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RIGHT FACTORIZATION OF A CLASS OF TIME-VARYING NONLINEAR SYSTEMS

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Abstract

We consider a class of nonlinear continuous-time time-varying plants with a state-space description which has a uniformly completely controllable linear part. For this class, we obtain by calculation a right factorization. In the case where the state is available for feedback, we obtain a normalized right-coprime factorization.

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Introduction

For plants with finite-dimensional linear state-space description, right and left coprime factorizations have been obtained in two cases: 1) the time-invariant stabilizable and detectable case in [Net.1, Vid.1,2], 2) the time-varying uniformly completely controllable and uniformly completely observable case in [Man.1]. Hammer has proved that a nonlinear time-invariant discrete-time recursive system with a continuous recursion function has a right coprime factorization [Ham.1].

In this paper we consider a class of nonlinear continuous-time time-varying plants with a state-space description which has a uniformly completely controllable linear part. For this class, we obtain by calculation a right factorization. Using input-output representation it can be shown that each of the subsystems of a class of S-stable feedback systems has a right factorization [Des.1]. In the case where the state is available for feedback, we obtain a normalized right-coprime factorization.

Notation : In this paper, the $\infty-$ norm for vectors in $\mathbb{R}^n$ and the corresponding induced norm for matrices are denoted by $\| \cdot \|$. For vector valued functions $x : \mathbb{R}_+ \to \mathbb{R}^n$, we also write

$\| x \| := \sup_{t \in \mathbb{R}_+} \| x(t) \|$, a slight abuse of notation.

The extended space $L^n_{\infty e}[0,\infty) := \{ x : \mathbb{R}_+ \to \mathbb{R}^n \mid \forall T \in \mathbb{R}_+ , \sup_{t \in [0,T]} \| x(t) \| < \infty \}$ is the causal extension of $L^n_{\infty}[0,\infty) := \{ x : \mathbb{R}_+ \to \mathbb{R}^n \mid \| x \| < \infty \}$.

A causal map $H : L^{n_i}_{\infty e}[0,\infty) \to L^{n_o}_{\infty e}[0,\infty)$ is said to be *S-stable* iff for all $\alpha > 0$ there exists a $\beta > 0$ such that $\| x \| < \alpha$ implies that $\| Hx \| < \beta$. An S-stable map need not be continuous. Note that the composition and the sum of S-stable maps are S-stable.

A causal map $P : L^{n_i}_{\infty e}[0,\infty) \to L^{n_o}_{\infty e}[0,\infty)$ is said to have a *right factorization* $(N_p , D_p ; L^{n_i}_{\infty e}[0,\infty))$ iff there exist causal S-stable maps N_p , D_p , such that

(i) $D_p : L^{n_i}_{\infty e}[0,\infty) \to L^{n_i}_{\infty e}[0,\infty)$ is bijective and has a causal inverse, and (ii) $N_p : L^{n_i}_{\infty e}[0,\infty) \to L^{n_o}_{\infty e}[0,\infty)$, with $N_p[L^{n_i}_{\infty e}[0,\infty)] = P[L^{n_i}_{\infty e}[0,\infty)]$,

and (iii) $P = N_p D_p^{-1}$ [Vid.3, Ham.1].

$(N_p, D_p; L^{n_i}_{\infty e}[0,\infty))$ is said to be a *normalized right-coprime factorization* of $P : L^{n_i}_{\infty e}[0,\infty) \to L^{n_o}_{\infty e}[0,\infty)$ iff

(i) $(N_p, D_p; L^{n_i}_{\infty e}[0,\infty))$ is a right factorization of P,

and (ii) there exist causal S-stable maps $U_p : L^{n_o}_{\infty e}[0,\infty) \to L^{n_i}_{\infty e}[0,\infty)$ and $V_p : L^{n_i}_{\infty e}[0,\infty) \to L^{n_i}_{\infty e}[0,\infty)$ such that

$$U_p N_p + V_p D_p = I \quad ,$$

where I denotes the identity map on $L^{n_i}_{\infty e}[0,\infty)$.

