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Linear Time Algorithm for Weak Parity Games ^{*}

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Abstract. We consider games played on graphs with the winning conditions for the players specified as *weak-parity* conditions. In weak-parity conditions the winner of a play is decided by looking into the set of states appearing in the play, rather than the set of states appearing infinitely often in the play. A naive analysis of the classical algorithm for weak-parity games yields a quadratic time algorithm. We present a linear time algorithm for solving weak-parity games.

1 Introduction

We consider two-player games on graphs with winning objectives formalized as a *weak-parity* objective. In a two-player game, the set of vertices or states are partitioned into player 1 states and player 2 states. At player 1 states player 1 decides the successor and likewise for player 2. We consider weak-parity objectives, where we have a priority function that maps every state to an integer priority. A *play* is an infinite sequence of states, and in a weak-parity objective the winner of a play is decided by considering the minimum priority state that appear in the play: if the minimum priority is even, then player 1 wins, and otherwise player 2 is the winner. The classical algorithm to solve weak-parity games with a naive running time analysis works in $O(d \cdot m)$ time, where d is the number of priorities and m is the number of edges of the game graph. Since d can be $O(n)$, in the worst case the naive analysis requires $O(n \cdot m)$ time, where n is the number of states. We present an improved analysis of the algorithm and show that the algorithm works in $O(m)$ time.

2 Definitions

We consider turn-based deterministic games played by two-players with *weak-parity* objectives; we call them weak-parity games. We define game graphs, plays, strategies, objectives and notion of winning below.

Game graphs. A *game graph* $G = ((S, E), (S_1, S_2))$ consists of a directed graph (S, E) with a finite state space S and a set E of edges, and a partition (S_1, S_2) of the state space S into two sets. The states in S_1 are player 1 states, and the states in S_2 are player 2 states. For a state $s \in S$, we write $E(s) = \{t \in S \mid (s, t) \in E\}$ for the set of successor states of s . We assume that every state has at least one out-going edge, i.e., $E(s)$ is non-empty for all states $s \in S$.

Plays. A game is played by two players: player 1 and player 2, who form an infinite path in the game graph by moving a token along edges. They start by placing the token on an initial state, and then they take moves indefinitely in the following way. If the token is on a state in S_1 , then player 1 moves the token along one of the edges going out of the state. If the token is on a state

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in S_2 , then player 2 does likewise. The result is an infinite path in the game graph; we refer to such infinite paths as plays. Formally, a *play* is an infinite sequence $\langle s_0, s_1, s_2, \dots \rangle$ of states such that $(s_k, s_{k+1}) \in E$ for all $k \geq 0$. We write Ω for the set of all plays.

Strategies. A strategy for a player is a recipe that specifies how to extend plays. Formally, a *strategy* σ for player 1 is a function $\sigma: S^* \cdot S_1 \rightarrow S$ that, given a finite sequence of states (representing the history of the play so far) which ends in a player 1 state, chooses the next state. The strategy must choose only available successors, i.e., for all $w \in S^*$ and $s \in S_1$ we have $\sigma(w \cdot s) \in E(s)$. The strategies for player 2 are defined analogously. We write Σ and Π for the sets of all strategies for player 1 and player 2, respectively. An important special class of strategies are *memoryless* strategies. The memoryless strategies do not depend on the history of a play, but only on the current state. Each memoryless strategy for player 1 can be specified as a function $\sigma: S_1 \rightarrow S$ such that $\sigma(s) \in E(s)$ for all $s \in S_1$, and analogously for memoryless player 2 strategies. Given a starting state $s \in S$, a strategy $\sigma \in \Sigma$ for player 1, and a strategy $\pi \in \Pi$ for player 2, there is a unique play, denoted $\omega(s, \sigma, \pi) = \langle s_0, s_1, s_2, \dots \rangle$, which is defined as follows: $s_0 = s$ and for all $k \geq 0$, if $s_k \in S_1$, then $\sigma(s_0, s_1, \dots, s_k) = s_{k+1}$, and if $s_k \in S_2$, then $\pi(s_0, s_1, \dots, s_k) = s_{k+1}$.

