ON THE MULTIPLE-INPUT MULTIPLE-OUTPUT DISCRETE SYSTEMS

by

E. I. Jury, S. M. Ahn and B. W. Lee

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

ABSTRACT

A study is made of the absolute stability of multi-variable, multi-non-linear discrete time systems by generalizing the Kalman-Szegö lemma. In so doing, two restrictions — complete controllability and principality — have been removed from the assumptions of the lemma.

INTRODUCTION

For discrete time systems with one nonlinearity, the connection between frequency domain stability criteria and corresponding Lyapunov functions is afforded by the Kalman-Szegö lemma. In this paper this result is generalized to cover systems with multiple inputs, multiple outputs, and multiple nonlinearities. In addition, the requirements for complete controllability and principality are removed. The removal of the complete controllability requirement from the Kalman-Yacubovich lemma for continuous system had been effected by Meyer [3] and Popov [4].

The basic results are not entirely new, since they can be anticipated from the work of Popov. However, the approach employed is of interest, the results establish for discrete time systems the correspondence between the frequency domain inequalities and certain Lyapunov functions without restrictions on either side.

1. Existence of Lyapunov function

Consider a system with the following state equation:

$x(t_{h+1})=Ax(t_h)- Bu(t_h)$ $h=0,1,2,...$

x ; $(n \times 1)$ state vector

A ; $(n \times n)$ matrix

u ; $(m \times 1)$ input vector

B ; $(n \times m)$ matrix

It is shown that in case (A,B) is not completely controllable, we can decompose A,B and x by a linear transformation into the following form †

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(A_{11},B_1) ; completely controllable pair

A_{11} ; $(n_1\times n_1)$ matrix

A_{22} ; $(n_2\times n_2)$ matrix

B_1 ; $(n_1\times m)$ matrix

x_1 ; $(n_1\times 1)$ controllable state vector

Theorem 1

If there exist

K^{-1} ; $(m \times m)$ hermitian matrix

D ; $(n \times m)$ matrix

M ; $(n \times n)$ matrix

such that

$$H(z)\triangleq K^{-1} + D^*(zI-A)^{-1}B + B^* (z^{-1}I-A^*)^{-1}D \tag{1}$$

†Unlike continuous case, the sampling period must be such that the system is controllable.

$$- B^*(z^{-1}I-A^*)^{-1}MM^* (zI-A)^{-1}B$$

is a positive semi-definite matrix for all $|z| = 1$ except when z is an eigenvalue of A, then

there exist

Q ; $(m \times m)$ matrix

R ; $(n \times m)$ matrix

N ; $(n \times n)$ positive definite hermitian matrix

such that

$$K^{-1} = B^* NB + QQ^* \tag{2}$$

$$A^* NB = D+RQ \tag{3}$$

$$A^* NA - N = - MM^* - RR^* \tag{4}$$

(*) denotes the complex conjugate transpose.

Assertion

1) H(z) depends only on the completely controllable part (A_{11}, B_1) i.e.,

$$H(z) = K^{-1} + D_1^* (zI_1-A_{11})^{-1}B_1 + B_1^* (z^{-1}I_1-A_{11}^*)^{-1}D_1$$

$$- B_1^* (z^{-1}I_1-A_{11}^*)^{-1} (M_{11}M_{11}^* + M_{12}M_{12}^*) (zI-A_{11})^{-1}B_1$$

2) If the whole system is described by

$$x(t_{h+1}) = Ax(t_h)-Bu(t_h)$$

$$y(t_h) = 2c' x(t_h) \qquad h=0,1,2,\ldots$$

and $D^* = c'$, $M = 0$, where $y(t_h)$ is the output and all quantaties are real, then

(1) becomes Tsypkin's inequality generalized to multiple-input and multiple-output systems.

