists, requires prior specific permission.

# A PROPERTY OF CERTAIN MULTISTAGE LINEAR PROGRAMS - SOME APPLICATIONS

by

J. B. Lasserre

Memorandum No. UCB/ERL M79/7

13 February 1979

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

# A PROPERTY OF CERTAIN MULTISTAGE LINEAR PROGRAMS - SOME APPLICATIONS

J. B. Lasserre†

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

In this paper we present a property of certain linear multistage problems. To solve them, a method which takes this property into account, is presented. It requires the resolution of 2N-1 subproblems if there are N stages in the original problem. A sufficient condition is given on the matrix of the constraints for the property to be true.

When only a submatrix has this property we propose to use the Dantzig Wolfe decomposition. We then can solve the subproblem with the proposed method.

Applications to linear and nonlinear programming are presented.

KEY WORDS: Linear programming, Multistage linear programming, Dantzig Wolfe decomposition principle

Research sponsored by the National Science Foundation Grant INT78-09263.

†Currently on leave from L.A.A.S. 7 avenue du Colonel Roche, 31400 Toulouse (FRANCE)

## I. Introduction

General methods exist to solve multistage linear programming problems (see [6] for example). Here we focus our attention on a special property of some problems of this family. Provided a certain hypothesis is true we give an efficient method to solve them.

In section I we describe the method and give a sufficient condition on the matrix of the constraints for the hypothesis to be true.

In section II we show how to use this method when only a submatrix satisfies the sufficient condition. Examples of applications are given.

## II. The Method

Let us consider the following multistage linear program

$$
(1)\quad \begin{cases} A_1x_1 = b_1 \\[2ex] A_1x_i + \sum_{j=1}^{i-1} L_{ij}x_j = b_i \quad i = 2,\dots,M \\[2ex] x_i \geq 0 \quad i = 1,\dots,M \\[2ex] \text{MAX} \sum_{i=1}^{M} C_i^T x_i \end{cases}
$$

the matrix of the constraints is shown below

$$
\begin{array}{|c|c|c|c|}
\hline
A_1 & & & \\
\hline
L_{21} & A_2 & & \\
\hline
 & & & \\
\hline
L_{M1} & & L_{M,M-1} & A_M \\
\hline
\end{array}
\;=\;
\begin{array}{|c|}
\hline
b_1 \\
\hline
 \\
\hline
b_M \\
\hline
\end{array}
$$

$$
\begin{array}{|c|c|c|}
\hline
C_1 & & C_M \\
\hline
\end{array}
$$

We call $(P_i)$ the following subproblem

$$(2)\quad \begin{cases} A_i x_i = b_i - \sum_{j=1}^{i-1} L_{ij} x_j^* \quad (\text{if } i \geq 2) \\ x_i \geq 0 \\ \text{MAX } \tilde{C}_i^T x_i \end{cases}$$

where $x_j^*$ is given $j = 1, \ldots, i-1$ such that the first $i-1$ stage constraints are satisfied.

We call $\mathcal{D}_i$ the dual problem of (2)

$$(3)\quad \begin{cases} A_i^T \lambda_i \geq \tilde{C}_i \\ \text{MIN } \lambda_i^T (b_i - \sum_{j=1}^{i-1} L_{ij} x_j^*) \end{cases}$$

HYPOTHESIS

The solution of (3) is (when it exists) independent of $x_j^*$ $(j=1,\ldots,i-1)$. With this hypothesis we can solve (1) with the following algorithm.

Step (0) Set $i = M$, $\tilde{C}_M = C_M$

Step (1) Solve $\mathcal{D}_i$; get $\bar{\lambda}_i$ (the optimal solution) then compute

$$\tilde{C}_{i-1} = C_{i-1} \sum_{j=i+1}^{M} L_{ji}^T \bar{\lambda}_j$$

Set $i = i-1$

if $i = 1$ go to step 2 else go to step 1.

Step (2) Solve $P_i$; the right hand side is $b_1$ if $i = 1$.

$$b_i - \sum_{j=1}^{i-1} L_{ij} \bar{x}_j \text{ , otherwise}$$

get $\bar{x}_i$; set $i = i+1$

if $i = M+1$ stop else go to step 2.

Proposition

i) $\bar{x} = (\bar{x}_1\bar{x}_2,\ldots,\bar{x}_M)$ is optimal solution of (1)

ii) $\bar{\lambda} = (\bar{\lambda}_1,\bar{\lambda}_2,\ldots,\bar{\lambda}_M)$ is " " of the dual of (1)

Proof:

- $\bar{x}$ is admissible for (1) by construction

- $\bar{\lambda}$ is admissible for the dual of (1) by construction also

for we have

$$\begin{cases} A_M^T\bar{\lambda}_M \geq \tilde{C}_M = C_M \\ A_i^T\bar{\lambda}_i \geq \tilde{C}_i = C_i - \sum_{j=i+1}^{M} L_{ji}^T\bar{\lambda}_j \\ A_i^T\bar{\lambda}_i + \sum_{j=i+1}^{M} L_{ji}^T\bar{\lambda}_j \geq C_i \quad i = 1,\ldots,M-1 \end{cases}$$

These constraints satisfied by $\bar{\lambda}_j$ $(j=1,\ldots,M)$ are the constraints of the dual problem of (1) by definition.

