Geometric interpretation of signals: background

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The opportunity

A number of problems in signal processing can be formulated within a common geometric framework. This offers several related opportunities: to observe commonalities among seemingly distinct contexts, to contribute to intuition through geometric reasoning, and to quickly identify solutions to common problems.

Vector spaces

Let $\Im$ be a field. For our purposes, there are two fields of interest, $\Re$, the field of real numbers, and $\mathbb{C}$, the field of complex numbers. In the following we will consistently assume that $\Im = \mathbb{C}$; that is, complex-valued scalar fields.

A vector space $\mathbb{V}$ is a set upon which two binary operations are defined, addition of two vectors ("+") and multiplication of a vector by a scalar ("•"). Specifically, $+ : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ and $\bullet : \mathbb{C} \times \mathbb{V} \to \mathbb{V}$ must satisfy, for all $\vec{U}, \vec{V}, \vec{W} \in \mathbb{V}$ and for all $\alpha, \beta \in \mathbb{C}$:

$$(\vec{U} + \vec{V}) + \vec{W} = \vec{U} + (\vec{V} + \vec{W})$$

$$\vec{U} + \vec{V} = \vec{V} + \vec{U}$$

There exists a $\vec{0} \in \mathbb{V}$ such that $\vec{U} + \vec{0} = \vec{U}$

There exists a $(-\vec{U}) \in \mathbb{V}$ such that $\vec{U} + (-\vec{U}) = \vec{0}$

$$\alpha \cdot (\beta \cdot \vec{U}) = (\alpha\beta) \cdot \vec{U}$$

$$1 \cdot \vec{U} = \vec{U}$$

$$\alpha \cdot (\vec{U} + \vec{V}) = \alpha \cdot \vec{U} + \alpha \cdot \vec{V}$$

$$(\alpha + \beta) \cdot \vec{U} = \alpha \cdot \vec{U} + \beta \cdot \vec{U}$$

Example. Let $\mathbb{C}^n$ be the space of *n*-dimensional complex-valued vectors under the normal rules of linear algebra. That is, each column vector of dimension *n* is associated with a vector,

$$\vec{Z} \leftrightarrow \begin{bmatrix} z_1 \\ z_2 \\ \cdots \\ z_n \end{bmatrix}$$

and multiplication by a scalar can be defined as

$$\alpha \cdot \vec{Z} \leftrightarrow \begin{bmatrix} \alpha \cdot z_1 \\ \alpha \cdot z_2 \\ \cdots \\ \alpha \cdot z_n \end{bmatrix}$$

A finite-time discrete-time complex-valued signal can thus be modeled as a vector in space $\mathbb{C}^n$, which is a linear space.

It is important to note the notation, in which $\vec{Z}$ is a vector, and the operator '$\leftrightarrow$' associates that vector with a mathematical object, such as a column vector or discrete-time signal.

Subspaces

A subspace $\mathbb{M}$ of a vector space $\mathbb{V}$ is a subset $\mathbb{M} \subseteq \mathbb{V}$ which is itself a vector space, and hence is closed with respect to all vector space operations.

Example: The easiest way to define a subspace is to choose a set of linearly-independent basis vectors, and then define the subspace as all linear combinations of those vectors. Consider $\mathbb{C}^3$ and define two vectors

$$\vec{Z_1} \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \vec{Z_2} \leftrightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then $\mathbb{M} = \left\{ \vec{Z} : \vec{Z} = \alpha \cdot \vec{Z_1} + \beta \cdot \vec{Z_2} \right\}$ for any scalars α, β is a two-dimensional subspace of $\mathbb{C}^3$.

Metric, sequences, and convergence

A *metric* $d(\vec{U},\vec{V})$ has the interpretation of a distance between vectors $\vec{U}$ and $\vec{V}$; thus, it adds a geometric interpretation to a vector space. Specifically, for vector space $\mathbb{V}$, a metric $d(\cdot,\cdot):\mathbb{V}\times\mathbb{V}\rightarrow\Re$ (note that a metric is real-valued) must satisfy, for all $\vec{U},\vec{V}\in\mathbb{V}$,

$d(\vec{U},\vec{V})=0$ iff $\vec{U}=\vec{V}$
$d(\vec{U},\vec{V})=d(\vec{V},\vec{U})$
Triangle inequality: $d(\vec{U},\vec{V})+d(\vec{V},\vec{W})\geq d(\vec{U},\vec{W})$

A metric $d(\cdot,\cdot)$ imposes a set of topological properties, such as open and closed sets and the convergence of sequences of vectors.

