LIMITS ON POWER INJECTIONS FOR
POWER FLOW EQUATIONS TO HAVE SECURE SOLUTIONS

by

F. F. Wu and S. Kumagai

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

Limits On Power Injections For Power Flow Equations To Have Secure Solutions

Felix F. Wu and Sadatoshi Kumagai†

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

Power systems in steady-state are described by a set of nonlinear equations known as the power flow equations. The system has to be operated within the operating limits of the equipments. This is described by a set of inequality constraints, known as the security constraints. Explicit limits on the amount of power generation and load demand within which the system can be operated with all the security constraints satisfied are obtained. The problem is approached as one of determining the existence of solutions to the power flow equations in the region defined by the security constraints. Leray-Schauder fixed-point theorem and concepts from the degree of mapping are used in the derivation.

†On leave from Department of Electronics Engineering, Osaka University, Suita, Osaka, Japan

I. Introduction

The steady-state operation of an electric power supply system requires that the power supply and the load demand must be balanced. This is described by a set of nonlinear equations known as the power flow or load flow equations [1-3]. Furthermore, the system has to be operated within the designed limits of the equipments. This is described by a set of inequality constraints, sometimes referred to as the security constraints. The basic problem in steady-state analysis of a power system is to determine for a given system and a set of load demand and available generation whether the system can be operated within the security constraints. The conventional approach to this problem is to solve the power flow equations numerically and then check whether security constraints are satisfied [4]. Galiana [5] is the first one to investigate analytically the properties of the image of the power flow map. The analytic approach is of great current interest [14,15]. We propose to approach this problem as one of determining the existence of a solution to the power flow equations in the region defined by the security constraints.

The formulation of the problem is presented in Section II and the results are presented in Section III. We have obtained explicit limits on power generation and load demand to guarantee that the system can be operated with all the security constraints satisfied. The approach that we have taken to tackle the problem is to divide it into two steps. We first consider two simpler problems using the approximate formulation of decoupled power flow equations. The results that we have obtained for them are then used for solving the original problem. The analytic tools that we have utilized for the study are the Leray-Schauder fixed-point theorem and concepts from the degree of mapping [6]. For ease of reference a summary of the definition of the degree and some of its properties that we have used are included in the Appendix. The degree theory has previously been applied successfully to the investigation of the existence-of-solution problems in nonlinear circuit theory [7-9].

Because the limits on power injection that we have obtained for the power flow equations to have secure solutions are explicitly dependent on network parameters, our results can be applied to various steady-state power system analysis problems that deal with line and generator removals, such as security assessment and VAR allocation. The security assessment problem is to determine for a list of possible disturbances, each of which will result in the removal of a line of a generator, whether the system can be operated within security constraints under each of the disturbances. The VAR allocation problem is to determine how to allocate the reactive power sources (synchronous condensers, shunt capacitors, and static VAR generators) so that for a list of possible disturbances the voltage magnitudes at all buses can always be maintained within desired range.

The proofs are included in the text because we believe that they enhance the understanding of the results and they also suggest where possible improvements can be made. Standard notations are used in the paper: Q_k denotes the k-th component of the vector $\underline{Q}$, Y_{ki} denotes the ki-th element of the matrix Y, $\underline{V} \leq \underline{V}^M$ means $V_i \leq V_i^M$ for all i, A:=B means that A is defined by the expression B, and $\| \cdot \|$ denotes the Euclidean norm.

II. Formulation

1. Power Flow Equations

The branches of a power network represent transmission lines, transformers, etc., which are modeled as linear time-invariant RLC elements. The nodes of the network other than the ground node are called buses. They correspond to generation stations and load-center substations. For steady-state analysis the network is considered as in sinusoidal steady-state.

Consider a power network with N+1 buses. Let [Y] denote the node (bus) admittance matrix of the network and $Y_{ki} = G_{ki} + jB_{ki}$ be its ki-th element. Using the standard models of transmission lines and transformers [3, p.189 and p.122], we have†

<u>Fact 1</u> $G_{kk} > 0$, $B_{kk} < 0$; $G_{ki} \leq 0$ and $B_{ki} \geq 0$ for $i \neq k$.

$$|B_{kk}| \geq \sum_{\substack{i=0\\ i\neq k}}^{N} B_{ki} \text{ and } G_{kk} \geq \sum_{\substack{i=0\\ i\neq k}}^{N} |G_{ki}|.$$

Let E_k denote the bus voltage phasor of bus k and $S_k = P_k + jQ_k$ denote the injected complex power at bus k. Let $\underline{E}$ and $\underline{P}$ be the vectors of complex voltages and power injections, respectively. For convenience, we introduce a diagonal matrix $[E] = \text{diag}\{E_1, E_2, \ldots, E_N\}$. Then we have

$$\underline{S}^* = [E^*][Y]\underline{E} \qquad (1)$$

where superscript * denotes complex conjugate. There are three types

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†In our model, no load is represented as a shunt impedance.

of buses:

(i) Slack bus: a bus whose voltage magnitude and phase angle are specified.

(ii) PQ bus: a bus where the injected real and reactive power are specified.

(iii) PV bus: a bus where the injected real power and the voltage magnitude are specified.

Normally PQ buses are load buses and PV buses and the slack bus are generator buses. We let subscript 0 correspond to the slack bus, subscripts $\{1,2,\ldots N_Q\}$ correspond to PQ buses, and subscripts $\{N_Q+1,\ldots,N\}$ correspond to PV buses. Let $E_k = V_k e^{j\theta_k}$ and $\theta_{ki} = \theta_k - \theta_i$. We may express (1) as

$$\begin{cases} \sum_{i=0}^{N} V_k V_i \,(G_{ki} \sin \theta_{ki} - B_{ki} \cos \theta_{ki}) = Q_k & k=1,2,\ldots N_Q \quad (2) \\ \sum_{i=0}^{N} V_k V_i \,(G_{ki} \cos \theta_{ki} + B_{ki} \sin \theta_{ki}) = P_k & K=1,2,\ldots N \quad (3) \end{cases}$$

where $\underline{V} = (V_1, V_2, \ldots V_{N_Q})^T$ and $\underline{\theta} = (\theta_1, \theta_2, \ldots \theta_N)^T$ are the unknown variables. Equations (2) and (3) are known as the <u>power flow equations</u> [1-3].