Description of the system : Consider a nonlinear time-varying system whose input-output map $P : L^{n_i}_{\infty e}[0,\infty) \to L^{n_o}_{\infty e}[0,\infty)$ is specified by the following state-space description:

$$P : u \mapsto y \quad \begin{cases} \dot{x} = A(t)x + f(t,x) + B(t)u & \text{(1a)} \\ y = h(t,x,u) & \text{(1b)} \\ x(0) = 0 \ , & \text{(1c)} \end{cases}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_i}$ and $y(t) \in \mathbb{R}^{n_o}$, $\forall t \in \mathbb{R}_+$.

On the functions $A(\cdot)$, $B(\cdot)$, $f(\cdot,\cdot)$ and $h(\cdot,\cdot,\cdot)$, we impose these assumptions.

Assumptions

I. For the initial condition (1c) and for all inputs $u \in L^{n_i}_{\infty e}[0,\infty)$, the differential equation (1a) has a unique solution. (Consequently, $P : u \mapsto y$ is a function.)

II. The nonlinearity f is bounded on $\mathbb{R}_+ \times \mathbb{R}^n$, more precisely there exists $m>0$ such that $\sup_{t \in \mathbb{R}_+, x \in \mathbb{R}^n} \| f(t,x) \| \le m$. (This assumption implies that $f(t,x)$ does not have a linear part in x .)

III. For any causal S-stable map $H_x : L^{n_i}_{\infty e}[0,\infty) \to L^{n}_{\infty e}[0,\infty)$, $H_x : u \mapsto x$, the causal map $H_y : u \mapsto y$ defined by $y(t) = h(t, (H_x u)(t), u(t))$ is an S-stable map, where

$h : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{n_i} \rightarrow \mathbb{R}^{n_o}$.

IV. The pair $(A(\cdot), B(\cdot))$ is uniformly completely controllable; equivalently, there exist $\Delta > 0$, $w_{\max} \geq w_{\min} > 0$ such that for all $t \in \mathbb{R}_+$

$$w_{\min} I \leq W(t, t+\Delta) \leq w_{\max} I \ , \tag{2}$$

where $W(t, t+\Delta)$ is the controllability gramian [Bro.1].

$$W(t, t+\Delta) := \int_t^{t+\Delta} \Phi(t,\tau) B(\tau) B^T(\tau) \Phi^T(t,\tau) d\tau \ , \tag{3}$$

and $\Phi(\cdot,\cdot)$ is the state-transition function of $\dot{x} = A(t)x$.

V. $B(\cdot)$ is bounded on $\mathbb{R}_+$; more precisely, there exists $b > 0$ such that $\sup_{t \in \mathbb{R}_+} \| B(t) \| \leq b$.

We now construct a right factorization of P : namely, we construct a causal S-stable bijective map D with a causal inverse and a causal S-stable map N such that $P = ND^{-1}$.

Proposition 1 : Let the nonlinear map P be described by (1a-c) and satisfy Assumptions I-V. Then P has a right factorization.

Proof : The proof is in two steps: 1) using Assumption I, we obtain a causal bijective map $D : L^{n_i}_{\infty e}[0,\infty) \rightarrow L^{n_i}_{\infty e}[0,\infty)$ with a causal inverse D^{-1} and a causal map $N : L^{n_i}_{\infty e}[0,\infty) \rightarrow L^{n_o}_{\infty e}[0,\infty)$ such that $P = ND^{-1}$; 2) using Assumptions II-V, we show that both N and D are S-stable maps.

Step 1 : Define the causal map $D : L^{n_i}_{\infty e}[0,\infty) \rightarrow L^{n_i}_{\infty e}[0,\infty)$ by

$$D : v_1 \mapsto u_1 \quad \begin{cases} \dot{x}_1 = (A + BK)(t) x_1 + f(t, x_1) + B(t) v_1 & \text{(4a)} \\ u_1 = K(t) x_1 + v_1 & \text{(4b)} \\ x_1(0) = 0 \ , & \text{(4c)} \end{cases}$$

for some piecewise continuous $K : \mathbb{R}_+ \rightarrow \mathbb{R}^{n_i \times n}$. We claim that D has a causal inverse $D^{-1} : L^{n_i}_{\infty e}[0,\infty) \rightarrow L^{n_i}_{\infty e}[0,\infty)$; indeed D^{-1} is given by

$$D^{-1}: u_2 \mapsto v_2 \quad \begin{cases} \dot{x}_2 = A(t)x_2 + f(t, x_2) + B(t)u_2 & (5a) \\ v_2 = -K(t)x_2 + u_2 & (5b) \\ x_2(0) = 0 \ . & (5c) \end{cases}$$