I) We will show that the theorem holds for (A_{11},B_1) part when it is a principal case. That is to say, there exist Q_1 R_1, and N_{11} such that

$$K^{-1} = B_1^* N_{11} B_1 + QQ^* \tag{5}$$

$$A_{11}^* N_{11} B_1 = D_1 + R_1 Q \tag{6}$$

$$A_{11}^* N_{11} A_{11} - N_{11} = - M_{11} M_{11}^* - M_{12} M_{12}^* - R_1 R_1^* \tag{7}$$

II) We will prove it when A has one eigenvalue at z = 1 i.e., when it is a particular case.

III) We will extend it to the uncontrollable part.

i) Existence of Q matrix.

Since H(z) is a positvie semi-definite hermitian matrix

$$H(z) = H_1(z)\, H_1(z^{-1})$$

Define

$$Q \triangleq \lim_{|z|\to\infty} H_1(z)$$

Let

$$\eta(z) \triangleq H_1(z) - Q$$

ii) Existence of R_1 matrix

R_1 is defined as the solution of

$$\eta(z) = R_1^* (zI_1 - A_{11})^{-1} B_1$$

Expanding both sides into series in the region $\Omega \triangleq \left\{ z \,\middle|\, |z| > \|A_{11}\| \right\}$

we get

$$\eta(z) = \eta_1 z^{-1} + \eta_2 z^{-2} + \dots + \eta_{n_1} z^{-n_1} + \dots$$

$$R_1^* (zI_1 - A_{11})^{-1} B_1 = R_1^* (I_1 B_1 z^{-1} + A_{11} B_1 z^{-2} + \dots + A_{11}^{n_1 - 1} z^{-n_1} + \dots)$$

Let

$$\hat{\eta} \triangleq \left[\eta_1 \vdots \eta_2 \vdots \cdots \vdots \eta_{n_1}\right]$$

$$S \triangleq \left[IB_1 \vdots A_{11}B_1 \vdots \quad \vdots A_{11}^{n_1-1} B_1\right]$$

Since (A_{11},B_1) is completely controllable part, S is of rank n_1.

Hence
$$R_1^* = (\hat{\eta}\, S^*)(SS^*)^{-1}$$

iii) <u>Proof of (7)</u>

$$N_{11} = \sum_{k=0}^{\infty} (A^*_{11})^k (M_{11}M_{11}^* + M_{12}^*M_{12}^* + R_1R_1^*)(A_{11})^k$$

The convergency of the infinite sume is justified by the principality and since each term is at least positive semidefinite hermitian, N_{11} is positive definite hermitian.

iv) <u>Proof of (5) and (6)</u>

Q and R_1 satisfy the following,

$$K^{-1} + D_1^*(zI_1-A_{11})^{-1}B_1 + B_1(z^{-1}I_1-A_{11}^*)^{-1}D_1$$
$$-B_1^*(z^{-1}I_1-A_{11}^*)^{-1}(M_{11}M_{11}^* + M_{12}M_{12}^*)(zI_1-A_{11})^{-1}B_1$$
$$\equiv(Q^*-B_1^*(z^{-1}I_1-A_{11}^*)^{-1}R_1)(Q-R_1^*(zI_1-A_{11})^{-1}B_1)$$

and (7) can be rewritten as follows,

$$(z^{-1}I_1-A_{11}^*)N_{11}(zI-A_{11}) + (z^{-1}I_1-A_{11}^*)N_{11}A_{11} + A_{11}^*N_{11}(zI_1-A_{11})$$
$$= M_{11}M_{11}^* + M_{12}M_{12}^* + R_1R_1^*$$

By means of some algebraic manipulation,

$$K^{-1}-B_1^*N_{11}B_1-QQ^* = \mathcal{H}\left\{(A_{11}^*N_{11}B_1 - D_1-R_1Q)(zI-A)^{-1}B_1\right\}$$

where

$$\mathcal{H}\left\{x\right\} \triangleq \left(x+x^*\right)$$

It can be shown that from complete controllability and principality

$$A_{11}^* N_{11} B_1 - D_1 - R_1 Q = 0$$

$$K^{-1} - B_1^* N_{11} B_1 - QQ^* = 0$$

Let

v) A_{11} has one eigenvalue at $z = 1$.