2. Decoupled Power Flow

Suppose that we make the following simplifying assumptions:

<u>(SA1)</u>. The line resistances are negligible, i.e., $G_{ki} = 0$.

<u>(SA2)</u>. The phase angles across the branches $\theta_{ki} = \theta_k - \theta_i$ are small

so that the second and higher order terms in the sin θ_{ki} and cos θ_{ki} series are negligible, i.e., cos $\theta_{ki} \approx 1$, sin $\theta_{ki} \approx \theta_{ki}$.

Then the power flow equations (2)(3) become

$$Q_k = \tilde{Q}_k(\underline{V}) := -V_k \sum_{i=0}^{N} B_{ki} V_i \qquad k=1,2,\ldots N_Q \quad (4)$$

$$P_k = \tilde{P}_k(\underline{V}, \underline{\theta}) := V_k \sum_{i=0}^{N} B_{ki} V_i (\theta_k - \theta_i) \qquad k=1,2,\ldots N \quad (5)$$

Equations (4) and (5) may be written in a compact matrix form

$$-[V]\{[B]\underline{V} + [B°]\underline{V}°\} = \underline{Q} \quad (6)$$

$$[A(\underline{V})]\underline{\theta} = \underline{P} \quad (7)$$

where $\underline{V}° = (V_0, V_{N_Q+1}, \ldots V_N)^T$, $\underline{Q} = (Q_1, \ldots Q_{N_Q})^T$, $\underline{P} = (P_1, P_2 \ldots P_N)^T$, the ki-th element of [B] is B_{ki}, $k, i \in \{1,2,\ldots N_Q\}$ and the elements of [B°] are B_{ki}, $k \in \{1,\ldots N_Q\}$, $i \in \{0, N_Q+1, \ldots N\}$.

$[A(\underline{V})]$ is an NxN matrix whose elements are functions of $\underline{V}$. Its diagonal and off-diagonal elements are

$$[A(\underline{V})]_{kk} = V_k \left(\sum_{\substack{i=0 \\ i \neq k}}^{N} B_{ki} V_i \right) \qquad k=1,2,\ldots N \quad (8)$$

$$[A(\underline{V})]_{ki} = -V_k V_i B_{ki} \qquad k \neq i, \; k,i=1,2,\ldots N \quad (9)$$

Equations (6) and (7) are known as the decoupled power flow equations [10,11].

3. Security Constraints

Operating limits are imposed on the voltage magnitudes of PQ buses, i.e.,

$$\underline{V}^m \leq \underline{V} \leq \underline{V}^M \tag{10}$$

Let us denote the region inside the limits by R_V, i.e.,

$$R_V: = \{\underline{V} | \; \underline{V} \leq \underline{V} \leq \underline{V}^M\} \tag{11}$$

Thermal considerations limit the amount of current flowing through transmission lines and transformers. The current I_j through branch j connecting bus k and bus i may be approximated as follows:

$$\begin{aligned} I_j &= -Y_{ki} \; (E_k - E_i) \\ &= -jB_{ki} \; (V_k e^{j\theta_k} - V_i e^{j\theta_i}) \\ &= -jB_{ki} V_k e^{j\theta_i} \; (\cos \theta_{ki} + j \sin \theta_{ki} - \frac{V_i}{V_k}) \\ &\simeq -jB_{ki} V_k e^{j\theta_i} \; (j\theta_{ki}) \end{aligned}$$

Where we used the approximations $\cos\theta_{ki} \simeq 1$ and $\frac{V_i}{V_k} \simeq 1$.†

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†This is true when per-unit system is used.

Hence

$$|I_j| \approx |B_{ki}||V_k||\theta_k - \theta_i| \tag{12}$$

Therefore the line flow constraints may be expressed in terms of $\theta_k - \theta_i$, i.e.,

$$-\delta_j \leq \theta_k - \theta_i \leq \delta_j \qquad \text{Whenever } j \text{ is a branch connecting buses } k \text{ and } i \tag{13}$$

where $\delta_j < \frac{\pi}{2}$. Or we may use the incidence matrix A of the network and write (13) as

$$\underline{-\delta} \leq A^T\underline{\theta} \leq \underline{\delta} \tag{14}$$

Let us denote the corresponding region in $\underline{\theta}$ by R_θ:

$$R_\theta : = \{\underline{\theta} | \ -\underline{\delta} \leq A^T\underline{\theta} \leq \underline{\delta}\} \tag{15}$$

We are going to use the approximate expression (14) for line flow constraints even when the full-fledged power flow equations are used. The justification for this is that unlike the "hard" constraints on power generation due to equipment limitation that will be introduced shortly, the line flow constraints are "soft" constraints for which approximation is usually adequate.

Physical limitation imposes constraints on the amount of real and reactive power that can be generated at PV buses, as well as the slack bus. The real power constraints on PV buses are

$$p_k^m \leq P_k \leq p_k^M \qquad k=N_Q+1,\ldots N \tag{16}$$

The real power generation from the slack bus is a function of $(\underline{V}, \underline{\theta})$ so the constraints are of the form

$$p_0^m \leq P_0(\underline{V},\underline{\theta}) \leq p_0^M \tag{17}$$

Let us denote the region in which (17) is satisfied to be

$$R_p: = \{(\underline{V},\underline{\theta}) |\ p_0^m \leq P_0(\underline{V},\underline{\theta}) \leq p_0^M\} \tag{18}$$

Under the assumption (SA1), $P_0 = -\sum_{i=1}^{N} P_i$, independent of $(\underline{V},\underline{\theta})$, (17) merely imposes a constraint on P_k, $k=1,2,\ldots N_Q$. The reactive power generation at the slack bus or a PV bus k is

$$Q_k(\underline{V},\underline{\theta}) = \sum_{i=0}^{N} V_k V_i \ (G_{ki} \sin \theta_{ki} - B_{ki} \cos \theta_{ki}) \tag{19}$$

$$k=0,N_Q+1,\ldots N$$

The reactive power constraints may be expressed as

$$q_k^m \leq Q_k\ (\underline{V},\underline{\theta}) \leq q_k^M \qquad K=0,N_Q+1,\ldots N \tag{20}$$