We show that D is bijective by showing that $D^{-1}D = DD^{-1} = I$. Consider $DD^{-1}: u_2 \mapsto u_1$; thus $v_1 = v_2 = -K(t)x_2 + u_2$ and from equations (4a-c) and (5a-c), we obtain

$$DD^{-1}: u_2 \mapsto u_1 \quad \begin{cases} \dot{x}_1 = (A + BK)(t)x_1 + f(t, x_1) + B(t)u_2 - (BK)(t)x_2 & (6a) \\ \dot{x}_2 = A(t)x_2 + f(t, x_2) + B(t)u_2 & (6b) \\ u_1 = K(t)(x_1 - x_2) + u_2 & (6c) \\ x_1(0) = x_2(0) = 0 \ . & (6d) \end{cases}$$

For any input u_2, using Assumption I it is easy to check that $(x_1(t) \equiv x_2(t), x_2(t))$ is *the* solution of the system of differential equations (6a-b) under the initial conditions (6d). Hence by equation (6c), we get $u_1 = u_2$ and $DD^{-1} = I$ on $L_{\infty e}^{n_i}[0,\infty)$. Similarly, consider $D^{-1}D : v_1 \mapsto v_2$; then $u_1 = u_2 = K(t)x_1 + v_1$, and by equations (4a-c) and (5a-c) we obtain

$$D^{-1}D : v_1 \mapsto v_2 \quad \begin{cases} \dot{x}_1 = (A + BK)(t)x_1 + f(t, x_1) + B(t)v_1 & (7a) \\ \dot{x}_2 = A(t)x_2 + f(t, x_2) + (BK)(t)x_1 + B(t)v_1 & (7b) \\ v_2 = K(t)(x_1 - x_2) + v_1 & (7c) \\ x_1(0) = x_2(0) = 0 \ . & (7d) \end{cases}$$

For any input v_1, $(x_1(t), x_2(t) \equiv x_1(t))$ is *the* solution of the system of differential equations (7a-b) under (7d). Hence by equation (7c), we obtain $v_1 = v_2$; so $D^{-1}D = I$ on $L_{\infty e}^{n_i}[0,\infty)$. Hence D is bijective and D^{-1} defined by (5a-c) is the causal inverse of D.

Now define $N : L^{n_i}_{\infty e}[0,\infty) \to L^{n_o}_{\infty e}[0,\infty)$ by equations (8a-c) for the same $K(\cdot)$ in equations (4a-b).

$$N : v \mapsto y \quad \begin{cases} \dot{x}_3 = (A + BK)(t)x_3 + f(t, x_3) + B(t)v & \text{(8a)} \\ y = h(t, x_3, K(t)x_3 + v) & \text{(8b)} \\ x_3(0) = 0 \ , & \text{(8c)} \end{cases}$$

From equations (5a-c) and (8a-c), we obtain

$$ND^{-1} : u \mapsto y \quad \begin{cases} \dot{x}_2 = A(t)x_2 + f(t, x_2) + B(t)u & \text{(9a)} \\ \dot{x}_3 = (A + BK)(t)x_3 + f(t, x_3) + B(t)u - (BK)(t)x_2 & \text{(9b)} \\ y = h(t, x_3, K(t)(x_3 - x_2) + u) & \text{(9c)} \\ x_2(0) = x_3(0) = 0 \ . & \text{(9d)} \end{cases}$$

For any input u, by Assumption I, $(x_3(t) \equiv x_2(t), x_2(t))$ is *the* solution of the system of differential equations (9a-b) under (9d). Hence, equations (9a-d) is an equivalent description of P as ND^{-1}.