$$A_{11} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & 1 \end{bmatrix} \qquad \tilde{A}_{11}\ ; \qquad (n_1-1 \times n_1-1) \text{ matrix}$$

$$M_{11}M_{11}^* + M_{12}M_{12}^* = \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{bmatrix} \qquad \tilde{M}\ ; \quad (n_1-1 \times n_1-1) \text{ matrix}$$

$$R_1 = \begin{bmatrix} \tilde{R}_1 \\ \tilde{R}_2 \end{bmatrix} \qquad R_1\ ; \qquad (n_1-1 \times m) \text{ matrix}$$

$$N_{11} = \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_{22} \end{bmatrix} \qquad \tilde{N}_{11}\ ; \qquad (n_1-1 \times n_1-1) \text{ matrix}$$

Consider the following when $z \to 1$

$$H(z) = K^{-1} + D_1^* (zI_1 - A_{11})^{-1} B_1 + B_1^* (z^{-1}I_1 - A_{11}^*)^{-1} D_1$$

$$- B_1^* (z^{-1}I_1 - A_{11}^*)(M_{11}M_{11}^* + M_{12}M_{12}^*)(zI_1 - A_{11})^{-1} B_1$$

and

$$H(z) = (Q^* - B_1^* (z^{-1}I_1 - A_{11}^*)^{-1} R_1)(Q - R_1^* (zI_1 - A_{11})^{-1} B_1)$$

where

$$H(z) \geq 0 \text{ for all } |z| = 1, \text{ except } z=1$$

It is necessary that

$$\tilde{M}_{12} = 0 \quad \tilde{M}_{21} = 0 \quad \tilde{M}_{22} = 0$$

$$\tilde{R}_2 = 0 \quad \tilde{N}_{12} = 0 \quad \tilde{N}_{21} = 0$$

to satisfy the above two equations, since otherwise H(z) may become negative. Now, we only have to show that $\tilde{N}_{22}$ is a positive number such that $\tilde{N}_{22}\,\tilde{B}_2 = \tilde{D}_2$

where $$B_1 = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} \quad \tilde{B}_1 = (n-1 \times m) \text{ matrix}$$

$$D_1 = \begin{vmatrix} \tilde{D}_1 \\ \tilde{D}_2 \end{vmatrix} \quad \tilde{D}_1 = (n-1 \times m) \text{ matrix}$$

The stability-in-the-limit (see Appendix) gurantees that $\tilde{B}_2\tilde{D}_2^*$ is a positive number.

<u>Remark</u>

The above generalization shows that the Lyapunov function for the particular cases can be of the form

$$V(x) = x'Hx + \beta\int_0^\sigma \phi(s)ds$$

In [6] Szegö and Pearson made use of $V(x) = x'Hx + \gamma\rho^2y^2 + \delta\int_0^\sigma \phi(s)ds$ where x is (n-1) dimensional vector which leads to a more complicated ΔV.

vi) <u>Proof for the uncontrollable part</u>

It suffices to show that the following three equations have unique solution R_2, N_{12} and N_{22}.

$$A_{12}^{*}N_{11}B_{11} + A_{22}^{*}N_{12}B_1 = D_2 + R_2Q \qquad (8)$$

$$A_{11}^{*}N_{11}A_{12} + A_{11}^{*}N_{12}N_{22} - N_{12} = -(M_{11}M_{21}^{*} + M_{12}M_{22}^{*} + R_1R_2^{*}) \qquad (9)$$