Let us denote the region in which (20) is satisfied as R_q

$$R_q: = \{(\underline{V},\underline{\theta}) |\ q_k^m \leq Q_k(\underline{V},\underline{\theta}) \leq q_k^M,\ k=0,N_Q+1,\ldots N\} \tag{21}$$

Under the assumptions (SA1)(SA2) we can approximate the reactive power

(19) by

$$\tilde{q}_k(\underline{V}) = -V_k \sum_{i=0}^{N} B_{ki} V_i \qquad k=0, N_Q+1, \ldots N \qquad (22)$$

The corresponding constraints (20) become

$$q_k^m \leq \tilde{q}_k(\underline{V}) \leq q_k^M \qquad k=0, N_Q+1, \ldots N \qquad (23)$$

Note that eq. (23) is a set of linear constraints on $\underline{V} = (V_1, \ldots V_{N_Q})^T$, because V_k, $k=0$, $N_Q+1, \ldots N$ of the PV buses are known. Let us denote the region in $\underline{V}$-space for which (23) is satisfied to be R'_q.

$$R'_q : = \{\underline{V} | \; q_k^m \leq \tilde{q}_k \; (\underline{V}) \leq q_k^m, \; k=0, \; N_Q+1, \ldots N\} \qquad (24)$$

We call the constraints (10) (14) (17) and (20) the <u>security constraints</u>, and the set in $(\underline{V}, \underline{\theta})$-space

$$R: = (R_V \times R_\theta) \cap R_p \cap R_q$$

the <u>security region</u>. If the approximate expression for the reactive power constraints (23) is used, and the slack bus generation con= straints are satisfied, then the security region becomes

$$R' : = (R_V \cap R'_q) \times R_\theta$$

4. Problem Formulation

The steady-state analysis problem is to determine for a given set of load demand and generation pattern, can the system be operated so that all the equipments are loaded within their operating limits. In terms of the power flow model this problem may be stated as follows:

(P) Given a set of power injections ($\underline{P}$, $\underline{Q}$), determine whether there exists a solution to the power flow equations (2) - (3) that lies in the security region R.

We shall first consider the simpler model of the decoupled power flow equations. Using this model, because of the decoupling of Q-V and P-θ in equations (6) - (7), we may split the problem into two, namely:

(P1) Given a set of reactive power injection $\underline{Q}$, determine whether there exists a solution $\underline{V}$ to eq. (6) that lies in $R_V \cap R_q'$.

(P2) Suppose that the answer to (P1) is affirmative and given a set of real power injection $\underline{P}$, determine whether there exists a solution $\underline{\theta}$ to eq. 7 that lies in R_θ.

We are going to present sufficient conditions on $\underline{Q}$ and $\underline{P}$ to guarantee the existence of solution to (6) and (7) in $R_V \cap R_q'$ and R_θ, respectively. We then use the result to solve (P).

III. Existence Theorems

1. Secure Solution of Decoupled QV Equations

Consider Problem (P1), i.e., to determine the existence of a solution $\underline{V}$ to

$$- [V] \{[B]\underline{V} + [B^\circ]\underline{V}^\circ\} = \underline{Q} \tag{25}$$

in $R_V \cap R_q'$, i.e., $\underline{V}^m \le \underline{V} \le \underline{V}^M$, and $\underline{q}^m \le \tilde{\underline{q}}(v) \le \underline{q}^M$. We make two assumptions on the power system and its security constraints.

Assumption (A1): $[B]\ \underline{V}^m \geq [B]\ \underline{V}^M$

Assumption (A2): For any k such that

$$([B]\underline{V}^m)_k = ([B]\underline{V}^M)_k$$

there exists a j such that $B_{kj} \neq 0$ and

$$([B]\underline{V}^m)_j > ([B]\underline{V}^M)_j \tag{26}$$

Remark 1. Consider the special case where the range of voltage magnitude is the same at all buses, i.e., $V_k^M - V_k^m = \alpha > 0$, for all k. The components of $[B]\ (\underline{V}^M - \underline{V}^m)$ in this case are α times the negative of the row sums of [B], which are nonpositive by Fact 1, hence assumption (A1) is satisfied. Moreover $([B]\underline{V}^M)_k = ([B]\underline{V}^m)_k$ if and only if the PQ bus k is not connected to any PV bus. Thus assumption (A2) implies, in this case, that for any PQ bus k that is not connected to a PV bus there is a neighboring bus j that does. If, however, assumption (A2) is not satisfied, we may use other alternative assumption (A2)' or (A2)", which will be presented after the proof of Theorem 1. (Remark 2).

Theorem 1 (Existence of Secure Solution to Decoupled QV Equation).

Suppose that the assumptions (A1) and (A2) are satisfied. If the reactive power injection Q_k at any PQ bus k satisfies the following condition

$$\text{(C1)} \qquad \tilde{Q}_k(\underline{V}^m) \leq Q_k \leq \tilde{Q}_k(\underline{V}^M) \qquad k=1,2,\ldots N_Q.$$

and the reactive power limits at any PV bus j satisfies the following condition

$$\text{(C2)} \qquad \tilde{q}_j(\underline{V}^m) \leq q_j^M \text{ and } q_j^m \leq \tilde{q}_j(\underline{V}^M),\ j=0,\ N_Q+1,\ldots N.$$

then the decoupled reactive power flow equation (6) has a solution $\underline{V}$ in the secure region $R_V \cap R_q'$.

