DETERMINATION OF THE TRANSITION BETWEEN ADIABATIC AND STOCHASTIC MOTION

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DETERMINATION OF THE TRANSITION BETWEEN ADIABATIC AND STOCHASTIC MOTION*

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Abstract

For nonlinear dynamical problems with two degrees of freedom a transition occurs between regions of the phase space in which stochastic motion is closely bounded by invariant (Kolmogorov, Arnold, Moser or KAM) surfaces, and regions for which the stochastic motion is interconnected over large portions of the space. The mechanism by which this transition takes place is discussed, qualitatively, in terms of the growth of primary and higher order islands. These islands both change the topology of the phase plane and introduce stochastic bands near island separatrices. The relationship between the complete dynamical problem and a two-dimensional mapping is given. Various criteria for the transition are developed. Quantitative values of the strength of the perturbation parameter, required to produce connected stochasticity, are given for the various criteria, as applied to a standard mapping. In particular, the criteria of loss of linear stability of the primary islands, resonance overlap, growth of second order islands, improved overlap criteria, and loss of linear stability at rational iterates of the irrational rotation number connected with a given KAM surface, are considered. The latter two criteria, based on the work of Chirikov and Greene, respectively, are particularly emphasized.

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1. Introduction

It is well known that in most nonlinear systems with two degrees of freedom, a transition occurs between regions of the phase space in which stochastic motion is closely bounded by smooth surfaces, and regions for which the stochastic motion is interconnected over large portions of the space. The existence of the smooth surfaces derives from a theorem conjectured by Kolmogorov and proved under different restrictions by Arnold (1963) and Moser (1962), commonly known as the KAM Theorem. It states that, for mutiply periodic systems close to integrable ones (for which all trajectories lie on smooth surfaces), provided the initial conditions correspond to frequencies sufficiently far from system resonances, the perturbed system will also have trajectories lying on smooth surfaces. The problem of proving the KAM theorem is that a Fourier analysis of the fundamental frequencies generates a large set of closely spaced resonances, such that a resonance condition $\underset{\sim}{m}.\underset{\sim}{\omega} = 0$ is approximately satisfied everywhere. This requires careful adjusting of initial conditions at each iteration in a perturbative expansion in order to remain sufficiently far from the resonances. Consequently, it has been difficult to estimate the size of the perturbation at which a KAM curve between two resonances ceases to exist. It is also possible to show from the existence of homoclinic points (intersecting trajectories) near the separatrices of resonances that there are stochastic trajectories in these neighbourhoods (see, for example, Moser, 1973). Therefore the disappearance of a last KAM surface between

resonances implies that the stochasticity is then connected between the resonances.

It is possible to obtain a sufficient condition for this connected or global stochasticity by requiring that no trajectory be single valued in phase i.e. that all trajectories are either island trajectories or intersecting (stochastic) trajectories. This can be insured by an even stronger condition that all fixed points are unstable , (Lieberman and Lichtenburg, 1972). However, what we really would like is the weakest possible condition for connected stochasticity, i.e. one that is both necessary and sufficient. Most attempts to obtain such a sharp condition are at least partly numerical, and we therefore need efficient methods for numerically studying the topology of a trajectory.

A convenient way to study phase space trajectories, particularly in problems with two degrees of freedom, is by means of a surface of section. If we hold one coordinate of a four dimensional phase space constant, then the motion is defined on a three-dimensional surface within the four-dimensional space; that is, we look only at the discontinuous points of intersection of the trajectory with the surface. If, further, the energy is taken as a constant of the motion, then that constant restricts the motion to a three dimensional surface whose intersections in the surface of section are then two-dimensional. An existence of another constant of the motion then reduces the phase space trajectory to a one-dimensional discontinuous curve of the intersections with the two-dimensional surface of section. Furthermore, for multiply-periodic motion this curve is asymptotically dense, except for the set of rationals, which have zero measure. Thus we can examine the non-linear motion in two degrees of freedom, with constant Hamiltonian, and determine if an invariant exists, by determining if the trajectory lies on a closed curve in a surface of section. The technique, first introduced by Poincare (1892) has been invaluable for numerical determination of the existence of invariants.

For integrable systems the equations of a mapping, from the n to $n+1$ crossing, are (in action-angle form)

$$J_{n+1} = J_n \ , \quad \theta_{n+1} = \theta_n + 2\pi\alpha(J_o) \tag{1.1}$$

where $\alpha = \omega_1/\omega_2$ is the rotation number. The mapping given by (1.1) is called a twist mapping, for which circles map into circles, but with the rotation number dependent on the radius of the circle. Fixed points of the mapping occur for $\alpha = r/s$, where r and s are integers.

If we perturb the integrable Hamiltonian which led to the mapping in (1.1), the new perturbed mapping can be expressed as

$$\begin{aligned} J_{n+1} &= J_n + \varepsilon f(J_n, \theta_n) \\ \theta_{n+1} &= \theta_n + 2\pi\alpha(J_n) + \varepsilon g(J_n, \theta_n) \end{aligned} \tag{1.2}$$

where f and g are periodic and the entire transformation is area preserving. J_n is no longer a constant of the motion in the perturbed transformation. However, there has been considerable progress in understanding the behaviour of a mapping such as (1.2). First, if we examine irrational surfaces with $\alpha > r/s$ and $\alpha < r/s$, sufficiently far from the rational $\alpha = r/s$, the KAM theorem tells us that the surfaces retain their topology and are only slightly deformed from the unperturbed circles. On the rational surface $\alpha = r/s$, and in a neighbourhood about it the KAM theorem breaks down. However it can be shown that at least $2s$ fixed points remain after the perturbation, which become the elliptic and hyperbolic fixed points of an s-island chain. Mapping theory also demonstrates the existence of the homoclinic points which can be considered as the origin of the stochasticity.

We must look for a procedure, which can be applied directly to mappings, as well as to the complete systems from which they arise, to predict the disappearance of the last KAM curves between the principle system resonances. The earliest procedure, advanced by Chirikov (e.g. Chirikov, 1960), and later refined by him (Chirikov, 1979), to determine such a transition, is now known as the overlap criterion. In its simplest form it postulates

that the last KAM surface between two lowest order resonances is destroyed when the sum of the half-widths of the two islands formed by the resonances, but calculated independently of one another, just equal the distance between the resonances. The distances are measured either in action or frequency space, whichever is more convenient. This criterion has an intuitive appeal, since we know that regions near the separatrix are, in fact, stochastic. Rigorously, however, the overlap criterion is neither necessary or sufficient. One can imagaine the last KAM surface breaking up well before the islands overlap, due to the interaction of the slowly varying terms outside of the two separatrices. Alternatively, the solution of the complete problem may significantly modify the island widths so that they do not actually overlap when the single resonance calculations predict that they do. In actual fact, numerical results indicate that the overlap criterion is too severe a condition for stochasticity. It can, however, be made sharper by considering both the width of the stochastic region near the separatrix and the more important higher s-value resonances, between the main ones. This procedure, as developed by Chirikov (1979), is the main subject of sec.3. Without the added complexity of these modifications the simplest form of the criteria still serves as a rough estimate of the transition, and has been used in a wide variety of problems by Chirikov (1960, 1979), Ford and coworkers (e.g. Walker and Ford, 1969), Rosenbluth et al. (1966) and by many others (see Chirikov, 1979, for a more complete bibliography).