Weak-parity objectives. We consider game graphs with weak-parity objectives for player 1 and the complementary weak-parity objectives for player 2. For a play $\omega = \langle s_0, s_1, s_2, \dots \rangle \in \Omega$, we define $\text{Ocuur}(\omega) = \{s \in S \mid s_k = s \text{ for some } k \geq 0\}$ to be the set of states that occur in ω . We also define reachability and safety objectives as they will be useful in the analysis of the algorithms.

1. *Reachability and safety objectives.* Given a set $T \subseteq S$ of states, the reachability objective $\text{Reach}(T)$ requires that some state in T be visited, and dually, the safety objective $\text{Safe}(F)$ requires that only states in F be visited. Formally, the sets of winning plays are $\text{Reach}(T) = \{\langle s_0, s_1, s_2, \dots \rangle \in \Omega \mid \exists k \geq 0. s_k \in T\}$ and $\text{Safe}(F) = \{\langle s_0, s_1, s_2, \dots \rangle \in \Omega \mid \forall k \geq 0. s_k \in F\}$. The reachability and safety objectives are dual in the sense that $\text{Reach}(T) = \Omega \setminus \text{Safe}(S \setminus T)$.
2. *Weak-parity objectives.* For $d \in \mathbb{N}$, we let $[d] = \{0, 1, \dots, d-1\}$ and $[d]_+ = \{1, 2, \dots, d\}$. Let $p: S \rightarrow [d]$ be a function that assigns a *priority* $p(s)$ to every state $s \in S$. The weak-parity objective requires that the minimal priority occurring is *even*. Formally, the set of winning plays is $\text{WeakParityEven}(p) = \{\omega \in \Omega \mid \min(p(\text{Ocuur}(\omega))) \text{ is even}\}$. The complementary objective to $\text{WeakParityEven}(p)$ is $\text{WeakParityOdd}(p)$ defined as the set $\text{WeakParityOdd}(p) = \{\omega \in \Omega \mid \min(p(\text{Ocuur}(\omega))) \text{ is odd}\}$ of winning plays.

Winning strategies and sets. Given a game graph G and an objective $\Phi \subseteq \Omega$ for player 1, a strategy $\sigma \in \Sigma$ is a *winning strategy* for player 1 from a state s if for all player 2 strategies $\pi \in \Pi$ the play $\omega(s, \sigma, \pi)$ is winning, i.e., $\omega(s, \sigma, \pi) \in \Phi$. The winning strategies for player 2 are defined analogously. A state $s \in S$ is winning for player 1 with respect to the objective Φ if player 1 has a winning strategy from s . Formally, the set of *winning states* for player 1 with respect to the objective Φ in a game graph G is $W_1^G(\Phi) = \{s \in S \mid \exists \sigma \in \Sigma. \forall \pi \in \Pi. \omega(s, \sigma, \pi) \in \Phi\}$. Analogously, the set of winning states for player 2 with respect to an objective $\Psi \subseteq \Omega$ is $W_2^G(\Psi) = \{s \in S \mid \exists \pi \in \Pi. \forall \sigma \in \Sigma. \omega(s, \sigma, \pi) \in \Psi\}$. If the game graph is clear from the context we drop the game graph from the superscript. We say that there exists a memoryless winning strategy for player 1 with respect to the objective Φ if there exists such a strategy from all states in $W_1(\Phi)$; and similarly for player 2.

Theorem 1. *For all game graphs $G = ((S, E), (S_1, S_2))$, for all weak-parity objectives $\Phi = \text{WeakParityEven}(p)$ for player 1, and the complementary objective $\Psi = \Omega \setminus \Phi$ for player 2, the following assertions hold.*

1. *We have $W_1(\Phi) = S \setminus W_2(\Psi)$.*

2. *There exists a memoryless winning strategy for both players.*

Closed sets and attractors. Some notions that will play key roles in the analysis of the algorithms are the notion of *closed sets* and *attractors*. We define them below.