$$A_{12}^{*}N_{11}N_{12} + A_{22}^{*}N_{12}A_{12} + A_{12}^{*}N_{12}A_{22} + A_{22}^{*}N_{22}A_{22} - N_{22}$$

$$= -(M_{21}M_{21}^{*} + M_{22}M_{22}^{*} + R_2R_2^{*}) \qquad (10)$$

Consider the first two homogeneous equations

$$A_{22}^{*} N_{12} B_1 = R_2Q$$

$$A_{11}^{*} N_{12} A_{22} - N_{12} = - R_1R_2^{*}$$

From the known equation we start

$$(z^{-1}I_1 - A_{11}^{*}) N_{12} (zI_2 + A_{22}) = -(A_{11}^{*}N_{12}A_{22} - N_{12})$$

$$+ z^{-1} N_{12}A_{22} - A_{11}^{*}N_{12}z$$

premultiplying by $(z^{-1}I_1 - A_{11}^{*})^{-1}$

postmultiplying by $(zI_2 + A_{22})^{-1}$,

and substituting $A_{11}^{*}N_{12}A_{22} - N_{12} = - R_1R_2^{*}$

we get

$$N_{12} = (z^{-1}I_1 - A_{11}^{*})^{-1}R_1R_2^{*} (zI_2 + A_{22})^{-1}$$

$$+ (z^{-1}I_1 - A_{11}^{*})^{-1}z^{-1}N_{12}A_{22}(zI_2 + A_{22})^{-1}$$

$$- (z^{-1}I_1 - A_{11}^{*})^{-1}A_{11}^{*}N_{12}z(zI_2 + A_{22})^{-1}$$

We will follow the same process as we did before

A) $\underline{A_{11}\text{ has no zero eigenvalue}}$

The coefficient equation of z^{-1} is

$$-R_1R_2^* + \left(A_{11}^*\right) N_{12} A_{22} + N_{12} = 0$$

but $$R_1R_2^* + \left(A_{11}^*\right) N_{12} A_{22} - N_{12} = 0$$

Hence

if $A_{22} = 0$, then $R_2 = 0$ and $N_{12} = 0$ is trivial

if $A_{22} \neq 0$ from the above two equations, $R_2 = 0$ and $N_{12} = 0$.

B) <u>A_{11} has zero eigenvalues</u>

Equating the coefficients of z and constant terms of both sides gives the same result.

(10) has a unique solution since A_{22} is a principal case and stable the-limit.

Hence (4) is satisfied, $N = \begin{bmatrix} N_{11} & N_{12} \\ N_{12}^* & N_{22} \end{bmatrix}$ is a positive definite matrix.

This completes the proof of the whole theorem.

2. <u>Application</u>

Consider the following multi-nonlinear system in Fig. 1.

$\Phi : R^m \rightarrow R^m$ is a time-invariant memoryless nonlinearity which satisfies the following

$$\Phi(0) = 0$$

$$\phi'\sigma \leq \phi' K^{-1} \phi$$

Assume

$$k_{ij} = k_{ji}$$

The system is represented by the following difference equation

$$x_{h+1} = A\, x_h - B\phi(\sigma_h)$$

$$\sigma_h = 2C'\, x_h \qquad h=0,1,2,\ldots.$$

σ_h represents the value σ at $t = h$, not a compontent of σ.
A is $(n \times n)$ real matrix whose eigenvalues lie inside the unite circle except possibly one.

B and C are $(n \times m)$ real matrices. Assume that (A,C) is completely observable, $G_{ij}(z)$ is stable-in-the-limit and the system is asymtotically stable for all linear matrix gain M such that

$$0 < m_{ij} < k_{ij} \qquad \text{for } i,j = 1,2,\ldots m$$

<u>Theorem 2</u>

If there exists K^{-1} symmetric matrix with $K^{-1} + \frac{1}{2} \mathcal{H}\left\{G(z)\right\} > 0$, $\forall\ |z|=1$ then the system is absolutely stable for the matrix gain K.

<u>Proof</u>

By the theorem 1, there exist R, Q and a positive definite N such that

$$A'NA - N = -RR'$$

$$A'NB = C + RQ$$

$$K^{-1} - B'NB = QQ'$$

Let the Lyapunov function

$$V = x_h{}'Nx_h$$

then

$$\Delta V = -(Q'\phi - R'x_h)'(Q'\phi - R'x_h) - \phi'(\sigma - K^{-1}\phi) \qquad (11)$$

The asymptotic stability for the linear gain and complete observability ensure $V > 0$ and $\Delta V < 0$

<u>Corollary</u>

If every element of K^{-1} is positive, then the system is absolutely stable for the matrix gain $K = \left|\left|k_{ij}\right|\right|$

Proof

(11) becomes

$$\Delta V < -(Q'\phi - R'x_h)'(Q\phi - Rx_h) - \phi'(\sigma - K^{-1}\phi)$$

The above corollary is more applicable to practical use than the theorem 2 since we now can talk about the individual sector in which the nonlinear lies. This will be illustrated by an example.