A sequence $\{\vec{U}_k\}, k=1,2,...,\infty$ is a *Cauchy sequence* when for every $\varepsilon>0$ there exists an N such that $d(\vec{U}_m,\vec{U}_n)<\varepsilon$ for all $m,n>N$. A sequence $\{\vec{U}_k\}, k=1,2,...,\infty$ is *convergent* to vector $\vec{U}$ when for every $\varepsilon>0$ there exists an N such that $d(\vec{U}_m,\vec{U})<\varepsilon$ for all $m>N$. Every convergent sequence is a Cauchy sequence, but not the reverse.

Example: Consider a space $x\in(0,1]$ under the Euclidean metric (this is a metric space, but *not* a vector space) and the sequence $\left\{x_n=\frac{1}{n}\right\}$. This is a Cauchy sequence which does not converge to an element within the space, because the vector $\{0\}$ is missing.

A metric space is said to be *complete* when every Cauchy sequence of vectors in that space is convergent.

Normed spaces

Let $\mathbb{N}$ be a vector space. A *norm* has the interpretation as the length of a vector. Specifically, $\|\cdot\|: \mathbb{N} \to \Re$ (note that a norm is real-valued) has the properties, for all $\vec{U}, \vec{V} \in \mathbb{N}$ and all $\alpha \in \mathbb{C}$:

$\|\vec{V}\| \geq 0$ with equality iff $\vec{V} = 0$

$\|\alpha \cdot \vec{V}\| = |\alpha| \cdot \|\vec{V}\|$

Triangle inequality: $\|\vec{U} + \vec{V}\| \leq \|\vec{U}\| + \|\vec{V}\|$

A normed space is the pair $(\mathbb{N}, \|\cdot\|)$, a linear space plus a norm defined on that space.

Example. Consider $\mathbb{C}^n$ as defined earlier. If $\vec{Z} \leftrightarrow Z$ for a column vector Z, then a valid norm is

$$\|\vec{Z}\| = \sqrt{Z^T Z^*}$$

where Z^T is the matrix transpose of Z.

A norm *induces* a metric through the relation $d(\vec{U}, \vec{V}) \equiv \|\vec{U} - \vec{V}\|$. (This is easily verified from the definitions.) Thus, any normed space possesses all the topological properties of a metric space (including Cauchy and convergent sequences). A normed space that is complete (with respect to its induced metric) is called a *Banach space*.

Inner-product spaces

Let $\mathbb{V}$ be a vector space. An *inner product* is a type of multiplication operator on two vectors. Specifically, $\langle \cdot | \cdot \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ (note that an inner product is complex-valued) has the properties, for all $\vec{U}, \vec{V} \in \mathbb{V}$ and all $\alpha \in \mathbb{C}$,

$$\left\langle \vec{U} \middle| \vec{V} + \vec{W} \right\rangle = \left\langle \vec{U} \middle| \vec{V} \right\rangle + \left\langle \vec{U} \middle| \vec{W} \right\rangle$$
$$\left\langle \alpha \cdot \vec{U} \middle| \vec{V} \right\rangle = \alpha \cdot \left\langle \vec{U} \middle| \vec{V} \right\rangle$$
$$\left\langle \vec{U} \middle| \vec{V} \right\rangle = \left(\left\langle \vec{V} \middle| \vec{U} \right\rangle \right)^*$$
$$\left\langle \vec{U} \middle| \vec{U} \right\rangle \geq 0 \text{ with equality iff } \vec{U} = 0$$

From these properties, it is easily inferred that $\left\langle \vec{U} + \vec{W} \middle| \vec{V} \right\rangle = \left\langle \vec{U} \middle| \vec{V} \right\rangle + \left\langle \vec{W} \middle| \vec{V} \right\rangle$ and $\left\langle \vec{U} \middle| \alpha \cdot \vec{V} \right\rangle = \alpha^* \cdot \left\langle \vec{U} \middle| \vec{V} \right\rangle$. $\left\langle \vec{U} \middle| \vec{U} \right\rangle$ is real valued (this follows from the fourth property, since $\left\langle \vec{U} \middle| \vec{U} \right\rangle = \left(\left\langle \vec{U} \middle| \vec{U} \right\rangle \right)^*$).