- Moreover, since the hypothesis holds, we have at each iteration of the step 2.

$$\tilde{C}_i^T\bar{x}_i = \bar{\lambda}_i^T\left(b_i - \sum_{j=1}^{i-1} L_{ij}\bar{x}_j\right) \text{ for } i \geq 2 \text{ and } \tilde{C}_1^T\bar{x}_1 = \bar{\lambda}_1^Tb_1 \text{ for } i = 1.$$

by adding up in each side of the equalities we have

$$\sum_{i=1}^{M} \tilde{C}_i^T\bar{x}_i = \sum_{i=1}^{T} \bar{\lambda}_i^Tb_i - \sum_{i=2}^{M} \bar{\lambda}_i^T \sum_{j=1}^{i-1} L_{ij}\bar{x}_j$$

if we remember what is $\tilde{C}_i^T$ we then have

$$\sum_{i=1}^{M} C_i^T\bar{x}_i = \sum_{i=1}^{M} \bar{\lambda}_i^Tb_i + \sum_{i=1}^{m-1} \sum_{j=i+1}^{M} \bar{\lambda}_j^TL_{ji}\bar{x}_i - \sum_{i=2}^{M} \bar{\lambda}_i^T \sum_{j=1}^{i-1} L_{ij}\bar{x}_j$$

it is easy to see that the two last terms in the equality are equal and so the proof is complete.

Note:

It would not be difficult to show that if one $\mathcal{D}_i$ has not any solution then the original problem has no finite solution (provided a feasible one exists).

## II.1 A Sufficient Condition

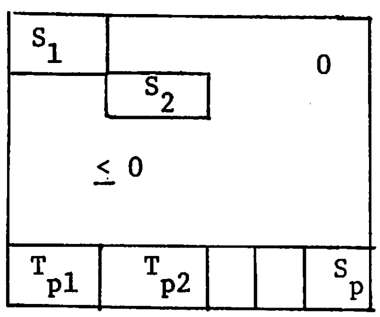
We now give a sufficient condition for the hypothesis to be true.

i) $b_i \geq 0 \quad \forall_i$

ii) $-L_{ij}$ is nonnegative $\quad \forall_i, \forall_j < i$

iii) $A_i$ may be reordered (by interchanging lines or columns) in a matrix $\tilde{A}$ described as below.

- $\tilde{A}$ is a triangular by blocks of one row each
- all the elements of the non-diagonal blocks are negative or equal to zero.




$\rightarrow \tilde{A}$

The problem $P_i$

$$\begin{cases} A_i x_i = b_i - \sum_{j=1}^{i-1} L_{ij} x_j^* \\ x_i \geq 0 \qquad\qquad\qquad\qquad (i \geq 2) \\ \text{MAX } \tilde{C}_i^T x_i \end{cases}$$

can be written now

$$\begin{cases} \tilde{A}y = \tilde{b} \\ \text{MAX } \tilde{d}^T y \qquad y \geq 0 \end{cases}$$

that is the same as

$$(4)\quad\begin{cases}\displaystyle\sum_{j=1}^{m_i} s_{ij}y_{ij} = \tilde{b}_i - \sum_{j=1}^{i-1} T_{ij}y_j \ ; \ \sum_{j=1}^{m_1} s_{1j}y_{1j} = \tilde{b}_1 \\ i = 2,\dots,p \\ \displaystyle\max \sum_{i=1}^{p}\sum_{j=1}^{m_i} \tilde{d}_{ij}y_{ij}\end{cases}$$

$m_i$ is the number of elements of $S_i$

p is the number of row of $\tilde{A}$

y is the vector $[y_1,\dots,y_i,\dots,y_p]$

and $y_i$ has $m_i$ elements $y_{ij}$.

(4) is still a multistage linear problem. We are going to see that the hypothesis is true for its subproblems.