Further restrictions on the non-linearities result the following theorem. Let Φ bé represented by a matrix $[\Psi_{ij}(\cdot)]$ where input to $\Psi_{ij}(\cdot)$ is σ_j and output ϕ_i

$$\Psi_{ij}(0) = 0$$

$$\Psi_{ij}(\cdot) = 0 \quad \text{if } i \neq j$$

$$0 \leq \sigma_j \Psi_{ij}(\sigma_j) \leq K_{ij} \sigma_j^2$$

$$-\infty < \frac{d\Psi_{ij}(\sigma_i)}{d\sigma_i} \leq \mu_{ii}$$

for $i,j = 1,2,\ldots m$.

Theorem 3

If there exists a $(m \times m)$ diagonal matrix $\beta = [\beta_{ii}]$ such that

$$K^{-1} + \frac{1}{2}\mathcal{H}\left\{(I + \beta(z-1))G(z)\right\} - \frac{1}{2}|z-1|^2 G^*(z)\mu|\beta|G(z) > 0 \qquad \forall |z| = 1 \qquad (12)$$

then the system is absolutely stabe for the matrix gain K.

μ is a $(m \times m)$ diagonal matrix with μ_{ii}.

Proof

Let the Lypunov function

$$V = x_h' Nx_h + \sum_{i=1}^{m} \beta_{ii} \int_0^{\sigma_{i_h}} \phi_i(\sigma_i) d\sigma_i$$

then ΔV becomes

$$\Delta V = x_{h+1}' Nx_{h+1} - x_h' Nx_h + \sum_{i=1}^{m} \beta_{ii} \int_{\sigma_{i_h}}^{\sigma_{i_{h+1}}} \phi_i(\sigma_i) d\sigma_i$$

Let $\gamma = \sqrt{2}\ (A-I)'C$

$$d = -\left(2\ \mathrm{sgn}\beta\ B'C\mu - I\right) C'(A-I)$$

$$\xi = C'B-B'C\ \mathrm{sgn}\beta\ \mu\ C'B$$

$$\Delta V = x_h' \left(A'NA-N + \gamma\mu|\beta|\gamma'\right) x_h$$

$$- 2\ \phi'(\sigma_h)\left(B'NA-C'-\beta d\right) x_h$$

$$- \phi'(\sigma_h)\left(-B'NB-K^{-1} + 2\beta\xi\right)\phi(\sigma_h)$$

$$- \phi'(\sigma_h)\left(\sigma_h-K^{-1}\phi(\sigma_h)\right)$$

The equation (12) can be rewritten as follows

$$K^{-1} + 2\beta\xi + \mathcal{H}\left\{(C' + \beta d)(Iz-A)^{-1}B\right\}$$

$$- \left(B' \ (Iz-A)^{-1}\right)' \gamma\mu|\beta|(Iz-A)^{-1}B>0 \qquad \forall|z|=1$$

By the theorem 1 there exist R and Q such that

$$A'NA-N + \gamma\mu|\beta|\gamma' = RR'$$

$$B'NA-C'-\beta d = 0$$

$$K^{-1}-B'NB-QQ' = 0$$

Hence

$$V \leq (R'x_h - Q\phi(\sigma_h))'(R'x_h - Q\phi(\sigma_h)) - \phi'(\sigma_h)(\sigma_h - K^{-1}\phi(\sigma_h))$$

The complete observability and the asymtotic stability for the linear gain ensure $V > 0$ and $\Delta V < 0$.