A related procedure, introduced by Jaeger and Lichtenberg (1972), calculates second order resonances between the island oscillations and the mapping frequency. They show that before resonance overlap the second order islands have grown to comparable relative size as the primary islands, and, by induction, higher order islands have attained similar amplitudes. At "overlap" the local winding number of the primary island about its fixed point is $\alpha = 1/4$, indicating 4 second order islands. The construction makes intuitively obvious the fact that an overlap criterion is too severe,

and numerically it was found that an island amplitude 2/3 the size, corresponding to an $n = 6$ island chain, is sufficient to destroy the last KAM curve between first order islands. The technique has been applied to a number of problems, including for example, Fermi acceleration (Lieberman and Lichtenberg, 1972) and cyclotron heating (Jaeger et al, 1972, Lieberman and Litchtenberg 1973), giving good agreement with numerical computations. The modified overlap criterian of Chirikov (1979) also makes use of second order islands to determine the stochastic width near the separatrix, but in his calculation the expansion is performed near the separatrix, rather than near the singular point.

A third method of determining the stochastic barrier returns to the examination of linear stability. The idea is that, although loss of stability of the lowest order islands ($s = 1$) is too strong a condition, the linear stability transition of the high k number islands close to a KAM curve may give a sharper criterian for the destruction of that surface. This proposition has been investigated, numerically, by Greene (1968,1979a), and found to be correct. More specifically, if a surface with an irrational winding number α is approximated more and more closely by a ratio of rationals, then the asymptotic stability of the motion about the fixed points is directly corrrelated to the existence of a KAM surface at the given α. In particular, between two lowest harmonic ($s = 1$) primary resonances the irrational number that is furthest away from neighboring rationals can be shown to be the golden mean $(\sqrt{5} - 1)/2$. Thus one expects that with increasing perturbation parameter the last KAM surface to disappear would be this one. Numerical calculations by Greene (1979) show this to be the case. Therefore the transition which destroys the last KAM barrier to stochastic wandering between $s = 1$ islands is found by determining the stability of the rational iterates of the golden mean. An interesting observation is that the transition occurs when the islands have <u>local rotation number</u> $\alpha = 1/6$. Since the primary island is the first

of the iterates, this is related directly to the observation of the transition using the other methods. It has also been shown to be possible to accurately calculate the stability of a KAM surface, directly, using variational techniques by Percival (1979).

If a system has more than two degrees of freedom, then, in addition to the phenomena described in this paper, a slow diffusion is possible. This is due to the connection throughout the phase space of narrow bands of stochasticity associated with the resonance separatrices. The mechanism was first described by Arnold and calculations have been made by Chirikov (1979). This diffusion is generally easily distinguishable from the global stochasticity described here.

Because of lack of space no attempt has been made in this review to reference the historical development of the subject, and the many contributions to a wide variety of examples. References to early work can be found in Lichtenberg (1969). For more recent developments, the reader should consult the review by Chirikov (1979).

2. Growth of Second Order Islands

2a. A Model for the Primary Resonances

We consider as a physical model for a mapping that of a ball bouncing between a fixed and an oscillating wall. The problem was originally examined by Fermi (1949) as an analogue to a cosmic ray acceleration mechanism. Numerical calculations gave conflicting results, depending on initial conditions, sometimes indicating oscillatory energy changes and sometimes indicating stochastic energy changes. Zaslaviskii and Chirikov (1965) partly resolved this contradiction by showing that for high velocities, in which the transit time is comparable to the wall-oscillation period, an adiabatic invariant exists to limit the energy excursions. A more complete treatment by Lieberman and Lichtenberg (1972) indicated that for smooth forcing functions the phase plane divides into three distinct regions. I. At low velocities all fixed points are unstable, leading to

stochastic motion over the entire region. II. At intermediate velocities islands of stability surrounding elliptic fixed points are embedded in a stochastic sea. III. At high velocities narrow bands of stochasticity near separatrices joining hyperbolic fixed points are bounded by regular orbits. The calculation of the transition between region II and region III, which is the boundary between contained and global stochasticity, is the subject matter of this review.

A simplified set of difference equations for the Fermi acceleration can be obtained by approximating the interaction as that in which the oscillating wall imparts momentum to the ball without physically changing its position in space. Assuming the wall oscillates with velocity $v_w = v_{wo} \sin\psi$ where $\psi = \omega t$, then defining a normalized ball velocity $u = v/v_{wo}$, the difference equations for the motion are

$$u_{n+1} = u_n + \sin\psi_n \tag{2.1}$$

and

$$\psi_{n+1} = \psi_n + 2\pi M/u_{n+1} \tag{2.2}$$

where $2\pi M/u = \omega(2\ell/v)$ with $2\ell/v$ the transit time between collisions. The mapping given by Eqs.(2.1) and (2.2) are area preserving and periodic in ψ. The behavior in the neighborhood of a particular value of velocity can be obtained by linearizing in velocity about the value of $u = u_o$ that makes the phase stationary for single iterations of the mapping ($s = 1$ fixed points). From (2.2) this condition can be seen to be

$$2\pi M/u_o = 2\pi m , \qquad m \text{ integer} \tag{2.3}$$

The mapping equations then take the form

$$I_{n+1} = I_n + K \sin\theta_n, \qquad \theta_{n+1} = \theta_n + I_{n+1} \tag{2.4}$$

where $\theta_n = \psi_n - 2\pi m$, $I_n = 2\pi M u_n/u_o^2$ and $K = -2\pi M/u_o^2$. The new mapping is periodic in both ψ and I, period 2π, and the nonlinearity, normalized to a given value of $u = u_o$ scales with K. Eqs.(2.4), which have been called the <u>standard mapping</u> by Chirikov, is locally equivalent to a large class of mappings.

The fixed points of either mapping are obtained by requiring in addition to (2.3) that the action is also stationary. For the purpose of this section we concern ourselves only with the $s = 1$ fixed points, which from (2.1) or (2.4) gives $\sin \psi_n = \sin \theta_n = 0$. Expanding about this fixed point we obtain the linearized matrix of the transformation

$$A = \begin{bmatrix} 1 & 1 \\ -K & 1-K \end{bmatrix} \tag{2.6}$$

for which $\det A = 1$ as required for area preserving mappings. The eigenvalues are obtained from

$$2 \cos \sigma = \mathrm{Tr}\, A \tag{2.7}$$

from which we have the stability condition, $|\cos \sigma| < 1$, or $(2 - K) < 2$. Thus for

$$K > 4 \tag{2.8}$$

the elliptic singular point changes to reflection hyperbolic, and there is no stable motion about single iteration fixed points.