Step 2 : We use a technique due to Cheng [Che.1] to show that there exists a $K : \mathbb{R}_+ \to \mathbb{R}^{n_i \times n}$ such that the causal map $H_x : L^{n_i}_{\infty e}[0,\infty) \to L^{n}_{\infty e}[0,\infty)$ defined by

$$H_x : v \mapsto x \quad \begin{cases} \dot{x} = (A + BK)(t)x + f(t, x) + B(t)v & \text{(10a)} \\ x(0) = 0 \ , & \text{(10b)} \end{cases}$$

is S-stable. Let

$$W_1(t, t+\Delta) := \int_t^{t+\Delta} e^{(t-\tau)}\Phi(t, \tau)B(\tau)B^T(\tau)\Phi^T(t, \tau)d\tau \ . \qquad (11)$$

Using (2), (3) and (11), for all $t \in \mathbb{R}_+$,

$$e^{-\Delta}w_{\min}I \le W_1(t, t+\Delta) \le w_{\max}I \ ,$$

hence for all $t \in \mathbb{R}_+$,

$$w_{\max}^{-1} I \le W_1^{-1}(t, t+\Delta) \le e^{\Delta} w_{\min}^{-1} I \quad . \tag{12}$$

Note that

$$\frac{d}{dt} W_1(t, t+\Delta) = e^{-\Delta}\Phi(t, t+\Delta)B(t+\Delta)B^T(t+\Delta)\Phi^T(t, t+\Delta)$$
$$- B(t)B^T(t) + W_1(t, t+\Delta) + A(t)W_1(t, t+\Delta)$$
$$+ W_1(t, t+\Delta)A^T(t) \quad . \tag{13}$$

For all $t \in \mathbb{R}_+$, let $K(\cdot)$ be defined as

$$K(t) := -B^T(t)W_1^{-1}(t, t+\Delta) \quad . \tag{14}$$

So $K : \mathbb{R}_+ \to \mathbb{R}^{n_i \times n}$ is bounded on $\mathbb{R}_+$. Let $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ be a Liapunov function candidate where

$$V(t, x) := x^T(t)W_1^{-1}(t, t+\Delta)x(t) \quad . \tag{15}$$

Differentiating equation (15) along the solution of (10a-b) we obtain for all $(t, x(t)) \in \mathbb{R}_+ \times \mathbb{R}^n$,

$$\frac{d}{dt} V(t, x(t)) \mid_{(10a)} = 2\dot{x}^T(t)W_1^{-1}(t, t+\Delta)x(t)$$
$$- x^T(t)W_1^{-1}(t, t+\Delta)\frac{d}{dt}\{W_1(t, t+\Delta)\}W_1^{-1}(t, t+\Delta)x(t) \quad . \tag{16}$$

For the time derivative of $V(\cdot, \cdot)$ along the solution of (10a) with $K(\cdot)$ given by (14) we obtain

$$\frac{d}{dt} V(t, x(t)) = 2x^T(t)A^T(t)W_1^{-1}(t, t+\Delta)x(t)$$
$$- 2x^T(t)W_1^{-1}(t, t+\Delta)B(t)B^T(t)W_1^{-1}(t, t+\Delta)x(t)$$
$$+ 2f^T(t, x(t))W_1^{-1}(t, t+\Delta)x(t) + 2v^T(t)B^T(t)W_1^{-1}(t, t+\Delta)x(t)$$
$$- x^T(t)e^{-\Delta}W_1^{-1}(t, t+\Delta)\Phi(t, t+\Delta)B(t+\Delta)B^T(t+\Delta)\Phi^T(t, t+\Delta)W_1^{-1}(t, t+\Delta)x(t)$$
$$+ x^T(t)W_1^{-1}(t, t+\Delta)B(t)B^T(t)W_1^{-1}(t, t+\Delta)x(t)$$
$$- x^T(t)W_1^{-1}(t, t+\Delta)x(t) - 2x^T(t)A^T(t)W_1^{-1}(t, t+\Delta)x(t) \quad . \tag{17}$$