An *inner product space* is a pair ($\mathbb{V}, \langle \cdot | \cdot \rangle$), a vector space together with an inner product defined on that space.

Example. Let $\mathbb{V}=\mathbb{C}^n$, the space of complex-valued column vectors of dimension *n*, and let H be an $n \times n$ positive definite Hermitian matrix ($H^T = H^*$). Then for any $\vec{U}, \vec{V} \in \mathbb{V}$, define an inner product

$$\left\langle \vec{U} \middle| \vec{V} \right\rangle_H = u^T H v^*$$

It is easy to verify that this satisfies all the conditions to be an inner product, so ($\mathbb{V}, \langle \cdot | \cdot \rangle$) is an inner product space. (The subscript H can be used to eliminate any confusion over which inner product is in play.)

The following result has wide applicability to optimization over inner product spaces.

Theorem (Schwarz inequality): For an inner product space ($\mathbb{V}, \langle \cdot | \cdot \rangle$) and arbitrary vectors $\vec{X}, \vec{Y} \in \mathbb{V}$,

$$\left| \left\langle \vec{X} \middle| \vec{Y} \right\rangle \right| \leq \left\| \vec{X} \right\| \cdot \left\| \vec{Y} \right\|$$

with equality iff $\vec{X} = \alpha \cdot \vec{Y}$ for some scalar α.

Proof:

$$\left\| \vec{X} - \alpha \cdot \vec{Y} \right\|^2 = \left\| \vec{X} \right\|^2 + \alpha \cdot \alpha^* \cdot \left\| \vec{Y} \right\|^2 - \alpha^* \cdot \left\langle \vec{X} \middle| \vec{Y} \right\rangle - \alpha \cdot \left(\left\langle \vec{X} \middle| \vec{Y} \right\rangle \right)^* \geq 0$$

with equality iff $\vec{X} = \alpha \cdot \vec{Y}$. This inequality is true for any scalar α, but it conveys the most information if we choose α to minimize the left side. Differentiating this expression w.r.t. α^* and setting to zero, a stationary point is at

$$\alpha = \frac{\langle \vec{X} | \vec{Y} \rangle}{\|\vec{Y}\|^2}.$$

Substituting this α into the inequality, we get

$$\|\vec{X}\|^2 - \frac{|\langle \vec{X} | \vec{Y} \rangle|^2}{\|\vec{Y}\|^2} \geq 0.$$

Theorem. An inner product *induces* a norm through the relation

$$\|\vec{U}\| \equiv \sqrt{\langle \vec{U} | \vec{U} \rangle}.$$

Proof: That the triangle inequality is satisfied is the only non-trivial step in establishing that $\|\vec{U}\|$ defined in this way is a norm. Note that for every $\vec{X}, \vec{Y} \in \mathbb{I}$,

$$\|\vec{X} + \vec{Y}\|^2 = \|\vec{X}\|^2 + \|\vec{Y}\|^2 + 2 \cdot \mathrm{Re}\{\langle \vec{X} | \vec{Y} \rangle\} \leq \|\vec{X}\|^2 + \|\vec{Y}\|^2 + 2 \cdot |\langle \vec{X} | \vec{Y} \rangle|$$

The inequality follows from an inequality for complex variables,

$$|x + i \cdot y|^2 = x^2 + y^2 \geq x^2 \text{ or } |x + i \cdot y| \geq |x| \geq x.$$

Invoking the Schwarz inequality, to further upper bound the third term,

$$\|\vec{X} + \vec{Y}\|^2 \leq \|\vec{X}\|^2 + \|\vec{Y}\|^2 + 2 \cdot \|\vec{X}\| \cdot \|\vec{Y}\| = (\|\vec{X}\| + \|\vec{Y}\|)^2.$$

Thus, every inner product space is also a normed space and a metric space. An inner product space that is complete under the induced metric is called a *Hilbert space*. Every Hilbert space is implicitly a Banach space.

