REALIZABILITY OF COMMUNICATION NETS:

AN APPLICATION OF THE ZADEH CRITERION

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ABSTRACT

The concepts of pseudo-Boolean matrix multiplication and pseudo-Boolean matrix adjoint are reviewed. For a given symmetric matrix T to be realizable as the terminal capacity matrix of an unoriented communication net, it is necessary and sufficient for T to be idempotent with respect to multiplication. Another equivalent condition is for T to be self-adjoint.

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Introduction

Let $T = (t_{ij})$ be the matrix of terminal capacities[1,2] of an n-node (oriented or unoriented) communication net G. By convention we set $t_{ii} = \infty$ for $i = 1,2,\ldots,n$. It is known[3,4] that the following "triangle inequality" holds

$$t_{ij} \geq \min\{t_{ik}, t_{kj}\} \qquad \text{for all } i,j,k = 1,2,\ldots,n. \qquad (*)$$

This same condition is also sufficient for any n x n symmetric matrix T with infinite diagonal elements and nonnegative real off-diagonal elements to be the matrix of terminal capacities of some <u>unoriented</u> net.

In a paper dealing with a different topic, Zadeh[5] described a simple test for the condition (*) within specified ranges. We make appropriate extensions here, and point out a further equivalent test.

Preliminaries

By the <u>extended real numbers</u> we mean the reals together with two symbols $-\infty, \infty$. By the <u>nonnegative</u> extended reals we mean the nonnegative reals together with ∞. For our purpose we need only know how to perform the following operations on the extended real numbers.

If α is an extended real number, we define

$$\min\{\alpha,\infty\} = \alpha$$

$$\min\{\alpha,-\infty\} = -\infty$$

$$\max\{\alpha,\infty\} = \infty$$

$$\max\{\alpha,-\infty\} = \alpha$$

Let $A = (a_{ij})$, $B = (b_{ij})$, and $C = (c_{ij})$ be n x m, m x p, and n x p matrices respectively with extended real entries. We define the <u>pseudo-</u>

Boolean product of A and B (written AoB) by

$$C = A \circ B \Leftrightarrow c_{ij} = \max_k \min\{a_{ik}, b_{kj}\}$$

Various properties of this operation are mentioned in [5].

Let T be an n x n matrix with extended real entries. The pseudo-Boolean determinant of T is

$$|T| = \max_{\sigma} \min\{t_{1i_1}, t_{2i_2}, \ldots, t_{ni_n}\}$$

where the maximum is taken over all possible permutations

$$\sigma = \begin{pmatrix} 1, & 2, & 3, & \ldots, & n \\ i_1, & i_2, & i_3, & & i_n \end{pmatrix}$$

Example If $T = \begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix}$, then

$$|T| = \begin{vmatrix} 5 & 2 \\ 4 & 3 \end{vmatrix} = \max\left|\min\{5,3\}, \min\{4,2\}\right| = 3.$$

We will denote by T_{ij} the matrix obtained from T by eliminating row i and column j. The pseudo-Boolean adjoint of T is the matrix adj $T = (\tau_{ij})$ defined by

$$\tau_{ij} = |T_{ji}| \text{ for all } i,j.$$

In the following, entries in a matrix will always be extended real numbers.

Results

The following lemma is an extension of a result in [5].

Lemma 1 Let $T = (t_{ij})$ be an n x n square matrix.

If $t_{ii} \geq \max\{t_{ij}, t_{ji}\}$ for all $i,j = 1,2,\ldots,n$, then we have

$$\left(\begin{array}{l} t_{ij} \geq \min\{t_{ik}, t_{kj}\} \\ \text{for all } \underline{\text{distinct}}\ i,j,k \\ \text{taking values in the} \\ \text{set } \{1,2,\ldots,n\} \end{array}\right) \Longleftrightarrow \left(\begin{array}{l} t_{ij} = \max\limits_{k} \min\{t_{ik}, t_{kj}\} \\ \text{for all } i,j \text{ taking values} \\ \text{in the set } \{1,2,\ldots,n\} \end{array}\right) \quad (\dagger)$$

Proof

($\Leftarrow$) Trivial.

($\Rightarrow$) Due to the restriction $t_{ii} \geq \max\{t_{ij}, t_{ji}\}$ on the diagonal elements of T, consideration of cases shows that the left-hand side of ($\dagger$) is equivalent to

$$t_{ij} \geq \min\{t_{ik}, t_{kj}\} \text{ for } \underline{\text{all}}\ i,j,k, = 1,2,\ldots\ldots,n.$$

Thus we have

$$t_{ij} \geq \max_{k} \min\{t_{ik}, t_{kj}\} \tag{1}$$

To show the reverse inequality, we note that

$$\max_{k} \min\{t_{ik}, t_{kj}\} \geq \min\{t_{ii}, t_{ij}\} \tag{2}$$

But $\qquad t_{ii} \geq \max\{t_{ij}, t_{ji}\}$

So $\qquad \min\{t_{ii}, t_{ij}\} = t_{ij} \qquad (3)$

Hence (2) and (3) yields

$$\max_{k} \min\{t_{ik}, t_{kj}\} \geq t_{ij} \tag{4}$$

(1) and (4) yields the desired equality. □

Lemma 2[3] An $n \times n$ symmetric matrix T with $t_{ii} = \infty$ for $i = 1,2,\ldots,n$ is the terminal capacity matrix of an unoriented communication net if and only if

$$t_{ij} \geq \min\{t_{ik}, t_{kj}\}$$

for all $i,j,k = 1,2,\ldots,n$.

Lemma 3[3,4] If T is the matrix of terminal capacities of an oriented communication net, then

$$t_{ij} \geq \min\{t_{ik}, t_{kj}\}$$

for all $i,j,k = 1,2,\ldots,n$.

Combining lemmas 1, 2, and 3, we arrive at the following two theorems.

Theorem 1 An $n \times n$ symmetric matrix T with $t_{ii} = \infty$ for $i = 1,2,\ldots,n$ is the terminal capacity matrix of an unoriented communication net if and only if $T = T \circ T$.

Theorem 2 If T is the matrix of terminal capacities of an oriented communication net, then $T = T \circ T$.

A further characterization is possible. This is due to the following result of A.G. Lunts[6,7] (see also [8,9]).

Lemma 4[6,7] Let T be $n \times n$ with $t_{ii} = \infty$ for $i = 1,2,\ldots,n$. Then

$$T = T \circ T \iff T = \operatorname{adj} T.$$

Applying this lemma to theorems 1 and 2, we get the following two results.

Theorem 3 An $n \times n$ symmetric matrix T with $t_{ii} = \infty$ for $i = 1,2,\ldots,n$

is the terminal capacity matrix of an unoriented communication net if and only if T = adj T.

<u>Theorem 4</u> If T is the matrix of terminal capacities of an oriented communication net, then T = adj T.

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