Performing the appropriate cancellations and neglecting nonpositive terms in (17), we obtain

$$\frac{d}{dt}V(t,x(t)) \le -x^T(t)W_1^{-1}(t,t+\Delta)x(t) + 2f^T(t,x(t))W_1^{-1}(t,t+\Delta)x(t)$$
$$+2v^T(t)B^T(t)W_1^{-1}(t,t+\Delta)x(t) \ . \tag{18}$$

By assumptions II, IV, V and (12), (18), we obtain

$$\frac{d}{dt}V(t,x(t)) \le -w_{\max}^{-1}\|x(t)\|_2^2$$
$$+2(m+\sqrt{n_i}b\,\|v\|)e^{\frac{\Delta}{2}}w_{\min}^{-\frac{1}{2}}\|x(t)\|_2 \ . \tag{19}$$

Let $\rho(\|v\|) := 2(m+\sqrt{n_i}b\,\|v\|)e^{\frac{\Delta}{2}}w_{\min}^{-\frac{1}{2}}w_{\max}$. From inequality (19) we conclude that

$$\forall t \in \mathbb{R}_+ \ , \ \|x(t)\|_2 \le \rho(\|v\|) \ . \tag{20}$$

Since all norms are equivalent in $\mathbb{R}^n$, by (20), we conclude that for any $\gamma > 0$ there exists $\Gamma > 0$ such that $\|v\| \le \gamma$ implies that $\|H_x v\| \le \Gamma$. Hence H_x defined by (10a-b) is S-stable.

For the choice of $K(\cdot)$ as in (14), by Assumption V and (12), there exists $\alpha > 0$ such that

$$\sup_{t\in\mathbb{R}_+}\|K(t)\| \le \alpha \ . \tag{21}$$

Then by Assumption III, (21) and the S-stability of H_x, the causal map N is S-stable. By (21) and the S-stability of H_x, the causal map D is S-stable. Hence $(N, D; L_{\infty e}^{n_i}[0,\infty))$ is a right factorization of P.

□

Proposition 2 : Let the map P be described by (1a-c) and satisfy Assumptions I, II, IV, V and let $h(t,x(t),u(t)) \equiv x(t)$. Then P has a normalized right-coprime factorization.

Proof : By Proposition 1, $(N, D; L_{\infty e}^{n_i}[0,\infty))$ given by (4a-c), (8a-c) is a right factorization of P for $K(\cdot)$ as in (14). We claim that $(N, D; L_{\infty e}^{n_i}[0,\infty))$ is a normalized right-coprime factorization of P when $h(t,x,u) \equiv x$. Let the causal S-stable maps $U : L_{\infty e}^{n_o}[0,\infty) \to L_{\infty e}^{n_i}[0,\infty)$ and $V : L_{\infty e}^{n_i}[0,\infty) \to L_{\infty e}^{n_i}[0,\infty)$ be defined as $V = I$ and

$$U : y \mapsto v \quad \begin{cases} v(t) := -K(t)y(t) \end{cases} .$$

Then, using (4a-c), (8a-c) and $h(t,x,u)=x$, we obtain

$$UN+VD : v_1 \mapsto v_2 \quad \begin{cases} \dot{x}_1=(A+BK)(t)x_1+f(t,x_1)+B(t)v_1 & \text{(22a)} \\ \dot{x}_3=(A+BK)(t)x_3+f(t,x_3)+B(t)v_1 & \text{(22b)} \\ v_2=K(t)(x_1-x_3)+v_1 & \text{(22c)} \\ x_1(0)=x_3(0)=0 \ . & \text{(22d)} \end{cases}$$

For any v_1, under (19d), the solutions of (19a-b) are identical. Hence, by (19c), $v_2=v_1$. So $UN+VD$ is the identity map and $(N,D;L_{\infty e}^{n_i}[0,\infty))$ is a normalized right-coprime factorization of P when $h(t,x,u)=x$.

□

In the time-invariant case, Assumptions I and II can be replaced by " $f(\cdot)$ is Lipschitz and does not have a linear part " and the second step in the proof of Proposition 1 can be simplified by selecting the eigenvalues of $(A+BK)$.

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