In signal processing, we call a Hilbert space of possible signals a *signal space*. The following theorem has numerous applications to signal processing.

Projection theorem: Let Hilbert space $\mathbb{H}$ have a closed subspace M and let $\vec{X} \in \mathbb{H}$ but $\vec{X} \notin M$. Then there exists a unique $\vec{P} = \vec{P}(X : M) \in M$ with the following two equivalent properties:

Closeness: $\left\|\vec{X} - \vec{P}\right\| \le \left\|\vec{X} - \vec{Y}\right\|$ for every $\vec{Y} \in M$

Orthogonality principle: $\left\langle \vec{X} - \vec{P} \middle| \vec{Y} \right\rangle = 0$ for every $\vec{Y} \in M$

In words, $\vec{P}(\vec{X} : M)$ is called the *projection* of $\vec{X}$ on M, and that projection is the vector within M that is closest to $\vec{X}$. Also, the vector $\vec{X} - \vec{P}$ is orthogonal to M (meaning that it is orthogonal to every vector in M, including $\vec{P}$).

In signal processing, $\vec{P}$ is often interpreted as an best estimate or approximation to $\vec{X}$ in subspace M with respect to metric $\|\cdot\|$, and thus $\vec{X} - \vec{P}$ is an *error vector* and the magnitude of the error is $\left\|\vec{X} - \vec{P}\right\|$. The orthogonality principle restated: for the optimum estimate of $\vec{X}$ based upon a vector in M, the error vector is orthogonal to the subspace M (orthogonal to every vector in M).

A convenient relation for the norm of this error vector follows from the Pythagorean theorem,

$$\left\|\vec{X}\right\|^2 = \left\|\vec{X} - \vec{P} + \vec{P}\right\|^2 = \left\|\vec{X} - \vec{P}\right\|^2 + \left\|\vec{P}\right\|^2 \text{ (because } \left\langle \vec{X} - \vec{P} \middle| \vec{P} \right\rangle = 0\text{)}$$

or

$$\left\|\vec{X} - \vec{P}\right\|^2 = \left\|\vec{X}\right\|^2 - \left\|\vec{P}\right\|^2 .$$

Partial proof of projection theorem:

Existence: This can be done by constructing a Cauchy sequence that converges to the projection and invoking completeness (details omitted).

Uniqueness: Assume that two vectors $\vec{Y}_1 \in M$ and $\vec{Y}_2 \in M$ both have orthogonal errors,

$$\left\langle \vec{X} - \vec{Y}_1 \middle| \vec{Y} \right\rangle = \left\langle \vec{X} - \vec{Y}_2 \middle| \vec{Y} \right\rangle = 0 \text{ for all } \vec{Y} \in M .$$

By linearity of the inner product, it follows that

$$\left\langle \vec{X} - \vec{Y}_1 - \vec{X} + \vec{Y}_2 \middle| \vec{Y} \right\rangle = \left\langle \vec{Y}_2 - \vec{Y}_1 \middle| \vec{Y} \right\rangle = 0 .$$

Since $\left(\vec{Y}_2 - \vec{Y}_1\right) \in M$, this must be true for $\vec{Y} = \left(\vec{Y}_2 - \vec{Y}_1\right)$, or

$$\left\|\vec{Y}_2 - \vec{Y}_1\right\|^2 = 0 \Rightarrow \left(\vec{Y}_1 - \vec{Y}_2\right) = 0 .$$

Closeness property $\Rightarrow$ orthogonality principle: Assume that

$$\left\|\vec{X} - \vec{P}\right\| \le \left\|\vec{X} - \vec{Y}\right\| \text{ for every } \vec{Y} \in M .$$