Proof: We first claim that for any $\underline{V} \in R_V$, (C2) implies $\underline{V} \in R_q'$. Note that $B_{ki} \geq 0$ (Fact 1) and V_k, $k \in \{0, N_Q+1, \ldots N\}$, are fixed, so for any $\underline{V} = (V_1, V_2 \ldots V_{N_Q})^T$ such that $V_i^m \leq V_i \leq V_i^M$, $i \in \{1, \ldots N_Q\}$, we have

$$\tilde{q}_k(\underline{V}^M) \leq \tilde{q}_k(\underline{V}) \leq \tilde{q}_k(\underline{V}^m), \quad k=0, N_Q+1, \ldots N \tag{27}$$

(C2) and (27) imply that $\underline{V} \in R_q'$. Hence, in this case, $R_v \cap R_q' = R_v$.

The problem is then tackled by the application of Leray-Schauder fixed-point theorem [6, p. 162]. Let C be an open and bounded set in $\mathbb{R}^n$ containing the origin and $\underline{f} : C \to \mathbb{R}^n$ be a continuous function. If $\underline{f}(\underline{x}) \neq \lambda \underline{x}$ for $\lambda > 1$ and $\underline{x}$ on the boundary of C ($\underline{x} \in \partial C$), then $\underline{f}$ has a fixed point $\underline{x}^*$, $\underline{f}(\underline{x}^*) = \underline{x}^*$, in the closure of C ($\underline{x}^* \in \bar{C}$).

First we rewrite eq. (6) as

$$\underline{V} = [B]^{-1} \{-[V]^{-1}\underline{Q} - [B°]\,\underline{V}°\} \tag{28}$$

Note that [B] is irreducibly diagonally dominant, hence nonsingular [6, pp.48-49]. In order to have the set of interest R_V containing the origin, let us shift the origin to $\underline{V}^* = \dfrac{\underline{V}^m + \underline{V}^M}{2}$ and set $\underline{x} := \underline{V} - \underline{V}^*$. In terms of $\underline{x}$, eq. (28) becomes

$$\underline{x} = \underline{f}(\underline{x}) \triangleq [B]^{-1} \{-[x+V^*]^{-1}\,\underline{Q} - [B°]\,\underline{V}°\} - \underline{V}^* \tag{29}$$

and the constraint set R_v becomes

$$C = \{ \underline{x} \mid \frac{\underline{V}^m - \underline{V}^M}{2} < \underline{x} < \frac{\underline{V}^M - \underline{V}^m}{2} \} \tag{30}$$

Note that $\underline{x} \in C$ iff $\underline{V} \in R_v$ where $\underline{x} = \underline{V} - \underline{V}^*$.

The condition $\underline{f}(\underline{x}) \neq \lambda \underline{x}$ can be expressed as

$$\lambda \underline{\beta}(\underline{x}) + \underline{\gamma}(\underline{x}) \neq 0 \tag{31}$$

where $$\beta_k(\underline{x}) = ([B]\underline{x})_k \tag{32}$$

$$\gamma_k(\underline{x}) = ([B]\underline{V}^*)_k + ([B^\circ]\underline{V}^\circ)_k - (x_k + V_k^*)^{-1} Q_k \tag{33}$$

Clearly

$$\lambda\beta_k + \gamma_k \neq 0 \text{ for } \lambda > 1$$

iff $$(\lambda-1)\beta_k + (\beta_k + \gamma_k) \neq 0 \quad \text{for } \lambda > 1$$

iff $$\beta_k(\beta_k + \gamma_k) \geq 0 \quad \text{but excluding } \beta_k = \gamma_k = 0$$

since $(\lambda-1) > 0$.

We now claim that condition (C1), together with assumptions (A1) and (A2), implies that for any point $\underline{x}$ on the boundary of C there exists a k such that

$$\beta_k(\underline{x})\{\beta_k(\underline{x}) + \gamma_k(\underline{x})\} \geq 0 \text{ and } \beta_k(\underline{x}) \neq 0.$$

The boundary of C is the union of the boundary defined by each x_k. Let ∂C_k^+ and ∂C_k^- be the boundary of C defined by $x_k = \frac{V_k^M - V_k^m}{2}$ and $x_k = \frac{V_k^m - V_k^M}{2}$ respectively.

Consider any point $\underline{x} \in \partial C_k^+$. We have $x_j \leq \frac{V_j^M - V_j^m}{2}$ for all j. Also recall that $B_{kk} < 0$ and $B_{kj} \geq 0$ for $j \neq k$. Thus

$$\beta_k(\underline{x}) = ([B]\,\underline{x})_k$$

$$= \sum_{j=1}^{NQ} B_{kj}\, x_j$$

$$\leq \sum_{j=1}^{NQ} B_{kj}\, \left(\frac{V_j^M - V_j^m}{2}\right)$$

$$= \left([B]\,\frac{\underline{V}^M - \underline{V}^m}{2}\right)_k \leq 0 \qquad (34)$$

The last inequality is due to assumption (A1). Moreover,

$$\beta_k(\underline{x}) + \gamma_k(\underline{x}) \leq \left([B]\,\frac{\underline{V}^M - \underline{V}^m}{2}\right)_k + \gamma_k(\underline{x})$$

$$= ([B]\,\underline{V}^M + [B^\circ]\,\underline{V}^\circ)_k + \frac{1}{V_k^M} Q_k$$

$$= \frac{1}{V_k^M} \{-\tilde{Q}_k(\underline{V}^M) + Q_k\} \leq 0 \qquad (35)$$

The last inequality is due to condition (C1).