We can also construct a Hamiltonian for the motion in the neighborhood of the $s = 1$ fixed points by introducting a δ-function force into the equations of motion and then expanding the δ-function in a Fourier series (Lieberman and Lichtenberg, 1972). The result for the standard mapping is

$$H = \frac{I^2}{2} + K \cos \theta \sum_{m=-\infty}^{\infty} e^{i2\pi m\tau} \tag{2.9}$$

where τ is a time variable in units of the bounce time at the fixed point. we are generally interested in $\frac{d\theta}{d\tau} \ll 2\pi$. Anticipating that the only term with time variation that will contribute significantly is one with a slowly varying phase, we keep only terms with $m = 0$ and $m = \pm 1$ to obtain

$$H = \frac{I^2}{2} + K \cos \theta + 2K \cos \theta \cos 2\pi\tau \tag{2.10}$$

Assuming the third term on the right is a perturbation on the motion which tends to average to zero, we have an unperturbed Hamiltonian

$$H_o = \frac{I^2}{2} + K \cos \theta \tag{2.11}$$

which is just that of the nonlinear pendulum. Its phase space trajectories

are nearly ellipses around $\theta = \pi$, changing to a separatrix trajectory through $\theta = 0, 2\pi$. The motion is libration out to the separatrix, on which the period becomes infinite, and that of rotation beyond.

For our present purposes we expand (2.11) for small θ near the elliptic fixed point. The transformation to action-angle variables J_o, ϕ_o, to lowest order in θ and I (which is the linearized motion), is (e.g. Lichtenberg, 1969)

$$I = (2 J_o R)^{\frac{1}{2}} \cos \phi_o \tag{2.12}$$

$$\theta = (2 J_o/R)^{\frac{1}{2}} \sin \phi_o \tag{2.13}$$

such that the lowest order Hamiltonian

$$\overline{K}_o = \Omega_B(o)\, J_o \tag{2.14}$$

where

$$R = (\partial^2 H/\partial\theta^2)/(\partial^2 H/\partial I^2) \tag{2.15a}$$

is the aspect ratio of elliptic orbits in phase space, and

$$\Omega_B(o) = [(\partial^2 H/\partial\theta^2)(\partial^2 H/\partial I^2)]^{\frac{1}{2}} \tag{2.15b}$$

For the simple form of the pendulum Hamiltonian in (2.11)

$$R = K^{\frac{1}{2}} \quad \text{and} \quad \Omega_B(o) = K^{\frac{1}{2}} \tag{2.16}$$

The peak excursion of I as a function of K is found, approximately, from (2.11) by taking $\cos\theta = 1$ at $I = 0$ such that the maximum excursion at $\cos\theta = -1$ is

$$\Delta I_{Max} = 2K^{\frac{1}{2}} \tag{2.17}$$

Since the distance between primary resonances δI is, in the standard mapping approximation, just equal to the periodicity 2π, the ratio of the full island width to the distance between islands is

$$2\Delta I_{Max}/\delta I = 4K^{\frac{1}{2}}/2\pi \tag{2.18}$$

which can be related to the central angular frequency and therefore the local rotation number through (2.16)

$$2\Delta I_{Max}/\delta I = 4\Omega_B(o)/2\pi = 4/n \tag{2.19}$$

where $n = 1/\alpha$ is the number of second order elliptic fixed points of the mapping trajectory. Eq. (2.19) is a universal relation between all

neighboring island chains of any order, relating the relative island size to its rotation number. For example, the experimental observation that the transition to global stochasticity occurs with the appearance of a set of six secondary islands (n = 6) implies, from (2.19), that $2\Delta I_{Max}/\delta I = 2/3$.

2b. Second Order Islands near the Elliptic Singular Point

Because of the nonlinearity of the oscillation of the pendulum, the motion contains harmonic components of the fundamental frequency. These components can resonate with the fast motion to produce local distortions in the phase plane, or second order islands. As the complete representation is rather cumbersome, we separate the problem into expansions valid near the elliptic fixed point and near the separatrix, the latter to be presented in the next section.

Near the elliptic fixed point standard perturbation theory (e.g. Born, 1927) can be used to determine the action-angle representation of (2.11) to next higher order in the nonlinearity, obtaining a Hamiltonian

$$K(J, \phi) = \overline{K}_0 + \overline{K}_2 \tag{2.20}$$

where

$$\overline{K}_2 = \langle H_2 \rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{K}{4!} \left(\frac{2J}{R}\right)^2 \cos^4 \phi \, d\phi = \frac{K}{16} \left(\frac{J}{R}\right)^2 \tag{2.21}$$

and the corresponding frequency

$$\Omega_B = \Omega_B(o) + \frac{\partial \overline{K}_2}{\partial J} \tag{2.22}$$

Reintroducing the time-dependent term from (2.10), Eq. (2.20) becomes

$$\overline{K} = \overline{K}_0 + \overline{K}_2 + \Lambda$$

where

$$\Lambda = 2K \cos\left[(2J/R)^{\frac{1}{2}} \sin \phi\right] \cos 2\pi\tau$$

The first term can be expanded in a Fourier series in ϕ, to obtain

$$\Lambda = K \sum_{\ell=0}^{\infty} \mathcal{J}_{2\ell}\left[(2J/R)^{\frac{1}{2}}\right] \sin(2\ell\phi - 2\pi\tau) \tag{2.23}$$

where the $\mathcal{J}$ are the ℓ^{th} order Bessel functions of the first kind. The terms in Λ will average to zero with τ, except in the neighborhood of the action for which $2\ell\Omega_B = 2\pi$. For this term, $2\ell = n$, the sinusoid in

(2.23) is slowly varying. We then use the transformation of secular perturbation theory (see Jaeger and Lichtenberg, 1972) to a locally slow phase variable

$$\hat{\phi} = n\phi - 2\pi\tau \tag{2.24}$$

with corresponding action

$$\hat{J} = J/n . \tag{2.25}$$

Averaging over τ, as previously, all terms except $2\ell = n$ are approximately zero. In the remaining term we expand the action about the local fixed point with $\partial\overline{K}/\partial J = 0$ to obtain

$$\Delta\overline{K} = \frac{\partial^2\overline{K}}{\partial\hat{J}^2}\frac{(\Delta\hat{J})^2}{2} + \Lambda_n \sin\hat{\phi} \tag{2.26}$$

where

$$\Lambda_n = K\mathcal{J}_n((2J/R)^{\frac{1}{2}}) . \tag{2.27}$$

The Hamiltonian of (2.26), describing the second order islands, has the same form as the Hamiltonian of (2.10), describing the primary islands. With all second order resonances present the frequency separation is

$$\delta\Omega_B = \Omega_{Bn} - \Omega_{B\,n+1} = 2\pi/n - 2\pi/(n+1) \cong \Omega_B/n . \tag{2.28}$$