<u>Example</u>

The transfer function and K are given by Fig. 2

Let

$$G(z) = \begin{bmatrix} \dfrac{0.368z + 0.264}{(z-1)(z-0.368)} & \dfrac{1}{z-1} \\ \\ \dfrac{1}{z-1} & \dfrac{0.4}{z-1} - \dfrac{0.1}{z} \end{bmatrix}$$

If

$$K^{-1} = \begin{bmatrix} \dfrac{3}{2} & \dfrac{1}{2} \\ \\ \dfrac{1}{2} & \dfrac{3}{10} \end{bmatrix}$$

then $K^{-1} + \frac{1}{2}\mathcal{H}\left\{G(z)\right\} > 0$

Hence

$$|K| = \begin{bmatrix} \frac{15}{10} & \frac{5}{2} \\ \frac{5}{2} & \frac{15}{2} \end{bmatrix} \quad \text{is a stable gain matrix.}$$

Now let $\phi_{12}(\cdot) = 0$, $\phi_{21}(\cdot) = 0$,

$$\frac{d\phi_{11}(\sigma_1)}{d\sigma_1} < \mu_{11} \quad \text{and} \quad \frac{d\phi_{22}(\sigma_2)}{d\sigma_2} < \mu_{22}$$

We are to obtain K^{-1} and β such that

$$K^{-1} + \frac{1}{2}\mathcal{H}\left\{(I + \beta(z-1)G(z)\right\} - \frac{1}{2}\ |z-1|^2 G^*(z)\mu G(z) > 0 \qquad \forall |z|=1$$

Manual calculation is almost impossible. However, the problem is well within the capacity of computers available nowadays.

Appendix

Stability-in-the-limit

Definition

In the system is shown in Fig. 3 if there exist $\delta_0 > 0$ such that for any $\delta \in \left\{\delta' \middle| 0<\delta' \leq \delta_0\right\}$, the linearized system with $\gamma(y(t_h)) = \delta y(t_h)$ is asymtotically stable, we call the system to be stable-in-the-limit.

Theorem 3

$$x(t_{h+1}) = A\ x\ (t_h)\ -Bu(t_h) \tag{13}$$

$$y(t_h) = 2C'\ x(t_h) \qquad h = 0,1,2,\ldots$$

The system (13) is stable-in-the-limit if and only if there exists a constant $\delta_0 > 0$ such that $A - 2\delta_0 BC'$ is a stable matrix.

Theorem 4

If G(z) has only one pole on the unit circle at z=1, the system is stable-in-the-limit and only if the residue of G(z) is positive at z=1.

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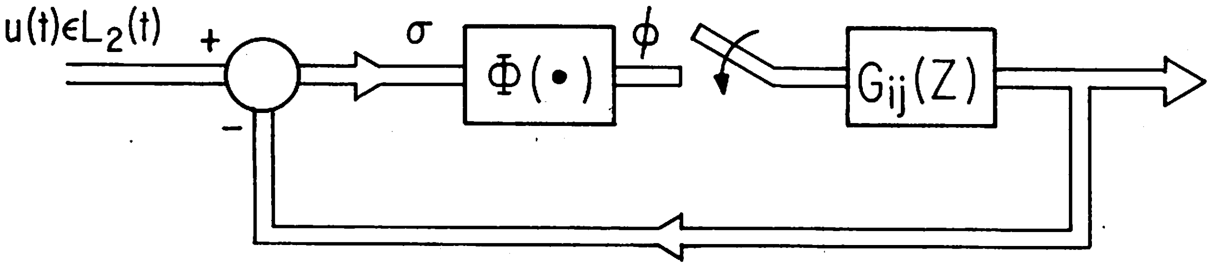