The subproblems of (4) (equivalent to the $P_i$ for the original problem) are now

$$(5)\quad\begin{cases}\displaystyle\sum_{j=1}^{m_i} s_{ij}y_{ij} = \tilde{b}_i - \sum_{j=1}^{i-1} T_{ij}y_j^* \quad \text{for } i \geq 2 \\ y_{ij} \geq 0 \\ \displaystyle\max \sum_{j=1}^{m_i} \tilde{\tilde{d}}_{ij}y_{ij}\end{cases}$$

(5) has only one constraint and $\mathcal{D}_i$ is

$$\begin{cases}\lambda_i \geq \tilde{\tilde{d}}_{ij}/s_{ij} \text{ if } s_{ij} > 0 \\ \lambda_i \leq \tilde{\tilde{d}}_{ij}/s_{ij} \text{ if } s_{ij} < 0 \\ \displaystyle\min \lambda_i \left(\tilde{b}_i - \sum_{j=1}^{i-1} T_{ij}y_j^*\right)\end{cases}$$

Since i) and ii) hold we have: $\tilde{b}_i - \sum_{j=1}^{i-1} T_{ij}y_j^* \geq 0 \ \forall y_j^*$. So if $\max_{S_{ij}>0} \tilde{\tilde{d}}_{ij}/s_{ij}$ $\leq \operatorname{MIN}_{s_{ij}<0} \tilde{\tilde{d}}_{ij}/s_{ij}$ ($+\infty$ if $\nexists\, s_{ij} < 0$) then $\bar{\lambda}_i = \operatorname{MAX}_{s_{ij}>0} \tilde{\tilde{d}}_{ij}/s_{ij}$ (independent of

$y_j^*$ and $b_i$) else the dual has no solution, and, as the primal has a feasible solution, it has no finite solution (we suppose it exists at least one $s_{ij} > 0$ in each diagonal block $S_i$, otherwise (2) has obviously no feasible solution).

So we have shown that the dual solution of (2) is independent of $x_j^*$ (assuming the sufficient condition to be true.)

It is easy to see that when there is a finite solution for (2) the optimal basis matrix is triangular with positive elements on the diagonal and negative elements in the triangle.

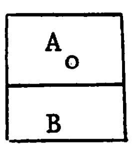
So the inverse is a nonnegative matrix.

Generally in most of the problems where constraints are linear the property is not true for the matrix of all the constraints. Only a submatrix has this property ; in next section we see how to use this fact.

## III. Applications

### III.1 Linear programming

When a submatrix (of the constraints matrix) has the property we studied in previous section we decompose the original matrix between two submatrices A and B as shown below.




We then apply the Dantzig Wolfe decomposition principle to this problem. $A_o$ will be the matrix of the "master program" and B the matrix of the subproblem to solve at each iteration.

As B has the property, the resolution of the subproblem is very fast. We need not to use a simplex algorithm; moreover there is not any numerical leeway and there is no matrix to store since we solve only one constraint linear programs! Then, practically, to solve the original problem is nearly the same as solving a problem in which the number of constraints is the row number of the matrix $A_o$.

Example 1. The dynamic Leontief model.

As described by LASDON (see [1] p. 115) the matrix of the dynamic Leontief model can be drawn as below.

$$
\begin{array}{|cccc|cccc|cccc|l}
\hline
I-A & & -B & -I & & & & & & & & & \\
I & I & -I & & & & & & & & & & \\
\hline
 & & & I & I-A & - & -B & -I & & & & & \\
 & & -I & & I & I & -I & & & & & & \\
\hline
 & & & & & & & I & I-A & & -B & -I & \\
 & & -I & & & & -I & & I & I & -I & & \\
\hline
 & & -I & & & & -I & & & & -I & & \text{etc...}
\end{array}
$$

By reordering the rows and the columns the submatrix B is then

$$
\begin{array}{|cc|c|cc|c|ccc|}
\hline
I & I & -I & & & & & & \\
\hline
 & & -I & I & I & -I & & & \\
\hline
 & & -I & & & -I & I & I & -I \\
\hline
 & & -I & & & -I & & & -I \\
\hline
\end{array}
$$

this is a multistage linear structure and the matrix of the subproblem $P_i$ (see (2) section I) is

$$\boxed{I \quad I \quad -I}$$

It is obvious that this matrix has the property. It can be reordered in diagonal blocks of one row. Each block contains 3 elements

+1 +1 −1

the matrix $L_{ij}$ are $\boxed{0 \quad 0 \quad -I}$ and then the sufficient condition is true. Note that the subproblem of the Dantzig Wolfe decomposition is then very easily solved and very fast. Now the number of constraints for the master program is exactly half as many as before. If A is triangular then we can consider B as the master program matrix and $A_o$ as the subproblem matrix.

By reordering, the matrix of the subproblem $P_i$ is now

$$\boxed{-B \quad I-A \quad I}$$

By reordering I-A and I as a block triangular matrix (since A is triangular) and since -B is negative the sufficient condition is still true.

Example 2

This example is a production planning problem. It is a middle-large linear program with 2200 constraints and 2700 variables. By using the Danztig Wolfe decomposition principle the master program has 470 constraints and the subproblem is made with five independent problems. For each problem we proved that the hypothesis as stated in section 1 is still true although the sufficient condition is no longer verified.