For the symmetry of (2.23) only even harmonics exist, and $\delta\Omega_B = 2\Omega_B/n$. In terms of the action

$$\delta\Omega_B = [\partial^2\overline{K}/\partial(n\hat{J})^2]\delta(n\hat{J}) . \tag{2.29}$$

Using the same procedure as with the first order islands we calculate the excursion of the first order action,

$$\Delta J_{Max} = 2[\Lambda_n/(\partial^2 K/\partial\hat{J}^2)]^{\frac{1}{2}} , \tag{2.30}$$

and the second order island frequency,

$$\Omega_{Bs}(o) = [\Lambda_n(\partial^2\overline{K}/\partial\hat{J}^2)]^{\frac{1}{2}} . \tag{2.31}$$

Eqs. (2.28) through (2.31) can be combined to give, as in (2.19) for the primary islands,

$$2\Delta\hat{J}_M/\delta\hat{J} = m\Omega_{Bs}(o)/\Omega_B(o) , \quad \begin{matrix} m = 4, \text{ no symmetry} \\ m = 2, \text{ even or odd symmetry} \end{matrix} \tag{2.32}$$

We now show that the second order island amplitude is small compared to

the distance between second order islands, except when the first order island size is also large. We first calculate $\Omega_{Bs}(o)$ explicitly. From (2.21) $\partial^2\overline{K}/\partial\hat{J}^2 = n^2K/8R^2 = \dot{n}^2/8$ which, when substituted in (2.31), together with Λ_n from (2.27) and $\Omega_B(o) = K^{\frac{1}{2}}$, gives, for (2.32)

$$2\Delta\hat{J}_M/\delta\hat{J} = 2[\mathcal{J}_n((2J/R)^{\frac{1}{2}})n^2/8]^{\frac{1}{2}} . \tag{2.33}$$

The argument of the Bessel function has its largest value at π (the separatrix trajectory). Thus for $n \gg \pi$ the Bessel function is exponentially small, and the 2^{nd} order islands are negligible. By a similar argument islands arising from higher order iteration of the mapping are also negligible. We are now in a position to calculate for what value of the perturbation the second order island perturbation is as important as that of the primary islands. Setting the ratio of (2.33) to (2.19) equal to unity we obtain for n ($n = 1/\alpha$)

$$n^4J_n(\pi) = 32 \tag{2.34}$$

Eq. (2.34) is satisfied for $n \approx 5$, a 5 island second order resonance. For $n = 5$ the effect of second order islands is of prime importance. (A previously quoted result, which indicated $n \approx 4$ at this crossover, resulted from the omission, there, of a small numerical factor (Lichtenberg, 1979)). By induction, the same result exists between second and third order islands, and consequently to all orders. Thus we obtain the remarkable result; at a critical value of rotation number of $\alpha = 1/5$ all higher order islands simultaneously become comparable to the distance between them. The result of the sharp transition to the destruction of that last KAM surface between islands is thus very physically plausible. From (2.16) the corresponding perturbation parameter is $K = 1.2$.

Detailed numerical calculations of the second order island structure have not been made for the standard mapping. However they have been performed for a number of two degree of freedom Hamiltonians in which an appropriate portion of the phase plane in a surface of section is similar to that of the standard mapping. Probably the closest correspondence is

that of a particle gyrating in a magnetic field and resonating with an obliquely propagating wave (Smith and Kaufman, 1975; Smith, 1977). Their Hamiltonian, in the wave frame, can be written in the form

$$H = \frac{P_\psi^2}{2M} - P_\psi \Omega + P_\phi \Omega - e\phi_o \sum_m \mathcal{J}_m[f(P_\phi)] \sin(\psi - m\phi).$$

Choosing P_ϕ such that $\mathcal{J}_m$ is of order of unity, a few nieghboring harmonics are analyzed near resonance. They chose $m = -1, 0, 1$ and with the appropriate choice of perturbation, amplitude $e\phi_o$ the result in Fig. 1 is obtained. Chains of five second order islands are seen around each

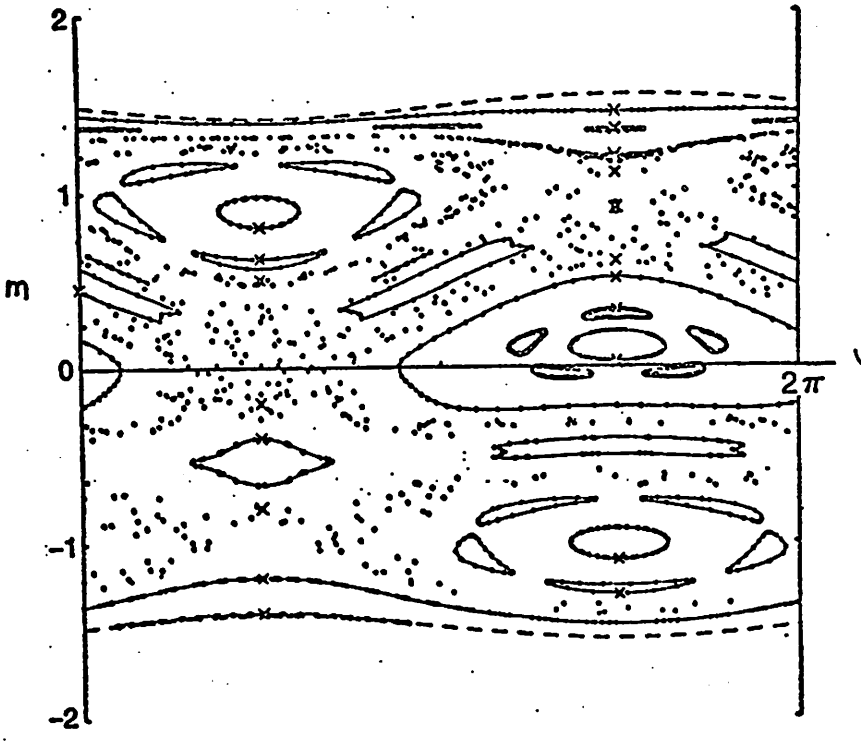

Fig. 1
Numerically computed trajectories in a surface of section for a gyrating particle resonating with an obliquely travelling wave (from Smith, 1977)

primary resonance. As expected, the last KAM surface between resonances is also seen to have disappeared, as the dots represent a single set of initial conditions whose trajectory wanders freely between the primary resonances.

