INTERNAL AND EXTERNAL STABILITY OF LINEAR SYSTEMS

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1. Introduction. This paper is concerned with the relationship between various types of stability in time-variable linear systems. Initially, we will consider equations of the type

$$z^{(n)} + \sum_{i=0}^{n-1} g_i \, z^{(i)} = 0 \tag{1}$$

where the g_i are real valued functions defined and bounded by some constant C on the entire real line. In Section 2 we will prove Theorem 1 stated below.

For any function f defined on a (possibly infinite) interval I let

$$\|f, I\|_p = \left[\int_I |f|^p\right]^{1/p}, \qquad 1 \leq p < \infty$$

and let

$$||f, I||_{\infty} = \sup_{I} |f| .$$

Theorem 1. Given any $\epsilon > 0$ and any p $(1 \leq p \leq \infty)$ there exists a constant K such that for each solution f of (1) defined on an interval I

$$||f^{(i)}, J||_p \leq K||f, J||_p, \qquad 1 \leq i \leq n$$

for all subintervals J of I of length at least ϵ.

In Section 3 this result is used to prove the equivalence of the external concept of bounded-input, bounded-output stability and the internal concept of exponential stability for a large class of linear systems, solving in part a question posed by Kalman [1].

An immediate corollary of Theorem 1 is a result of Esclangon [2] and Landau [3]:

Corollary 1. If f is a bounded solution of (1) on the entire real line then the $f^{(i)}$ $(1 \leq i \leq n)$ are also bounded.

This corollary was extended in various ways by von Neumann and Halperin [4] and Beckenback and Bellman [5]. However, in contrast to these earlier generalizations, note that in Theorem 1 above the derivatives $f^{(i)}$ are explicitly bounded by the solution f. Moreover, the bounding constant K is independent of particular solutions f and intervals I. This stronger form is essential for the results of Section 3.

2. Proof of Theorem 1.

Lemma 1. Let g and h be any functions defined on an interval I. If $||g, J_0||_\infty \leq ||h, J_0||_\infty$ for all subintervals J_0 of I of length ϵ_0, then $||g, J_1||_\infty \leq ||h, J_1||_\infty$ for all subintervals J_1 of I of length $\epsilon_1 > \epsilon_0$.

The proof of this lemma is a simple exercise and will be omitted.

Lemma 2. If g is differentiable on a closed interval I of length ϵ and $||g, I||_\infty > 2|g(u_1)|$ for some u_1 in I then $||g^{(1)}, I||_\infty > \frac{1}{2\epsilon} ||g, I||_\infty$.

Proof. Since I is compact there is a point u_0 in I such that $|g(u_0)| = ||g, I||_\infty$. Then, by the law of the mean

$$||g^{(i)}, I||_\infty \geq \frac{|g(u_0) - g(u_1)|}{|u_0 - u_1|} > \frac{1}{2\epsilon} ||g, I||_\infty .$$

Theorem 1a. Given any $\epsilon > 0$ there exists a constant K such that for each solution f of (1) defined on an interval I

$$||f^{(i)}, J||_\infty \leq K||f, J||_\infty, \qquad 1 \leq i \leq n$$

for all subintervals J of I of length at least ϵ.

Proof. Suppose the contrary. Then there is a smallest integer k $(1 \leq k < n)$ and an $\epsilon_1 > 0$ such that the theorem fails. Choose $\epsilon_0 > 0$

such that

$$(2) \qquad \frac{1}{\epsilon_0} > \max\{2nC, 2, \frac{1}{\epsilon_1}\}.$$

By definition of k there exists an $M > 0$ such that

$$||h^{(i)}, J||_\infty \leq M||h, J||_\infty, \qquad 0 \leq i < k$$

for any solution h of (1) defined on any interval J of length ϵ_0. Again by the definition of k there exists a solution f of (1) defined on an interval I of length ϵ_1, such that $||f^{(k)}, I||_\infty > N||f, I||_\infty$ where

$$N = \left[\frac{2^{n+1} n}{\epsilon_0}\right]^{n+1} 2M .$$

Therefore, by Lemma 1 there must be a closed subinterval J_0 of I of length ϵ_0 such that

$$(3) \qquad ||f^{(k)}, J_0||_\infty > N||f, J_0||_\infty .$$

Since $N > M$,

$$(4) \qquad ||f^{(k)}, J_0||_\infty > M||f, J_0||_\infty \geq ||f^{(i)}, J_0||_\infty, \qquad 0 \leq i < k .$$