The equations of one of the five problems are

$$\begin{cases} x(1,k) + xs(1,k) = x(1,0) \quad k = 1,\dots,T \\ x(\alpha,k) + xs(\alpha,k) - p(\alpha-1,k)\; xs(\alpha-1,k) = x(\alpha,k-1) \\ xs(\alpha,k) \le x(\alpha,k-1) + \sum_{j=1}^{n_\alpha} x(\alpha-j,k-1) \times \prod_{\ell=1}^{j} p(\alpha-\ell,k-1) \\ k = 1,\dots,T \\ 2 \le \alpha \le M \end{cases}$$

where $x(\alpha,k)$ and $xs(\alpha,k)$ are the variables

$\rho(\alpha,k)$ is a parameter

$n_\alpha$ is a parameter $n_\alpha \leq n_{\alpha-1}+1$

This is a multistage linear problem. The sufficient condition is not verified for the subproblems $P_i$ because of the inequality constraints. Nevertheless we could show that the hypothesis (as stated in section I) is still true. For details about the proof see [2].

Thus a problem of 550 constraints and 600 variables was solved with less than 2/100 sec. on an IBM 370/168 computer. Moreover there is no matrix to store, in solving this problem and no numerical leeway.

Then at each iteration of the Dantzig Wolfe method the resolution of the subproblems is quickly performed. It allowed us to solve the original problem within 5 mns on an IBM370/168.

In other examples of linear programming problems we can check that the property (as stated in section I) is true for a submatrix of the constraints matrix (see for example [3], [4] and [5]).

III.2 Application to nonlinear programming

Let us consider the problem

$$\begin{cases} \min f(x) \\ Ax = b \\ x \geq 0 \end{cases}$$

f is a nonlinear differentiable function and the set $\{x: x\geq 0,\ Ax=b\}$ is compact. Moreover A is the matrix of a multistage linear problem and the hypothesis, as stated in section I, is true.

Then if we use the FRANCK & WOLFE method to solve the problem, finding the direction descent at iteration k is equivalent to solving

$$\begin{cases} \min\langle \nabla f(x^k), \mu\rangle \\ A\mu = b \\ \mu \geq 0 \end{cases}$$

and since the hypothesis is true this problem can be solved very quickly

- Suppose now we have to solve

$$(P) \quad \begin{cases} \min f(x) \\ Ax = b \\ x \geq 0 \\ g_i(x) \leq 0 \quad i = 1,\ldots,m \end{cases}$$

We can apply an interior point unconstrained minimization method for the constraints $g_i(t) \leq 0$ (see [7]).

We then solve the sequence of problems $P_k$.

$$(p_k) \quad \begin{cases} \min f(x) - \sum_{i=1}^{m} \frac{r_k}{g_1(x)} \\ Ax = b \\ x \geq 0 \end{cases}$$

where $r_k$ is a positive decreasing sequence whose limit is zero as $k \to \infty$.

As before we can use the Franck & Wolfe method to solve $P_k$. Under certain assumptions the solution of $P_k$ converge towards the solution of (P). The assumptions could be

- f convex, $g_r(x)$ convex
- (P) can be solved by mixed interior point and exterior point algorithm.

With this method we don't apply the penalty function, to the equality constraints. So by solving $P_k$ we always have a feasible solution to the initial problem.

In the case $g_i(x)$ are linear we can also directly solve (P) with the Franck & Wolfe method or another feasible direction method.

The linear programming problem to solve in order to find the descent direction can be solved by the Dantzig Wolfe decomposition method where the matrix A would be the matrix of the subproblem. Solving the subproblem would be very easy since the property holds for A.

## References


[1] Lasdon, L. S., "Optimization for large scale systems MacMillan series for operations research, 1970.

[2] Lasserre, J. B., "Etude de la planification a moyen terme d'une unité de fabrication," These de docteur ingenieur, Universite Paul Sabatier, Toulouse, 1978.

[3] Mahey, P., "Etude de la planification d'une unité de fabrication en vue de sa gestion integree," These de eme cycle, Universite Paul Sabatier, Toulouse, 1978.

[4] MASDEN, Oli, "A case study in problem structuring," Technical University of Denmark, IMSOR, 1978.

[5] Machado, L., "Gestion a court terme des reserves hydroelectriques (avec constraintes de vallees) a l'aide de la programmation lineaire," These de docteur Ingenieur, Universite Paul Sabatier, Toulouse, 1976.

[6] Ho, J. K. and Manne, A. S., "Nested decomposition for dynamic models," Math. Programming, Vol. 6, No. 2 (1974).

[7] Fiacco, A. J. and McCormick, G. P., Nonlinear programming: Sequential unconstrained minimization techniques, Wiley, 1968.

[8] Luenberger, D. G., Introduction to linear and nonlinear programming, Addison Wesley.