3. Resonance Overlap Criteria

We know that the existence of homoclinic and heteroclinic points in the neighborhood of a separatrix lead to local stochastic motion. If the amplitudes of two neighboring islands become sufficiently large that the stochastic regions near their separatrices join, then we expect connected stochasticity between the two resonances. This is the rationale of an

overlap criterion. Starting from this picture Chirikov (1979) has constructed a sharp quantitative criterion for the transition to connected stochasticity, i.e. the value of perturbation required to destroy the last KAM surface between two $s = 1$ primary resonances. A numerical plot of the standard mapping of (2.4), with a value of $K = 0.97$, is shown for five orbits in Fig. 2. The shaded regions are explored by the stochastic wandering of a mapping trajectory with the blank regions (those forbidden by the existence of KAM trajectories) surrounding the main-island elliptic singularities. Here KAM orbits still exist, isolating the $s = 2$ trajectory from the $s = 1$ trajectory, but the $s = 4$ trajectory has been engulfed by the stochasticity near the $s = 1$ island separatrix. The amplitude of the $s = 2$ and, to a lesser extent the higher s harmonic islands, as well as the thickness of the stochastic regions near the separatrices of the islands, contributes toward the merging of the $s = 1$ stochastic regions.

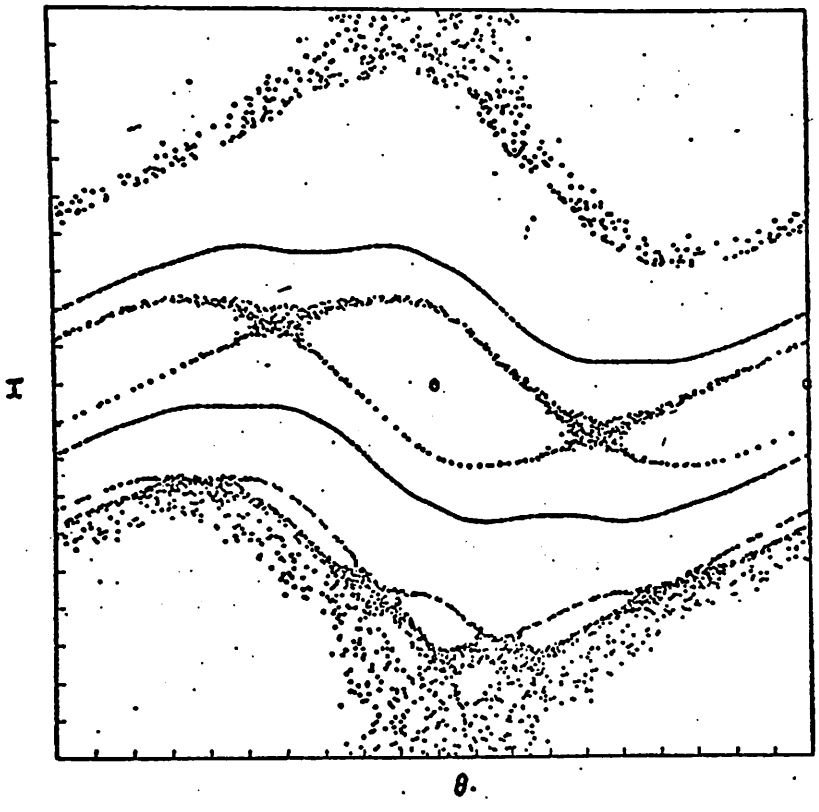

Fig. 2

Trajectories for the standard mapping $K = 0.97$ (from Greene, 1979a)

3a. Overlap of First Order Resonance

Using the averaged Hamiltonian of (2.11) we have found the maximum excursion of the primary island in (2.17) to be $\Delta I_{Max} = 2K^{\frac{1}{2}}$ and the

distance between resonances $\delta I = 2\pi$. Taking the ratio of $2\Delta I_{Max}/\delta I = 1$ we obtain the simplest overlap criterion for primary resonances

$$K = (\pi/2)^2 \tag{3.1}$$

We know this K is too large, so we improve on it by calculating the amplitude of the two iteration resonance ($s = 2$) and define the overlap criterion as

$$\Delta I_{1_{Max}} + \Delta I_{2_{Max}} = \delta I_{12} = \pi \tag{3.2}$$

where the subscripts 1 and 2 indicate the $s = 1$ and $s = 2$ resonances, respectively.

Following Chirikov (1979) we calculate the width of the $s = 2$ island by employing standard perturbation theory to eliminate terms in (2.10) linear in K. The Hamiltonian to order K^2 is

$$H_2 = \frac{I_2^2}{2} + \frac{K^2}{4} \sum_{\ell,m} \frac{\cos(2\theta - 2\pi(\ell + m)\tau) - \cos 2\pi(\ell - m)\tau}{(2\pi\ell - I)(2\pi n - I)} \tag{3.3}$$

The phase of the first term is slowly varying for $2\frac{d\theta}{d\tau} = 2\pi(\ell + m)$ and eliminating the integral resonances this is $\frac{d\theta}{d\tau} = (2p + 1)/2$, p integer. Since $I = \frac{d\theta}{d\tau}$, from the original mapping, $I = (2p + 1)/2$. Substituting this into (3.3) with the change in variable $2\hat{\theta} = 2\theta - 2\pi(\ell + m)\tau$, for a given p, $(2p + 1 = \ell + m)$, we obtain

$$H = \frac{I_2^2}{2} + \frac{K^2}{4} \cos 2\hat{\theta} \frac{1}{(2\pi)^2} \sum_n \frac{1}{(n - p - \frac{1}{2})^2} \tag{3.4}$$

where $2\hat{\theta}$ is slowly varying near the half integral resonances and $\hat{I}_2 = I_2$. The sum is independent of p and equal to π^2. When this value is substituted in (3.4) we obtain the Hamiltonian for the half integer resonances

$$H = \hat{I}_2^2/2 + (K/4)^2 \cos 2\hat{\theta} \tag{3.5}$$

which gives a peak amplitude of $\hat{I}$ at the separatrix

$$\hat{I}_{2_{Max}} = K/2 \tag{3.6}$$

The improved value of K at overlap is then obtained by substituting (3.6) into (3.2) giving

$$2K^{\frac{1}{2}} + K/2 = \pi, \quad \text{or} \quad K = 1.76, \tag{3.7}$$

as found by Chirikov. Eq. (3.7) still overestimates K. We can further improve the estimate by (1) taking account of the next higher harmonic (s = 3) resonance and (2) taking account of the finite width of the stochastic layer near the separatrix. Chirikov made both of these calculations, finding the dominant effect to be the stochastic layer width.