Letting α be a scalar, it follows, since $\vec{P} \in M$ that for every α and $\vec{Y} \in M$

$$\left\|\vec{X} - \vec{P}\right\| \le \left\|\vec{X} - \vec{P} - \alpha \cdot \vec{Y}\right\|$$

or

$$\left\|\vec{X} - \vec{P}\right\|^2 \le \left\|\vec{X} - \vec{P}\right\|^2 + |\alpha|^2 \cdot \left\|\vec{Y}\right\|^2 - 2 \cdot \mathrm{Re}\left\{\alpha^* \cdot \left\langle \vec{X} - \vec{P} \middle| \vec{Y} \right\rangle\right\}.$$

Letting $\alpha = \beta \cdot \left\langle \vec{X} - \vec{P} \middle| \vec{Y} \right\rangle$ for any real-valued β, this becomes

$$0 \le \left(\beta^2 \cdot \left\|\vec{Y}\right\|^2 - 2 \cdot \beta\right) \cdot \left|\left\langle \vec{X} - \vec{P} \middle| \vec{Y} \right\rangle\right|^2$$

Since for $\beta = \dfrac{1}{\left\|\vec{Y}\right\|^2}$ the first term is negative, it must be that $\left\langle \vec{X} - \vec{P} \middle| \vec{Y} \right\rangle = 0$.

Orthogonality principle $\Rightarrow$ closeness property: Suppose $\left\langle \vec{X} - \vec{P} \middle| \vec{Y} \right\rangle = 0$ for every $\vec{Y} \in M$. Then

$$\left\|\vec{X} - \vec{Y}\right\|^2 = \left\|\vec{X} - \vec{P} + \vec{P} - \vec{Y}\right\|^2 = \left\langle \vec{X} - \vec{P} + \vec{P} - \vec{Y} \middle| \vec{X} - \vec{P} + \vec{P} - \vec{Y} \right\rangle.$$

Expanding this term by term,

$$\left\|\vec{X} - \vec{Y}\right\|^2 = \left\|\vec{X} - \vec{P}\right\|^2 + \left\|\vec{P} - \vec{Y}\right\|^2 + 2 \cdot \mathrm{Re}\left\{\left\langle \vec{X} - \vec{P} \middle| \vec{P} - \vec{Y} \right\rangle\right\}$$

Since by assumption $\vec{P} \in M$ and $\vec{Y} \in M$ and hence $(\vec{P} - \vec{Y}) \in M$, the third term must be zero. Thus

$$\left\|\vec{X} - \vec{Y}\right\|^2 = \left\|\vec{X} - \vec{P}\right\|^2 + \left\|\vec{P} - \vec{Y}\right\|^2 \ge \left\|\vec{X} - \vec{P}\right\|.$$

End of proof.

Given N linearly independent vectors $\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_N}$, an N-dimensional subspace *M* is defined by all linear combinations of these vectors. Use the notation $M = \left\{\vec{X}_i, 1 \le i \le N\right\}$ to denote this subspace. It is more convenient to have a set of N *orthogonal* vectors $\overrightarrow{Y_1}, \overrightarrow{Y_2}, \ldots, \overrightarrow{Y_N}$ that also spans this subspace; that is, $M = \left\{\vec{Y}_i, 1 \le i \le N\right\}$ and $\left\langle \vec{Y_i} \middle| \vec{Y_j} \right\rangle = 0$ for $i \ne j$. (A set of orthogonal vectors is necessarily linearly independent.) The $\overrightarrow{Y_1}, \overrightarrow{Y_2}, \ldots, \overrightarrow{Y_N}$ is called an *orthogonal basis* for *M*, and it is easily generated by a *Gram-Schmidt orthogonalization procedure*. Actually, this is a straightforward application of the projection theorem as follows. Define $\vec{Y}_1 = \vec{X}_1$ and let

$$\vec{Y}_2 = \vec{X}_2 - P(\vec{X}_2 : \{\vec{Y}_1\}) = \vec{X}_2 - \frac{\left\langle \vec{X}_2 \middle| \vec{Y}_1 \right\rangle}{\left\|\vec{Y}_1\right\|^2} \cdot \vec{Y}_1.$$