Similarly for any point $\underline{x} \in \partial C_k^-$, we have

$$\beta_k(\underline{x}) \geq \left([B]\,\frac{\underline{V}^m - \underline{V}^M}{2}\right)_k \geq 0 \qquad (36)$$

and

$$\beta_k(\underline{x}) + \gamma_k(\underline{x}) \geq ([B]\frac{\underline{V}^m - \underline{V}^M}{2})_k + \gamma_k(\underline{x})$$

$$= \frac{1}{V_k^m}\{-\tilde{Q}_k(\underline{V}^m) + Q_k\} \geq 0 \qquad (37)$$

Now if for this k

$$([B]\underline{V}^m)_k > ([B]\underline{V}^M)_k \qquad (38)$$

then on ∂C_k we have $\beta_k(\beta_k + \gamma_k) \geq 0$ and $\beta_k \neq 0$. Our claim is thus established. However, if for this k

$$([B]\underline{V}^m)_k = ([B]\underline{V}^M)_k \qquad (39)$$

Then $\forall \hat{\underline{x}} \in \partial C_k^+$ consider a j for which $B_{kj} \neq 0$, either $\hat{x}_j < \frac{V_j^M - V_j^m}{2}$ then $\beta_k(\hat{\underline{x}}) < 0$ or $\hat{x}_j = \frac{V_j^M - V_j^m}{2}$, in this case we have

$$\beta_k(\hat{\underline{x}}) = ([B]\hat{\underline{x}})_k = ([B]\frac{\underline{V}^M - \underline{V}^m}{2})_k = 0 \qquad (40)$$

But in this case $\hat{\underline{x}} \in \partial C_j^+$ for all j such that $B_{kj} \neq 0$. Assumption (A2) guarantees that for one of these j's, $\beta_j(\hat{\underline{x}}) < 0$. Since $\hat{\underline{x}} \in \partial C_j^+$, $\beta_j(\beta_j + \gamma_j) \geq 0$. Therefore, our claim, though is not true for that k at $\hat{\underline{x}}$, still holds for this j at $\hat{\underline{x}}$. Similarly for the case $\hat{\underline{x}} \in \partial C_k^-$. The proof is thus completed. ¤

<u>Remark 2.</u> It is clear from the proof that assumption (A2) can be replaced by either one of the following:

(A2)': For any k such that

$$([B]\,\underline{V}^m)_k = ([B]\,\underline{V}^M),$$

the condition (c) is replaced by

$$\tilde{Q}_k\,(\underline{V}^m) < Q_k < \tilde{Q}_k\,(\underline{V}^M)$$

(A2)": For any k such that

$$([B]\,\underline{V}^m)_k = ([B]\,\underline{V}^M)_k$$

there exists a j such that $B_{kj} \neq 0$ and

$$\tilde{Q}_j\,(\underline{V}^m) < Q_j < \tilde{Q}_j\,(\underline{V}^M)$$

At this juncture we would like to present sufficient conditions under which the Jacobian of the decoupled reactive power flow map will have a positive determinant. This will be used later in Sec. III-3.

Assumption (A3): The power network is connected.

Assumption (A4): $2V_k > V_i \quad i,k = 1,2,\ldots,n.$

Assumption (A5):

$$\tilde{Q}_k\ (V_k=V_k^m;\ V_i=V_i^m \text{ for } i=1,\ldots,N_Q \text{ and } i\neq j) \geq -(\sum_{i=0,N_Q+1} B_{ki})(V_k^m)$$

and the strict inequality holds for at least one k.‡

Fact 2 If assumptions (A3)-(A5) hold, then the Jacobian $D\tilde{\underline{Q}}\ (\underline{V})$ of the decoupled reactive power flow map $\tilde{\underline{Q}}\ (\cdot)$ is nonsingular for all $\underline{V}$ in R_V. Furthermore, det $D\tilde{\underline{Q}}\ (\underline{V}) > 0 \quad \forall \underline{V} \in R_V$.

Proof: From the definition of $Q_k\ (\underline{V})$ in eq (4), we have

$$[DQ\ (\underline{V})]_{kk} = -2B_{kk}V_k - \sum_{\substack{i=0\\ i\neq k}}^{N} B_{ki}V_i, \quad k=1,2,\ldots,N_Q \tag{41}$$

$$[DQ\ (\underline{V})]_{ki} = -B_{ki}V_k, \quad i=1,2,\ldots,N_Q,\ \ i\neq k \tag{42}$$

Assumption (A3) implies that $D\underline{Q}$ is irreducible [6, pp 46-47]. Consider

$$|2B_{kk}V_k + \sum_{\substack{i=0\\ i\neq k}}^{N} B_{ki}V_i| - \sum_{\substack{i=1\\ i\neq k}}^{N_Q} |B_{ki}V_k| \tag{43}$$

$$\geq 2|B_{kk}V_k| - \sum_{\substack{i=0\\ i\neq k}}^{N} |B_{ki}V_i| - \sum_{\substack{i=1\\ i\neq k}}^{N_Q} |B_{ki}V_k| \tag{44}$$

‡ $\sum_{i=0,N_Q+1}^{N} B_{ki}$ denotes $B_{ko} + \sum_{i=N_Q+1}^{N} B_{ki}$.

Using Fact 1, $|B_{kk}| \geq \sum_{\substack{i=0 \\ i\neq k}}^{N} B_{ki}$, we have

$$\geq |B_{kk}V_k| + \sum_{\substack{i=0 \\ i\neq k}}^{N} B_{ki}V_k - \sum_{\substack{i=1 \\ i\neq k}}^{N_Q} B_{ki}V_k - \sum_{\substack{i=0 \\ i\neq k}}^{N} B_{ki}V_i \tag{45}$$

$$\geq (-B_{kk})V_k^m + (\sum_{i=0,N_Q+1}^{N} B_{ki})V_k^m - \sum_{\substack{i=1 \\ i\neq k}}^{N_Q} B_{ki}V_k - \sum_{i=0,N_Q+1}^{N} B_{ki}V_i \tag{46}$$

$$= \frac{1}{V_k^m} \{\tilde{Q}_k(V_k=V_k^m;V_i=V_i^M \text{ for } i = 1,\ldots,N_Q \text{ and } i \neq k)$$

$$+ (\sum_{i=0,N_Q+1}^{N} B_{ki})(V_k^m)^2 \} \tag{47}$$

$$\geq 0 \tag{48}$$

The last inequality is true because of assumption (A5). Note that we have used the fact that $\underline{V} \in R_V$ in the inequality from (45) to (46). Hence $D\tilde{Q}(\underline{V})$ is diagonally dominant for all $\underline{V}$ in R_V, and strict inequality on (45) holds for at least one k. Therefore, $D\tilde{\underline{Q}}(\underline{V})$ is nonsingular [6, pp. 48-49].