3b Second Order Island Overlap near the Separatrix

The Hamiltonian of (2.11) can be written in action-angle form, for arbitrary initial conditions, using the definition of J and the corresponding angle variable

$$J = \frac{1}{2\pi} \int_0^{2\pi} (2H_o + 2K \cos\theta)^{\frac{1}{2}} \, d\theta \tag{3.8}$$

$$\phi = \int_0^{\theta} \frac{d\theta}{(2H_o + 2K \cos\theta)^{\frac{1}{2}}} \tag{3.9}$$

Eqs. (3.8) and (3.9) can be written in terms of elliptic integrals (see Smith, 1977)

$$J = K^{\frac{1}{2}}(8/\pi)[E(\kappa) - (1-\kappa^2)D(\kappa)], \qquad \kappa < 1 \tag{3.10}$$

$$\phi = (\pi/2)[D(\kappa)]^{-1} F(\xi, \kappa), \qquad \kappa < 1 \tag{3.11}$$

Here $D(\kappa)$ and $E(\kappa)$ are the complete elliptic integrals of the first and second kind (the usual symbol for D is K), $F(\xi, \kappa)$ the incomplete form of D, $2\kappa^2 = 1 + H/K$, and $\xi = \theta/2$. The quantity κ is a measure of the normalized oscillator energy, with $\kappa = 1$ at $H/K = 1$, the separatrix energy. Setting $\phi = 2\pi$ in (3.11) we obtain the normalized frequency

$$\Omega_B(\kappa)/\Omega_B(o) = \pi/2D(\kappa) \tag{3.12}$$

with an asymptotic value for κ near unity of

$$\lim_{\kappa \to 1} \Omega_B(\kappa)/\Omega_B(o) = -(\pi/2)\ln[(1-\kappa^2)^{\frac{1}{2}}/4] \tag{3.13}$$

A general overlap condition for second order islands can be written from (2.32), after substituting for Ω_{Bs} from (2.31) as

$$2\Delta J/\delta J = 2(\Lambda_n |d\Omega_B/dJ|)^{\frac{1}{2}} \, n/\Omega_B = 1 \quad \text{(even symmetry)} \tag{3.14}$$

where the quantities are now evaluated for arbitrary values of the action.

From (3.12) and (3.10) we obtain

$$\left|\frac{d\,\Omega_B}{dJ}\right| = \frac{\pi^2}{16}\,\frac{1}{\kappa D^3}\,\frac{\kappa}{1-\kappa^2}\left[D - \frac{D-E}{\kappa^2}\right]\Omega_B \tag{3.15}$$

We obtain Λ_n from a Fourier analysis of the third term in (2.10). The calculation is rather complicated, involving complex integration to evaluate an improper integral. Smith and Pereira (1978, see their App. A) find

$$\Lambda_n = \frac{(\pi/D)^2 n q^{n/2}}{1-(-q)^n}\,, \quad q = \exp\left(-(1-\kappa^2)^{\frac{1}{2}}\right) \tag{3.16}$$

Fukuyama et al. (1977) using a simplified form of (3.16) have plotted the percentage of the primary action space for which second order islands occur, as a function of $N = n(o)$. They find a rapid increase in this percentage at $N = 5$, consistent with our other results. However, we know that a simple overlap criterion is too strong for merging of second order islands, just as it is for the primary islands.

To perform a self-consistent calculation we first simplify (3.14) by expanding (3.15) and (3.16) near the separatrix to obtain

$$2\Delta J_M/\delta J = 16\ N^3 \exp\left[-\pi N/2\right]/\pi(1-\kappa^2) \tag{3.17}$$

which can be written in terms of the separatrix thickness $h = (1 - H/K)$ by substituting $h/2 = 1 - \kappa^2$ in (3.17). We expand the fundamental Hamiltonian of (2.11) to obtain

$$\Delta I/I = h/4 \tag{3.18}$$

An improved overlap criterion, including the thickness of the separatrix is then

$$(1 + h/4)\ 2K^{\frac{1}{2}} + K/2 = \pi \tag{3.19}$$

For self consistency we can set $2\Delta J/\delta J = 2K^{\frac{1}{2}}/\pi$ in (3.17) and substituting for h from (3.18) and (3.17) with $N = 2\pi/K^{\frac{1}{2}}$, (3.19) can be written as an an equation for K, alone

$$\left[1 + (2\pi)^4/K^{5/2}\right)\exp\left(-\pi^2/K^{\frac{1}{2}}\right)\ ]\ 2K^{\frac{1}{2}} + K/2 = \pi \tag{3.20}$$

The solution obtained numerically is

$$K = 1.2\,, \tag{3.21}$$

or to the nearest rational island number $N \cong 5$. The calculation of the

self consistent value of K is a little different than, but in the spirit of, a calculation by Chirikov (1979). He derives a second order standard mapping by linearizing a mapping near the separatrix. He also introduces a weighting factor in second order to bring his analytic results into close agreement with a detailed numerical analysis of the transition which gives $K = 0.99$.

In the Fermi acceleration problem, renormalization of (2.1) and (2.2) gives the standard mapping with $K = u_o/2\pi M$. Numerically, for the fully nonlinear mapping, Lieberman and Lichtenberg (1972) found the barrier to global stochasticity, at $u_b = 2.8\sqrt{M}$. If one sets $u_b = u_o$ this gives a value of $K = 0.8$, significantly different from $K = 1.0$, found for the standard mapping. This difference can be readily explained by a difference in definitions between u_b and u_o. The value of u_b was taken at the limit of the stochastic motion, corresponding to the last KAM surface. However, the analogy to the standard mapping relates K to the value of u_o at the center of the primary island associated with the last KAM surface. This occurs at $u_o = 2.5\sqrt{M}$. The corresponding value of $K = 1.0$ agrees with results for the standard mapping. The nonlinearity associated with the Fermi acceleration, decreasing the perturbation strength K with increasing u, creates sufficient asymmetry to destroy the last KAM surface at $u < u_o$ but retain the one for $u > u_o$. In principle one might expect stronger nonlinearities to give more significant deviations from the standard mapping.

4. Stability of Fixed Points near Irrational Winding Number

4a. The Basic Elements of Greene's Method

The method developed by Greeńe (1968, 1979a) for finding the exact transition to global stochasticity postulates a correspondence between two properties of the system: the disappearance of a KAM surface having an irrational winding number α, with the destabilization of the elliptic singular points of the high harmonic rational iterates ($\alpha \approx r/s$, r, s relatively prime integers, with s large), which approach the irrational α

in the limit $s \to \infty$. The correspondence is justified numerically. We first explore some of the basic elements of the method.

The Mean Residue: The residue of a tangent mapping may be defined as

$$R = \tfrac{1}{4} (2 - \mathrm{Tr}\, A) . \tag{4.1}$$

Comparison with (2.7) shows that

$$R = \sin^2 \sigma/2 \tag{4.2}$$

where σ is the phase shift per iteration of the mapping. There is, therefore, stable motion about the fixed point for

$$0 < R < 1 . \tag{4.3}$$

Over a full cycle of the periodic orbit, the matrix of the linearized standard mapping of (4.3) is, for rational winding number $\alpha = r/s$,

$$A = \prod_{i=1}^{s} \begin{bmatrix} 1 - K \cos \theta_i & 1 \\ -K \cos \theta_i & 1 \end{bmatrix} \tag{4.4}$$

from which $\mathrm{Tr}\, A$ can be determined. When K is large it is intuitively obvious that $R \propto K^s$. Greene has also shown that to be true for small K and postulated it to be true for all K. This implies that the residue is exponential with the orbit length which is proportional to s. For orbits with $R > 0$ it has a transition, for large s, from values approaching zero for stable orbits, to very large values and thus unstable orbits. It is therefore natural to investigate the behaviour of a quantity proportional to the s^{th} root of R defined by

$$f = (|R|/\beta)^{1/s} \tag{4.5}$$

which Greene calls the mean residue. The constant β is chosen for rapid convergence, ie. reliable answers for relatively small s, and can always be set equal to one. The test for stability, then, is $f < 1$.