which we know (1) has a component in the direction of $\vec{X}_2$ and (2) by the orthogonality principle we know that this Y_2 is orthogonal to $\vec{Y}_1$. Now we can proceed by induction, defining $\vec{Y}_n$ in terms of $\vec{Y_1}, \vec{Y_2}, ..., \overrightarrow{Y_{n-1}}$,

$$\vec{Y}_n = \vec{X}_n - P(\vec{X}_n : \{\vec{Y}_1, \vec{Y}_2, ..., \vec{Y}_{n-1}\}) = \vec{X}_n - \sum_{k=1}^{n-1} \frac{\langle \vec{X}_n | \vec{Y}_k \rangle}{\|\vec{Y}_k\|^2} \cdot \vec{Y}_k \; .$$

The coefficients in this expansion have been determined by the orthogonality principle,

$$\left\langle \vec{X}_n - \sum_{k=1}^{n-1} \alpha_k \cdot \vec{Y}_k \middle| \vec{Y}_j \right\rangle = 0 \, , \; 1 \le j \le n-1$$

and exploiting the orthogonality of $\vec{Y_1}, \vec{Y_2}, ..., \overrightarrow{Y_{n-1}}$.

The following are important Hilbert spaces in signal processing applications. Details such as the exact meaning of integrals and proofs of completeness are left to the references.

Example. $\mathbb{H} = \mathbf{C}^n$ is a Hilbert space with the appropriate definition of inner product. Let $\mathbf{H}$ be an $n \times n$ positive-definite Hermitian matrix ($\mathbf{H}^{\mathbf{T}} = \mathbf{H}^*$), and for a vector $\vec{U} \leftrightarrow \mathbf{u}$ let $\mathbf{u}^{\mathbf{H}}$ denote the conjugate transpose of $\mathbf{u}$. Then two (equivalent possibilities) for an inner product are to let vectors correspond to $1 \times n$ (row) matrices with $\langle \vec{U} | \vec{V} \rangle = \mathbf{uHv^H}$ and to let vectors correspond to $n \times 1$ (column) matrices with $\langle \vec{U} | \vec{V} \rangle = \mathbf{u^T H v^*}$. When $\mathbf{H} = \mathbf{I}$ (identity matrix), this is ordinary complex-valued Euclidean space.

Example. Let $\mathbb{l}_2$ be the space of all double-infinite complex-valued time sequences of the form

$$\vec{Z} = \{..., z(-1), z(0), z(1), ...\}$$

where the sequence has finite energy,

$$\sum_{n=-\infty}^{\infty} |z(n)|^2 < \infty$$

and with the inner product defined as

$$\langle \vec{U} | \vec{V} \rangle = \sum_{-\infty}^{\infty} u(n) \cdot v^*(n) \, .$$

Then $\mathbb{l}_2$ is a Hilbert space. When the limits of summation are restricted to a finite interval, this is reverts to the earlier example.

Example. Let $\mathbb{L}_2$ be the space of all complex-valued continuous-time signals of the form

$$\vec{Z} \leftrightarrow \{z(t), -\infty < t < \infty\}$$

where each signal has finite energy,

$$\int_{-\infty}^{\infty} |z(t)|^2 \cdot dt < \infty$$

and with the inner product defined as

$$\langle \vec{U} | \vec{V} \rangle = \int_{-\infty}^{\infty} u(t) \cdot v^*(t) \cdot dt \, .$$

Then $\mathbb{L}_2$ is a Hilbert space. When the integrals are restricted to a finite or semi-infinite interval, this remains a Hilbert space.

Example. Let $(\Omega, \Im, P)$ be a probability space (Ω is the sample space, $\Im$ is the set of all events defined on that sample space, and $0 \le P \le 1$ is a probability measure defined over all events), and let $\mathbb{L}_2(\Omega)$ be the space of all complex-valued random variables Z with zero mean and finite second moments,

$$E[Z] = 0 \text{ and } E[|Z|^2] < \infty$$

with a vector associated with each such random variable,

$$\vec{Z} \leftrightarrow Z$$

and the inner product defined as

$$\langle \vec{U} | \vec{V} \rangle = E[U \cdot V^*] \, .$$

Then $\mathbb{L}_2(\Omega)$ is a Hilbert space.

References

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