Furthermore $[D\tilde{Q}(V)]_{ki} \leq 0$ and assumption (A4) implies $[D\tilde{Q}(V)]_{kk} > 0$, so $D\tilde{Q}(\underline{V})$ is in fact an M-matrix [6,p.55] and det $D\tilde{\underline{Q}}(\underline{V}) > 0$ [12]. ¤

2. Secure Solutions of Decoupled Pθ Equations

Consider Problem (P2), i.e., to determine the existence of a solution $\underline{\theta}$ to the equation

$$[A(\underline{V})]\ \underline{\theta} = \underline{P}$$

such that $\underline{\theta}$ is in $R_\theta = \{\underline{\theta} | -\underline{\delta} \leq A^T\underline{\theta} \leq \underline{\delta}\}$ given that $\underline{V}$ is in $R_V = \{\underline{V} | \underline{V}^m \leq V \leq V^M\}$. Equation (7) is linear in $\underline{\theta}$. We first establish the nonsingularity of the matrix $[A(\underline{V})]$. In fact $[A(\underline{V})]$ is more than nonsingular if we make the following assumption.

Assumption (A6). There is no phase-shifting transformers in the system, so that $B_{ki} = B_{ik}$, $ki = 1,\ldots,N$.

Fact 3. If assumption (A3) holds, then $[A(\underline{V})]$ is nonsingular and $\det[A(\underline{V})] > 0$. If assumptions (A3) and (A6) hold, then $[A(\underline{V})]$ is positive definite.

Proof: Consider the elements of $[A(\underline{V})]$ (eqs. (8)(9)). The network is connected implies that $[A(\underline{V})]$ is irreducible. Clearly

$$|[A(\underline{V})]_{kk}| \geq \sum_{\substack{i=1 \\ i \neq k}}^{N} |[A(\underline{V})]_{ki}| \tag{49}$$

and strict inequality holds for those buses k that are connected to the slack bus. Therefore $[A(\underline{V})]$ is nonsingular $\forall \underline{V} \in R_V$ [6,pp.48-49]. Furthermore, $[A(\underline{V})]_{kk} > 0$ and $[A(\underline{V})]_{ki} \leq 0$ for $i \neq k$, hence $[A(\underline{V})]$ is in fact an M-matrix [6,p.55] and $\det[A(\underline{V})] > 0 \; \forall \; \underline{V} \in R_V$ [12]. Furthermore, under assumption (A6), $[A(\underline{V})]$ is symmetric, hence a Stieltjes matrix [6,p.54] and is positive definite [6,p.55]. ¤

The region defined by R_θ is a polyhedron. For analysis, it is more convenient to consider the largest ball B_θ that is contained in R_θ. We therefore actually consider in what follows the modified problem to (P2), i.e., determining the existence of a solution of eq. (7) in B_θ.

Fact 4. Let us define the ball

$$B_\theta := \{\underline{\theta} \,|\, \|\underline{\theta}\| < r\} \tag{50}$$

where

$$r := \frac{1}{\sqrt{2}} \min_j (\delta_j) \tag{51}$$

then $B_\theta \subseteq R_\theta$.

Proof. The boundary of R_θ is defined by the pairs of hyperplanes of the form

$$\theta_k - \theta_i = \pm\delta_j$$

Clearly the ball $\{\underline{\theta} | \|\underline{\theta}\| < \frac{1}{\sqrt{2}} \delta_j\}$ lies inside the region $\{\underline{\theta} | -\delta_j \leq \theta_k - \theta_i \leq \delta_j\}$. By the definition of r, we have $B_\theta \subseteq R_\theta$. ⌑

Theorem 2. (Existence of Secure Solutions to Decoupled Pθ Equations)

Suppose that the assumptions (A3) and (A6) are satisfied. If the real power injection $\underline{P}$ satisfies the following condition

(C3) $\|\underline{P}\| < r\rho$

where

$$\rho := \min_{\underline{V} \in R_V} \left(\begin{array}{c} \text{smallest eigenvalue} \\ \text{of } [A(\underline{V})] \end{array} \right) \qquad (52)$$

then the decoupled real power flow equation (7) has a solution $\underline{\theta}$ in the region R_θ.

Proof. We are going to apply the translation invariance property of the degree [Appendix (D2)] to the linear map

$$\underline{\tilde{P}}(\underline{V},\cdot) : \underline{\theta} \rightarrow [A(\underline{V})]\underline{\theta} \qquad (53)$$

First note that $\underline{\theta} = 0$ is a solution in B_θ of (7) for $\underline{P} = 0$. By Fact 3, namely, $\det[A(\underline{V})] > 0$, we have $\deg(\underline{\tilde{P}}(\underline{V},\cdot), B_\theta, 0) = 1$ [Appendix].

For any $\underline{P}$ such that (C3) is satisfied, we claim that for any $\underline{V}$ in R_V.

$$\underline{\tilde{P}}(\underline{V},\underline{\theta}) \neq t\underline{P} \qquad t \in [0,1] \text{ and } \underline{\theta} \in \partial B_\theta \qquad (54)$$

Since $[A(\underline{V})]$ is positive definite (Fact 3) ρ defined in eq. (52) is positive and we have [13,p.312] $\forall \underline{V} \in R_V$

$$\|[A(\underline{V})]\underline{\theta}\| \geq \rho\|\underline{\theta}\| \qquad (55)$$

So $\|[A(\underline{V})]\underline{\theta}\| \geq r\rho$ on ∂B_θ. Combining this with condition (C3) on $\underline{P}$ we establish the claim (54).