We assert that there exists w_i in J_0 such that

(5) $$|f^{(i)}(w_i)| < \frac{1}{2} \| f^{(k)}, J_0 \|_\infty , \qquad k \leq i < n .$$

This assertion will be proved below. But first, by repeated application of Lemma 2 with $g = f^{(i)}$, $I = J_0$ and $u_1 = w_i$ we get

(6) $$\| f^{(i+1)}, J_0 \|_\infty > \frac{1}{2\epsilon_0} \| f^{(i)}, J_0 \| , \qquad k \leq i < n .$$

Since $\epsilon_0 < \frac{1}{2}$, (4) and (6) imply

(7) $$\| f^{(n-1)}, J_0 \|_\infty > \| f^{(i)}, J_0 \|_\infty , \qquad 0 \leq i < n .$$

Furthermore (6) and (7) imply

$$\| f^{(n)}, J_0 \|_\infty > \frac{1}{2\epsilon_0 n} (n \| f^{(n-1)}, J_0 \|_\infty)$$

$$> C \sum_0^{n-1} \| f^{(i)}, J_0 \|_\infty$$

$$\geq \left\| \left(\sum_0^{n-1} g_i f^{(i)} \right) , J_0 \right\|_\infty ,$$

a contradiction which proves the theorem.

We will now prove the assertion (5).

Case 1. $f^{(k)}$ has at least n zeros in J_0. Then by Rolle's Theorem $f^{(k+j)}$ has at least $n-j$ zeros in J_0, $1 < j < n$. Let w_i be a zero of $f^{(i)}$ in J_0 so that

$$|f^{(i)}(w_i)| = 0 < \frac{1}{2} ||f^{(k)}, J_0||_\infty , \qquad k \leq i < n .$$

Case 2. $f^{(k)}$ has less than n zeros in J_0. Then there exists a subinterval J_1 of J_0 of length ϵ_0/n such that $f^{(k)} \neq 0$ in J_1.

Let $\delta = \epsilon_0/n2^n$. By induction we will show that for $k \leq i < n$ there are finite subsets A_i of J_1 such that

(a) A_i consists of $2^{n+k-i-1}$ elements

(b) $|r-s| > \delta$ if r, s in A_i and $r \neq s$

(c) $|f^{(i)}(r)| < \frac{1}{2} ||f^{(k)}, J_0||_\infty \left(\frac{\delta}{2}\right)^{(n+k-i)}$ if r in A_i.

Suppose that $|f^{(k)}| \geq \frac{\delta^n}{2^{n+1}} ||f^{(k)}, J_0||_\infty$ on any subinterval $[u, u+\delta]$ of J_1. But then by the definition of M

$$(8) \quad 2M||f, J_0||_\infty \geq |f^{(k-1)}(u+\delta)| + |f^{(k-1)}(u)|$$

$$\geq |f^{(k-1)}(u+\delta) - f^{(k-1)}(u)| = \left| \int_u^{u+\delta} f^{(k)} \right| .$$

Since $f^{(k)} \neq 0$ on $[u, u+\delta]$

$$(9) \qquad \left|\int_u^{u+\delta} f^{(k)}\right| = \int_u^{u+\delta} |f^{(k)}| \geq \left[\frac{\epsilon_0}{2^{n+1}{}_n}\right]^{n+1} \left\|f^{(k)}, J_0\right\|_\infty .$$

From (3), (8) and (9), therefore, $||f, J_0||_\infty > ||f, J_0||_\infty$,

a contradiction.

Therefore,

$$|f^{(k)}| \not\geq \frac{\delta^n}{2^{n+1}} ||f^{(k)}, J_0||_\infty$$

on any subinterval of J_1 of length δ. Since the length of $J_1 = 2^n \delta$ there is a set A_k of 2^{n-1} points such that

$$|f^{(k)}(r)| < \frac{\delta^n}{2^{n+1}} ||f^{(k)}, J_0||_\infty$$

and $|r - s| > \delta$ for r, s in A_k, $r \neq s$. Hence (a), (b) and (c) hold for $i = k$.

Now assume there exists A_p $(k \leq p \leq n - 2)$, such that (a), (b) and (c) hold. Let $A_p = \{u_1, u_2, \cdots, u_q\}$ where $|u_{j+1} - u_j| > \delta$, $1 \leq j < q = 2^{n+k-p-1}$.