Fig. 1

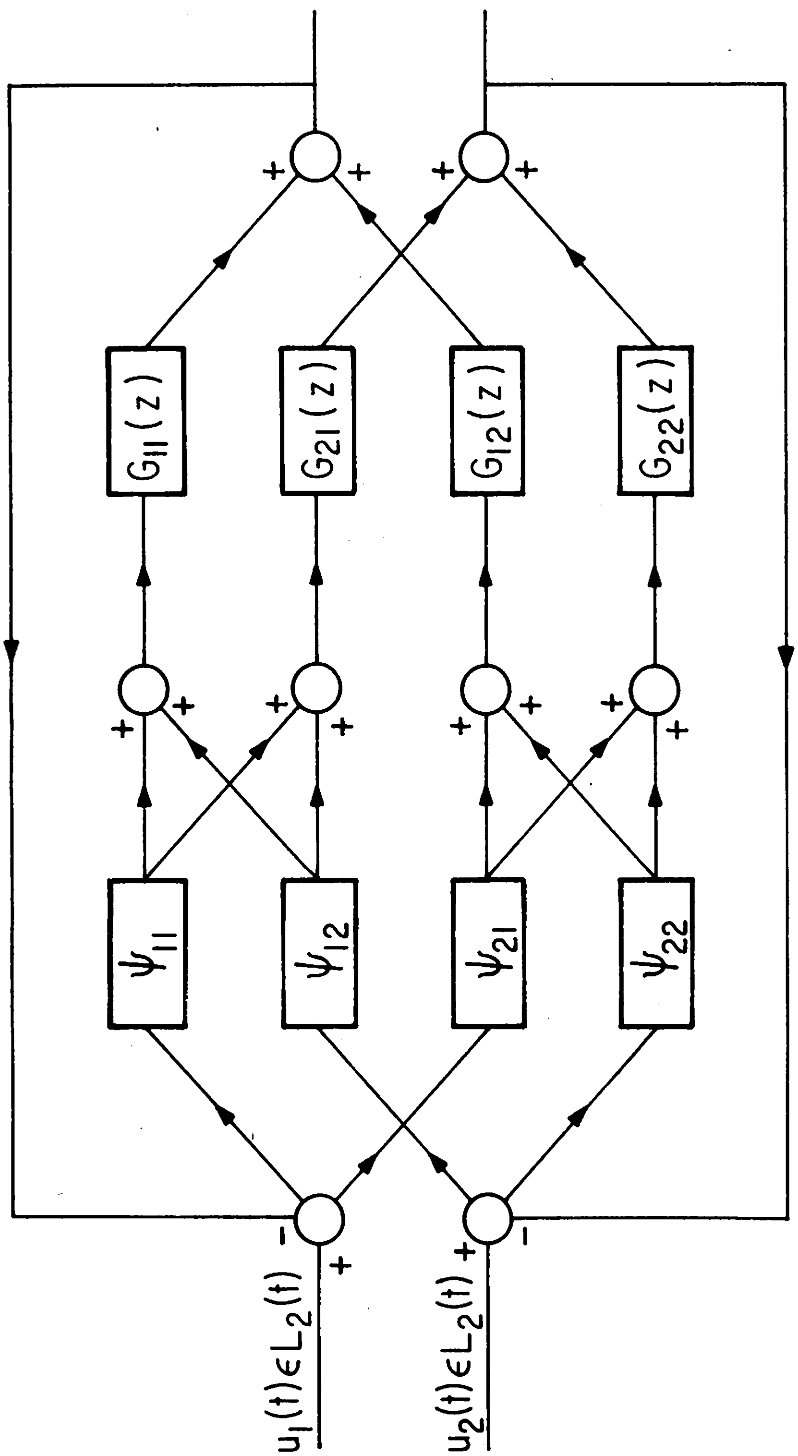

Fig. 2

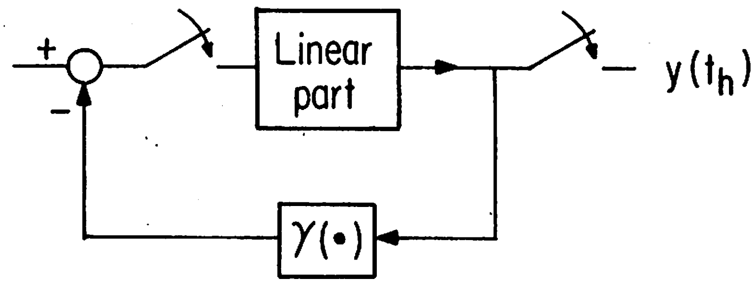

Fig. 3