By the translation invariance property of the degree we have $\deg(\underline{\tilde{P}}(\underline{V},\cdot),B_\theta,\underline{P}) = 1$ for any $\underline{P}$ satisfying (C3). Theorem 2 then follows from Kronecker theorem [Appendix (D1)]. ¤

3. Secure Solutions of Power Flow Equations

We are now ready to tackle Problem (P) utilizing the results we have obtained for the decoupled power flow equations. Let us define a vector $\underline{\varepsilon}$ which is an error bound on the approximation by the decoupled power flow equations.

$$\underline{\varepsilon}_k^Q := \sum_{\substack{i=0 \\ i\neq k}}^{N} V_k^M V_i^M \{B_{ki}(1-\cos\delta) - G_{ki}\sin\delta\} \qquad k = 1,2,\ldots,N_Q \qquad (56)$$

$$\varepsilon_k^P := \sum_{\substack{i=0 \\ i\neq k}}^{N} V_k^M V_i^M \{B_{ki}(\delta^M - \sin\delta) - G_{ki}\} + G_{kk}(V_k^M)^2 \qquad k = 1,2,\ldots,N \quad (57)$$

$$\varepsilon_k^q := \sum_{\substack{i=0 \\ i\neq k}}^{N} V_k^M V_i^M \{B_{ki}(1-\cos\delta) - G_{ki}\sin\delta\} \qquad k = 0,N_Q+1,\ldots,N \qquad (58)$$

where

$\delta = \max_j\{\delta_j\}$ and we set

$V_k^M = V_k$ for $k = 0,N_Q+1,\ldots,N$.

<u>Theorem 3</u>. (Existence of Secure Solutions to Power Flow Equations)

Suppose that the assumptions (A1)-(A6) are satisfied. Suppose, furthermore, that the reactive power limits at the stack bus or any PV bus j satisfying the following condition.

(C4) $\tilde{q}_j(\underline{V}^m) + \underline{\varepsilon}_j^q \leq q_j^M$ and $q_j^m \leq \tilde{q}_j(\underline{V}^M) - \varepsilon_j^q \qquad j = 0, N_Q+1, \ldots, N$

If the power injection $(\underline{Q}, \underline{P})$ satisfies the following conditions

(C5) For reactive power injection Q_k at any PQ bus k,

$$\tilde{Q}_k(\underline{V}^m) + \underline{\varepsilon}_k^Q < Q_k < \tilde{Q}_k(\underline{V}^M) - \underline{\varepsilon}_k^Q \qquad k = 1,2,\ldots,N_Q$$

(C6) For real power injection $\underline{P} = (P_1, P_2, \ldots, P_N)$,

$$\|\underline{P}\| < r_\rho - \|\underline{\varepsilon}^P\|$$

and

(C7) $p_0^m \leq (-\sum_{i=1}^{N} P_i) \leq p_0^M - L^M$

where

$$L^M := \max_{\underline{V} \in R_V} \{\sum_{k=0}^{N} V_k^2 G_{kk} + \sum_{k=0}^{N} \sum_{\substack{i=0 \\ i \neq k}}^{N} V_k V_i G_{ki} \cos \delta_j\} \tag{59}$$

then the power flow equations (2) (3) have a solution $(\underline{V}, \underline{\theta})$ in the secure region R.

<u>Proof</u>. First note that for $\underline{V} \in R_V$ and $\underline{\theta} \in R$

$$\tilde{q}_j(\underline{V}^M) - \underline{\varepsilon}_j^q \leq Q_j(\underline{V}, \underline{\theta}) \leq \tilde{q}_j(\underline{V}^m) + \underline{\varepsilon}_j^q \qquad j = 0, N_Q+1, \ldots, N \tag{60}$$

Condition (C4) implies that $(R_V x R_\theta) \subseteq R_q$.

From the power flow equations

$$P_0 + \sum_{i=1}^{N} P_i = \sum_{k=0}^{N} \sum_{i=0}^{N} V_k V_i G_{ki} \cos \theta_{ki} \tag{61}$$

Clearly L^M is an upper bound of the right hand expression (line losses) and 0 is a lower bound of it. Hence (C7) guarantees constraint (13).

Consider the decoupled power flow map

$$F:(\underline{V},\underline{\theta}) \to (\tilde{\underline{Q}}(\underline{V}),\tilde{P}(\underline{V},\underline{\theta}))^T \tag{62}$$

We are going to construct a trivial homotopy from F to the power flow map and apply the homotopy invariance property of the degree [Appendix (D3)].

First we claim that $\deg(F,R_V x B_\theta,y) \neq 0$ for any

$$y \in K := \{(\underline{Q},\underline{P})|\underline{Q} \text{ satisfies (C5) and } \underline{P} \text{ satisfies (C6)}\} \tag{63}$$

The Jacobian of F is given by

$$DF(\underline{V},\underline{\theta}) = \begin{bmatrix} D\tilde{Q}(\underline{V}) & 0 \\ * & [A(\underline{V})] \end{bmatrix} \tag{64}$$

$\det DF(\underline{V},\underline{\theta}) = \det D\tilde{Q}(\underline{V})\cdot\det[A(\underline{V})] > 0$ by Facts 2 and 3. This, together with Theorems 1 and 2, establishes the claim.

Let us introduce the following homotopy $H:(\underline{V},\underline{\theta}) \to (H^Q(\underline{V},\underline{\theta}),\ H^P(\underline{V},\underline{\theta}))^T$ from the decoupled power flow map F to the power flow map defined by eqs. (2) (3).

$$H((\underline{V},\underline{\theta}),t) := F(\underline{V},\underline{\theta}) + tG(\underline{V},\underline{\theta}) \qquad 0 \leq t \leq 1 \tag{65}$$

where $G:(\underline{V},\underline{\theta}) \to (G^Q(\underline{V},\underline{\theta}),\ G^P(\underline{V},\underline{\theta}))^T$ is defined by

$$G_k^Q(\underline{V},\underline{\theta}) = -V_k \sum_{i=0}^{N} B_{ki}V_i(\cos\theta_{ki}-1) + \sum_{i=0}^{N} V_kV_iG_{ki}\sin\theta_{ki} \tag{66}$$

$$k = 1,2,\ldots,N_Q$$

$$G_k^P(\underline{V},\underline{\theta}) = -V_k \sum_{i=0}^{N} B_{ki}V_i(\theta_{ki}-\sin\theta_{ki}) + \sum_{i=0}^{N} V_kV_iG_{ki}\cos\theta_{ki} \tag{67}$$

$$k = 1,2,\ldots,N.$$

We claim that for any $y \in K$,

$$H((\underline{V},\underline{\theta}),t) \neq y \tag{68}$$

for $t \in [0,1]$ and $(\underline{V},\underline{\theta}) \in \partial(R_V x B_\theta)$. We consider two cases

Case (i). $\underline{V} \in \partial R_V$.