By the law of the mean there exists ν_j in $[u_{2j-1}, u_{2j}]$ $(1 \leq j \leq \frac{q}{2})$ such that

$$|f^{(p+1)}(\nu_j)| = \frac{|f^{(p)}(u_{2j}) - f^{(p)}(u_{2j-1})|}{|u_{2j} - u_{2j-1}|} .$$

Since u_{2j} and u_{2j-1} are in A_p (b) and (c) imply

$$|f^{(p+1)}(\nu_j)| < \frac{1}{2}\left(\frac{\delta}{2}\right)^{n+k-(p+1)} ||f^{(k)}, J_0||_\infty .$$

Hence (a), (b) and (c) hold for $A_{p+1} = \{\nu_j : 1 \leq j \leq q/2\}$ and therefore the induction hypothesis is valid for $k \leq i < n$.

Since $\delta < 2$ (c) implies w_i can be any point of A_i. This establishes Case 2 of Assertion (5).

Theorem 1b. Given any $\epsilon > 0$ and any finite $p \geq 1$ there exists a constant K such that for each solution f of (1) defined on an interval I

$$||f^{(i)}, J||_p \leq K||f, J||_p, \qquad 1 \leq i \leq n$$

for all subintervals J of I of length at least ϵ.

Proof. Suppose the theorem is false for a smallest integer $i = k$ $(1 \leq k < n)$, $\epsilon = \epsilon_0$ and some $p = q$.

By Theorem 1a, there is a constant $N_1 \geq \frac{1}{\epsilon_0}$ such that

(10) $$||h^{(i)}, J||_\infty \leq N_1||h, J||_\infty , \qquad 1 \leq i \leq n$$

for any solution h of (1) defined on any interval J of length ϵ_0.

By the definition of k, there is a solution f of (1) defined on a closed interval J_0 of length ϵ_0 such that

$$(11) \qquad ||f, J_0||_q < Q\,||f^{(k)}, J_0||_q$$

where Q is a constant to be suitably chosen below.

Since $1/2N_1 < \epsilon_0$ there is a closed sub-interval J_1 of J_0 of length $1/2N_1$ such that $||f, J_0||_\infty = ||f, J_1||_\infty$.

Suppose that $|f| \geq \frac{1}{2}||f, J_0||_\infty$ on J_1 then if $P = (4N_1)^{-\frac{1}{q}}$, clearly

$$(12) \qquad P||f, J_0||_\infty \leq ||f, J_1||_q \leq ||f, J_0||_q ,$$

and

$$(13) \qquad ||f^{(k)}, J_0||_q \leq \epsilon_0^{\frac{1}{q}}\,||f^{(k)}, J_0||_\infty \,.$$

Combining (10), (11), (12) and (13) we get

$$P||f, J_0||_\infty < QN_1\,\epsilon_0^{\frac{1}{q}}\,||f, J_0||_\infty ,$$

a contradiction since Q can be chosen equal to $P/N_1\epsilon_0^{\frac{1}{q}}$.

Therefore, there exists u_1 in J_1 such that $\frac{1}{2}||f, J_1||_\infty > |f(u_1)|$. But then by Lemma 2

$$N_1\|f, J_0\|_\infty = N_1\|f, J_1\|_\infty < \|f^{(1)}, J_1\|_\infty \leq \|f^{(1)}, J_0\|_\infty,$$

a contradiction which establishes the theorem.

Theorem 1 indicates that the solutions of equations of type (1) cannot fluctuate too rapidly. The following corollary makes this notion precise. Furthermore, this corollary plays a key role in Section 3.

Corollary 2. Given any $\epsilon > 0$ there exists a $\delta > 0$ such that if f is any solution of (1) defined on an interval I and J_0 is any closed subinterval of I of length ϵ then there is a subinterval J_1 of J_0 of length δ such that $|f| \geq \frac{1}{2}\|f, J_0\|_\infty$ on J_1.

Proof. By Theorem 1 there is a constant K such that $\|f^{(1)}, J_0\|_\infty \leq K\|f, J_0\|_\infty$. Choose $\delta = \min\left\{\frac{1}{2K}, \epsilon\right\}$. Let t_0 be a point at which $|f|$ takes its maximum on J_0 and let J_1 be any neighborhood of t_0 of length δ. Then by the law of the mean if t is in J_1

$$\frac{|f(t_0)| - |f(t)|}{\delta} \leq \|f^{(1)}, J_0\|_\infty \leq K|f(t_0)| .$$

3. Stability in Linear Systems.

In this section, by application of Theorem 1, an equivalence between two basic types of stability in linear systems will be established. The class of systems to be considered are those which may be represented in the form

(14 a) $$x^{(1)}(t) = A(t)x(t) + b(t)u(t)$$

(14 b) $$y(t) = c(t)x(t)$$

where u and y are the system input and output respectively and x, an n-vector, the internal state of the system. The coefficient matrices A, b, and c of orders $n \times n$, $n \times 1$ and $1 \times n$, respectively, are bounded on $(-\infty, \infty)$.