Continued Fraction Approximation to Irrationals: It can be shown that the best way to approximate irrationals by rationals is with a continued fraction expansion. For an irrational between 0 and 1, this expansion is represented by a set of positive integers

$$\alpha = [a_1, a_2, a_3 \ \ldots] \tag{4.6}$$

where a_1 is determined by reciprocating α and taking the integer part, a_2 is determined by reciprocating the remainder and again taking the integral part, etc. The expansion is unique and the successive iterates of this continued fraction r_n/s_n, where a_n is the last term taken, best approximates α. The larger the a_n the more rapid the convergence. We expect the last remaining KAM surface to be that surface farthest from rationals, which implies the lowest value of the a_n's. Clearly that value of winding number is the one for which all $a_n = 1$ for which $\alpha_I = (\sqrt{5} - 1)/2$. α_I has long been known to be of special significance, and has been given the name of the golden mean. We therefore examine the stability of the iterates α_{In} to determine the transition to global stochasticity.

The above also suggests the procedure to find the disappearance of the last KAM surface in any localized region of the action space. We choose the appropriate irrational surface by requiring that its partial fraction expansion end up converging as slowly as possible, that is, $\alpha = [a_1 \ldots a_1, 1, 1 \ldots]$, where $a_1 - a_n$ are chosen to place the winding number in the appropriate portion of the space.

It is efficient, but not necessary, to use continued fraction expansions in employing the general technique. For example, Lunsford and Ford (1972), basing their calculation on the earlier work of Greene (1968), used the method to examine the disappearance of KAM orbits for the Hénon and Heiles potential $U(x,y) = \frac{1}{2}(x^2 + y^2 + 2x^2y - \frac{2}{3}y^3)$. Lunsford and Ford found that a choice of $\alpha^{-1} = k \pm 1/n$ where k is a set of harmonics chosen over the range of interest $k = 4, 5, 6, 7, 8$ etc. and n is allowed to run over a set of integers $1 < n < n_o$, proved a convenient (but not precise) method of determining the transition. Greene (1979b) has also obtained the transition for the Henon and Heiles potential, using the more precise version of the method described here.

4b Numerical Calculations

The Numerical Procedure: We briefly review here procedures to perform

the numerical calculations to be described in this section. The details of the procedures used by Greene are described more fully in his papers (Greene, 1968, 1979a, 1979b). (1) We consider that the problem has been represented as a mapping, eg. the standard mapping or the Fermi acceleration mapping. We note that for a complete two degree of freedom problem this requirement may present serious difficulties, as a mapping constructed by intersections with a surface of section can only be obtained numerically. Greene (1979b) has used natural symmetries to overcome this difficulty for the Hénon and Heiles potential. (2) Find the primary ($s = 1$) elliptic fixed points to high accuracy, analytically, if possible. (3) On a symmetry line of the phases, numerically compute the rotation number α as a function of the distance in action away from the fixed point. This must be done as a long time average. However the high harmonic fixed points $\alpha = r/s$, which have the same symmetry as the primary fixed points, will be accurately determined after s mappings. (4) Choose a set of iterates $\alpha_n = r_n/s_n$ near some irrational α where the test for stochasticity is to be made. If one is looking for the transition to global stochasticity, where the last KAM surface between adjacent $s = 1$ fixed points is destroyed, then the set of iterates are those found from the partial fraction expansion of the 'golden mean'. (5) Find the linearized mapping in the neighbourhood of the singular points, A, defined by

$$\underset{\sim}{x}^s - \underset{\sim}{x}_o = A(\underset{\sim}{x} - \underset{\sim}{x}_o)$$

where $\underset{\sim}{x}^s$ is the value of $\underset{\sim}{x}$ (near $\underset{\sim}{x}_o$) after s iterations of the mapping, and $\underset{\sim}{x}_o$ is the coordinates of the particular fixed point under investigation. This can normally be accomplished numerically from the second partial derivatives of the Hamiltonian evaluated at the fixed point (see Greene, 1979b, for details). (6) Calculate $f = |R|/\beta)^{1/s}$, as discussed in sec. 4.4a, to determine, for a given perturbation strength, whether $f > 1$.

Numerical Results: For the standard mapping, (2.4), Greene (1979a) used the method to explore the properties previously discussed. He

changed variables to transform the mapping to be periodic in the unit square, rather than in 2π, but this does not alter any results. Fig. 2 gave his numerical results for five orbits for $K = 0.97$, slightly smaller than that required to destroy the last KAM curve. Because of symmetry around $\alpha = 1/2$, there are two last KAM curves. Also, the $s = 1$ separatrix trajectory is probably diffusing very slowly beyond the limits shown, which could correspond to a near-adiabatic irrational surface. That is, due to the limits placed on the integration time, the fate of the orbits is not certain from this picture. Other slightly less stable KAM surfaces, corresponding to other irrational winding numbers, may still exist. However, at $K = 0.9716$, there is a clearly different behaviour of the KAM surfaces at the golden mean, and at $K = 0.975$ the invariant is certainly destroyed as trajectories are seen to diffuse through it (albeit very slowly).

Greene has calculated the values of f at the rational continued fraction iterates of the golden mean for $K = 0.9716$, and at $K = 0.9$, which we can compare. These results are shown in Table 1.1. We see a clear transition from $f < 1$ at $K = 0.9$ to $f \approx 1$, asymptotically, at $K = 0.9716$. We also note the dramatic change in R for long orbit lengths (large s), from the asymptotically small value for stable orbits to the value near $R = \pm 0.25$ at the transition.

Table 1.1

$K = 0.9$		$K = 0.971635$		
r_n/s_n	$f^{(+)}$	r_n/s_n	$f^{(+)}$	R
2/3	0.93896	1/1	0.971635	0.24291
3/5	0.91959	1/2	0.971635	0.23602
5/8	0.92775	2/3	1.014042	0.26068
⋮		3/5	0.993528	0.24201
34/55	0.92727	⋮		
55/89	0.92409	377/610	0.99999965	0.24995
89/144	0.92406	610/987	1.00000009	0.25002
144/233	0.92701	987/1597	0.9999970	0.24988