Suppose the boundary is defined by $V_k = V_k^M$. We have on this boundary

$$\tilde{Q}_k(\underline{V})\Big|_{V_k=V_k^M} \geq \tilde{Q}_k(\underline{V}^M) \tag{69}$$

and

$$G_k^Q(\underline{V},\underline{\theta}) \geq -\varepsilon_k^Q \tag{70}$$

Consequently,

$$H_k^Q((\underline{V},\underline{\theta}),t) \geq \tilde{Q}_k(\underline{V}^M) - \varepsilon_k^Q \tag{71}$$

Condition (C5) and equation (71) establish our claim (68). Similarly for $V_k = V_k^m$.

Case (ii). $\underline{\theta} \in \partial B_\theta$.

Since on ∂B_θ

$$\|\underline{\tilde{P}}(\underline{V},\underline{\theta})\| \geq r_P \tag{72}$$

and by definition

$$\|G^P(\underline{V},\underline{\theta})\| \leq \|\underline{\varepsilon}^P\| \tag{73}$$

Therefore

$$\|\underline{H}^P((\underline{V},\underline{\theta}),t)\| \geq r_P - \|\underline{\varepsilon}^P\| \tag{74}$$

Condition (C6) and equation (74) establish our claim (68).

By homotopy invariance property of the degree we have $\deg(F+G,R_V x B_\theta,y) \neq 0$ for any $y \in K$. Theorem 3 then follows from the Kronecker theorem. ¤

4. Example

The following simple example illustrates the fact that the numbers one obtains by applying our results are reasonable.

Consider a generator connected to a load through a transmission line (Fig. 1) with impedance $z = j0.1$.

Let $R_V = \{V_1 | 0.95 \leq V_1 \leq 1.05\}$,

$R = \{\theta_1 | -0.1745 \leq \theta_1 \leq 0.1745\}$, $R_q = \{(V_1,\theta_1) | -0.7 \leq Q_0(V_1,\theta_1) \leq 0.7\}$, $R_p = \{(V_1,\theta_1) | 0 \leq P_0(V_1,\theta_1) \leq 1.5\}$, and $V_0 = 1$.

For this example we have $\tilde{Q}_1(V_1^M) = 0.525$, $\tilde{Q}_1(V_1^m) = -0.475$, $\tilde{q}_0(V_1^M) = -0.5$, $\tilde{q}_0(V_1^m) = 0.5$, $r = 0.1745$, $\rho = 9.5$, $\varepsilon^Q = 0.1595$, $\varepsilon^P = 0.0093$, $\varepsilon^q = 0.1595$ and $L^M = 0$.

The inequality in assumption (A5) is satisfied because $\tilde{Q}_1(V_1=V_1^m) = -0.475 \geq -(B_{10})(V_1^m)^2 = -9.02$. The condition (C4) is satisfied since $\tilde{q}_0(V_1^m) + \varepsilon^q = 0.6595 < q_j^M = 0.7$ and $q_j^m = -0.7 < \tilde{q}_0(V_1^M) - \varepsilon^q = -0.6595$.

The limits on power injection imposed by condition (C5) are

$$-0.3155 < Q_1 < 0.3655$$

and that by conditions (C6)(C7) are

$$|P_1| < 1.6485, \quad -1.5 \leq P_1 \leq 0$$

Together they require

$$-1.5 \leq P_1 \leq 0.$$

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APPENDIX: DEGREE OF MAPPING

<u>Definition</u> [6, pp. 147-160] Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on the open set D, and C an open, bounded set such that its closure $\bar{C} \subset D$. If $y \notin F(\partial C) \cup F(S(\bar{C}))$, where ∂C denotes the boundary of C and $S(\bar{C}) := \{x \in \bar{C} \mid F'(x) \text{ singular}\}$. Let $\Gamma = \{x \in C \mid Fx = y\}$. The <u>degree of F on C with respect to y</u> is defined as

$$\deg(F,C,y) = \begin{cases} \sum_{j=1}^{m} \operatorname{sgn} \det F'(x^j) & \text{if } \Gamma = \{x^1,\ldots,x^m\} \\ 0 & \text{if } \Gamma \text{ is empty.} \end{cases}$$

<u>Properties</u> [6, pp. 156-164]

(D1) <u>Kronecker Theorem</u>: Let $F : \bar{C} \to \mathbb{R}^n$ be continuous, and $y \notin F(\partial C)$. If $\deg(F,C,y) \neq 0$, then $Fx = y$ has a solution in C.

(D2) <u>Translation Invariance</u>: Let $F : \bar{C} \to \mathbb{R}^n$ be continuous. Suppose $y^0, y^1 \in \mathbb{R}^n$ are any two points which can be connected by a continuous path p, i.e., $p(0) = y^0$, $p(1) = y'$, and $p(t) \notin F(\partial C)$ for $t \in [0,1]$. Then $\deg(F,C,y^0) = \deg(F,C,y^1)$.

(D3) <u>Homotopy Invariance</u>: Let $H : \bar{C} \times [0,1] \to \mathbb{R}^n$ be continuous. If $H(x,t) \neq y$ for all $(x,t) \in \partial C \times [0,1]$, then $\deg(H(\cdot,t),C,y)$ is a constant for all $t \in [0,1]$.

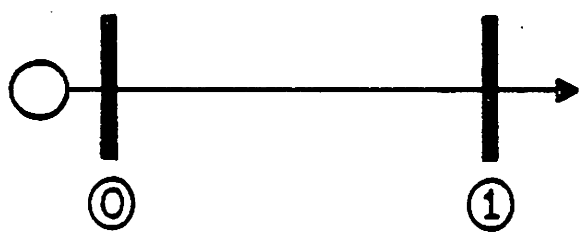

Fig. 1 Example of a two-bus system.