The system (14) is said to be bounded-input, bounded-output (BIBO) stable if for each constant K_1 there is a constant K_2 such that if $|u| < K_1$ on any interval (t_0, ∞) then $|y| < K_2$ on (t_0, ∞). A well known [6, 7] necessary and sufficient condition for BIBO stability when $x(t_0) = 0$ is that a constant M exist such that for all t,

(15) $$\int_{-\infty}^{t} |h(t,\tau)|\, d\tau \leq M$$

where

$$h(t, \tau) = c(t)\, X(t)\, X^{-1}(\tau)\, b(\tau)$$

with $X(t)$ a fundamental matrix for the homogenious part of (14 a).

In contrast to the external constraint of BIBO stability are various types of Lyapunov stability, which are concerned with the behavior of the internal state in the absence of an input. We are interested in relating BIBO stability to the exponential stability of the state. The system (14) is said to be exponentially stable, if there exist constants $K_3 > 0$ and

$K_4 > 0$ such that for any vector function f satisfying the homogenious part of (14 a)

$$||f(t)|| \leq K_3 ||f(t_0)|| e^{-K_4(t-t_0)}$$

for all t_0 and for all $t \geq t_0$, where $||\cdot||$ denotes the euclidean norm.

A necessary and sufficient condition for exponential stability [8] is that there exist constants P and N such that for all τ, and for all $t \geq \tau$

$$(16) \qquad || X(t) X^{-1}(\tau) || \leq P e^{-N(t-\tau)} .$$

The connection between the two types of stability is well understood in the case where A, b and c are constant matrices [8]. If the system is completely controllable [9] and completely observable [9], then (15) and (16) are equivalent. For time-variable systems no such result is available. In fact, as shown by Kalman [1], there are simple examples of completely controllable and completely observable systems in which the two types of stability are in no way related. It will be shown below, however, that under conditions somewhat stronger than complete controllability and complete observability (15) and (16) are equivalent.

Let us first introduce the controllability and observability matrices [10,11] of system (14) :

$$Q_c = [p_0 p_1 \cdots p_{n-1}]$$

where

$$p_{k+1} = -Ap_k + d/dt\, p_k \,; \qquad p_0 = b$$

and

$$Q_o = [r_0\, r_1 \cdots r_{n-1}]$$

where

$$r_{k+1} = A'r_k + d/dt\, r_k \,; \qquad r_0 = c'$$

(' = transpose). The main result to be established is summarized in the following theorem.

Theorem 2. If Q_c and Q_o are Lyapunov transformations [8] on $(-\infty, \infty)$ then the stability criteria (15) and (16) are equivalent.

Remark. For constant systems the conditions on Q_c and Q_o reduce to complete controllability and complete observability. For time variable systems, if the first n derivatives of the matrices A, b and c are bounded the conditions are equivalent to uniform controllability and observability in the sense that the determinants of Q_c and Q_o are bounded away from zero.

Before proving Theorem 2 several preliminary results will be established. Let $\psi(t) = c(t)X(t)$ and $\theta(t) = X^{-1}(t)b(t)$.

Lemma 3. If Q_o is a Lyapunov transformation on $(-\infty, \infty)$ then the elements of ψ form a set of linearly independent solutions of an equation of type (1).

Proof. Consider the transformation of coordinates $w = Q'_o x$ mapping (A, b, c) into $(\bar{A}, \bar{b}, \bar{c})$. Since Q_o is a Lyapunov transformation the matrices $\bar{A}$, $\bar{b}$, and $\bar{c}$ must be bounded on $(-\infty, \infty)$. Furthermore it is easily shown [10] that $\bar{A}$ and $\bar{c}$ have the canonical form

$$\bar{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}, \qquad \bar{c} = [1 \ 0 \cdots 0]$$

Since ψ is invariant under a transformation of coordinates it is clear that the elements of ψ satisfy an equation of the type (1) with $a_i = g_i$, $0 \leq i \leq n-1$.