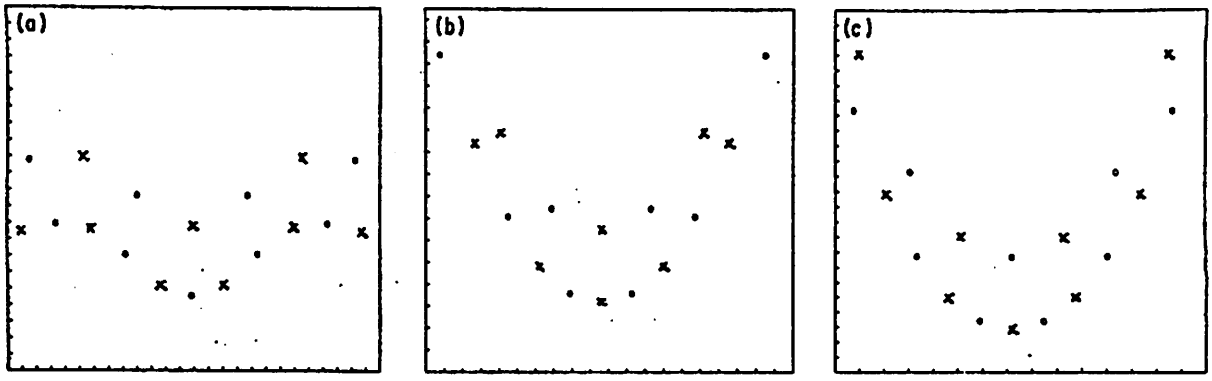

Fig. 3A

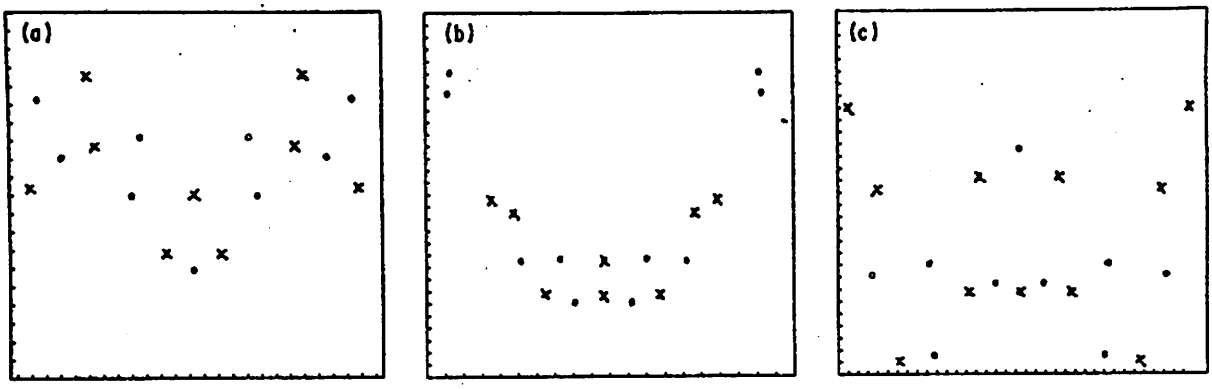

Fig. 3B

Fig. 3 Fixed points of the standard mapping for successively better approximations to the golden mean. 3A: K = 0.95; 3B: K = 0.9716. The dots are elliptic points, the crosses hyperbolic points (from Greene, 1979).

The physical picture of this breakup of the KAM surface at the golden mean can be explored by plotting the physical positions of the fixed points at successive pairs of continued fraction iterates. This is done in Fig. 3, comparing K = 0.95 for which $f\left(\lim_{n\to\infty}\alpha_n\right) \cong 0.977$ with the case of K = 0.9716 for which, as we have seen, $f\left(\mathrm{Lim}_{n\to\infty}\alpha_n\right) \cong 1.000$. Because each successive golden mean iterate has approximately $1/\alpha$ more fixed points than the preceeding one, and because two iterates are used to bracket the actual α, the horizontal scale is expanded by $(1/\alpha)^2$ to keep the same number of points in successive frames (labelled (a), (b), (c). To preserve the aspect ratio where the fixed points lie roughly on a parabola, the vertical scale is expanded by $(1/\alpha)^4$. For K = 0.95, in Fig. 3A, we see that successive iterates appear to converge uniformly down to the KAM surface. This is contrasted with the results for K = 0.9716, in Fig. 3B, where the successive

sets of fixed points appear to have an underlying variability that has structure at every scale. This observation can be explained in terms of our previous results that a winding number of $\alpha = 1/6$ implies local island amplitudes that are 2/3 of the distance between islands. Thus the island structure on the previous scale is always sufficiently large to disrupt the next finer scale.

5. Summary of Transition Criteria

In Table 2 we summarize the various criteria for the transition between stochastic regions closely bounded by KAM surfaces and connected or global stochasticity, for the characteristic problem of the standard mapping. The criteria are arranged from the strongest (most overstating the condition on the perturbation amplitude required for stochasticity), to the sharpest (both necessary and sufficient). Because no fully analytic theory of the transition exists, the sharper the criterion, the more important the numerical element becomes. The criterion is also presented in terms of the winding number $\alpha = \Omega_B(o)/2\pi$ of the primary resonance. The third item in the table is from Lieberman and Lichtenberg (1972) and has not been discussed in detail in the text. The last item in the table gives the numerically determined transition for the fully nonlinear Fermi mapping for comparison with the standard mapping which is linear in momentum.

Table 2

Criteria for Transition to Global Stochasticity

Mapping: $I_{n+1} = I_n + K\sin\theta_n \; ; \; \theta_{n+1} = \theta_n + I_{n+1}$

Physical Criterion	Mathematical Criterion	K	$\Omega_B(o)/2\pi$	Characterization
linear stability of primary resonance	$\cos\Omega_B(o) = \frac{1}{2}\,\mathrm{Tr}(2 + K) = 1$	4	$\frac{1}{2}$ †	sufficient (very strong)
overlap of primary resonance	$\frac{2\Delta I_{max}}{\zeta I} = \frac{2K^{\frac{1}{2}}}{2\pi} = 1$	$\left(\frac{\pi}{2}\right)^2$	$\frac{1}{4}$ *	(strong)
no single-valued functions $I(\theta)$	$U_w + U_\theta \leq -2 \quad ; \; w = \frac{2\pi M}{u}$ $U = -w^2 \sin\theta/(2\pi M + w\sin\theta)$	2	$\frac{1}{4}$ †	sufficient
second order islands important	$\frac{\Delta J/\delta J}{\Delta I/\delta I} = \left(\frac{J_n(\pi)n^2/2}{16/n^2}\right)^{\frac{1}{2}} = 1$	1.2	$\frac{1}{5}$ *	(sharp)
improved "overlap" criterion	$\left(2 - \frac{H}{H_{sx}}\right)\Delta I_{1M} + \Delta I_{2M} = \pi$	1.2	$\frac{1}{5}$ *	(sharp)
loss of stability at rational iterates of golden mean	$f = \lvert 2 - \mathrm{Tr}A\rvert^{1/s} \; ; \; \lim_{s\to\infty} \frac{r}{s} = \frac{\sqrt{5}-1}{2}$	0.9716	$\frac{1}{6}$ †	(very sharp)
numerically determine from fully nonlinear Fermi acceleration	For Eqs.(2.1) and (2.2) $L_i c = 2.5\, M^{\frac{1}{2}}$	1.0	$\frac{1}{6}$ †*	(very sharp)

† calculated from tangent map * calculated from averaged Hamiltonian

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