Similarly, it may be shown that the following lemma is true.

Lemma 4. If Q_c is a Lyapunov transformation on $(-\infty, \infty)$ then the elements of θ form a set of linearly independent solutions of an equation of type (1).

Lemma 5. Let D_1 and D_2 be equations of type (1). For each t, let $h(t, \tau)$ be a solution of D_1 and for each τ, let $h(t, \tau)$ be a solution of D_2. If there exists a constant M such that for all t

$$\int_{-\infty}^{t} |h(t, \tau)| \, d\tau \leq M ,$$

then there is a constant N such that for all t

$$\int_{-\infty}^{t} |h_{ij}(t, \tau)| \, d\tau \leq N , \qquad 0 \leq i, j \leq n$$

where

$$h_{ij}(t, \tau) = \frac{\partial^i}{\partial t^i} \frac{\partial^j}{\partial \tau^j} h(t, \tau) .$$

Proof. For each τ let h_τ be the function such that $h_\tau(t) = h(t, \tau)$ for all t. Then by Theorem 1, there is a constant K such that for all τ

$$||h_\tau^{(i)}, I||_\infty \leq K||h_\tau, I||_\infty , \qquad 1 \leq i \leq n$$

on every interval I of length 1. Therefore,

$$\int_{-\infty}^{t} |h_\tau^{(i)}| d\tau \leq K \int_{-\infty}^{t} ||h_\tau, I_t||_\infty d\tau \tag{17}$$

where $I_t = [t, t+1]$. By Corollary 2 there exists a $\delta > 0$ and subintervals J_t of I_t of length δ such that $|h_\tau(t)| \geq \frac{1}{2} \| h_\tau, I_t \|_\infty$ on J_t. Let k be the smallest integer such that $0 \leq k\delta \leq 1 < (k+1)\delta$. But then (17) implies

$$\int_{-\infty}^{t} |h_\tau^{(i)}(t)| \, d\tau \leq K \int_{-\infty}^{t} 2 |h_\tau(t + \ell\delta)| \, d\tau$$

for some integer ℓ such that $0 \leq \ell \leq k$. Hence, reintroducing $h(t, \tau)$, we have

$$\text{(18)} \qquad \int_{-\infty}^{t} |h_{i0}(t, \tau)| \, d\tau \leq K \int_{-\infty}^{t} \sum_{0}^{k} 2 |h(t + j\delta, \tau)| \, d\tau$$

$$\leq 2K(k+1)M.$$

It follows immediately from Theorem 1 with $p = 1$ that a constant K_1 exists such that

$$\text{(19)} \qquad \int_{-\infty}^{t} |h_{0j}(t, \tau)| \, d\tau \leq K_1 M$$

Now observe that if $h(t, \tau)$ is a solution of D_1 for all t, then $h_{0j}(t, \tau)$ is also a solution of D_1 for all t and for all $j \geq 1$. It follows immediately from (18) and (19) that

$$\int_{-\infty}^{t} |h_{ij}(t, \tau)| \, d\tau \leq 2 K K_1 (k+1) M = N$$

for $0 \leq i, \quad j \leq n$.

We are now ready to prove Theorem 2.

As a consequence of Lemmas 3 and 4 the function $h(t, \tau)$ as defined in (15) satisfies the hypothesis of Lemma 5. Hence, if (15) holds there is a constant M_1 such that for all t

$$\int_{-\infty}^{t} ||\Gamma(t, \tau)|| \, d\tau < M_1$$

where $\Gamma(t, \tau)$ is the $n \times n$ matrix with elements $h_{ij}(t, \tau)$. It may be readily verified [12] that

$$\Gamma(t, \tau) = Q_o'(t) X(t) X^{-1}(\tau) Q_c(\tau).$$

Therefore, since Q_o and Q_c are Lyapunov transformations

$$\int_{-\infty}^{t} ||X(t) X^{-1}(\tau)|| \, d\tau < M_2$$

for some constant M_2. Since A of (14) is bounded, it follows [8] that (16) holds for some P and N.

Clearly (16) implies (15), and this completes the proof of Theorem 2.

It should be noted that if system (14) satisfies condition (16) then the output is bounded for all initial states and all bounded inputs. Thus, an immediate corollary of Theorem 2 is the following.

<u>Corollary 3.</u> If system (14) satisfies the hypothesis of Theorem 2 then it is BIBO stable if and only if it is exponentially stable.

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