Closed sets. A set $U \subseteq S$ of states is a *closed set* for player 1 if the following two conditions hold: (a) for all states $u \in (U \cap S_1)$, we have $E(u) \subseteq U$, i.e., all successors of player 1 states in U are again in U ; and (b) for all $u \in (U \cap S_2)$, we have $E(u) \cap U \neq \emptyset$, i.e., every player 2 state in U has a successor in U . A player 1 closed set is also called a *trap* for player 1. The closed sets for player 2 are defined analogously. Every closed set U for player ℓ , for $\ell \in \{1, 2\}$, induces a sub-game graph, denoted $G \upharpoonright U$.

Proposition 1. *Consider a game graph G , and a closed set U for player 2. For every objective Φ for player 1, we have $W_1^{G \upharpoonright U}(\Phi) \subseteq W_1^G(\Phi)$.*

Attractors. Given a game graph G , a set $U \subseteq S$ of states, and a player $\ell \in \{1, 2\}$, the set $Attr_\ell(U, G)$ contains the states from which player ℓ has a strategy to reach a state in U against all strategies of the other player; that is, $Attr_\ell(U, G) = W_\ell(\text{Reach}(U))$. The set $Attr_1(U, G)$ can be computed inductively as follows: let $R_0 = U$; let

$$R_{i+1} = R_i \cup \{s \in S_1 \mid E(s) \cap R_i \neq \emptyset\} \cup \{s \in S_2 \mid E(s) \subseteq R_i\} \quad \text{for all } i \geq 0;$$

then $Attr_1(U, G) = \bigcup_{i \geq 0} R_i$. The inductive computation of $Attr_2(U, G)$ is analogous. For all states $s \in Attr_1(U, G)$, define $rank(s) = i$ if $s \in R_i \setminus R_{i-1}$, that is, $rank(s)$ denotes the least $i \geq 0$ such that s is included in R_i . Define a memoryless attractor strategy $\sigma \in \Sigma$ for player 1 as follows: for each state $s \in (Attr_1(U, G) \cap S_1)$ with $rank(s) = i$, choose a successor $\sigma(s) \in (R_{i-1} \cap E(s))$ (such a successor exists by the inductive definition). It follows that for all states $s \in Attr_1(U, G)$ and all strategies $\pi \in \Pi$ for player 2, the play $\omega(s, \sigma, \pi)$ reaches U in at most $|Attr_1(U, G)|$ transitions.

Proposition 2. *For all game graphs G , all players $\ell \in \{1, 2\}$, and all sets $U \subseteq S$ of states, the set $S \setminus Attr_\ell(U, G)$ is a closed set for player ℓ .*

Notation. For a game graph $G = ((S, E), (S_1, S_2))$, a set $U \subseteq S$ and $\ell \in \{1, 2\}$, we write $G \setminus Attr_\ell(U, G)$ to denote the game graph $G \upharpoonright (S \setminus Attr_\ell(U, G))$.

Computation of attractors. Given a game graph $G = (S, E)$ and a set $T \subseteq S$ of states let us denote by $A = Attr_\ell(T, G)$ the attractor for a player $\ell \in \{1, 2\}$ to the set T . A naive analysis of the computation of attractor shows that the computation can be done in $O(m)$ time, where m is the number of edges. An improved analysis can be done as follows. For every state $s \in S \setminus T$ we keep a counter initialized to 0. Whenever a state t is included for the set of states in A , for all states s such that $(s, t) \in E$ we increase the counter by 1. For a state $s \in S_\ell$ if the counter is positive, then we include it in A , and for a state $s \in S \setminus S_\ell$ if the counter equals the number of edges $|E(s)|$, then we include it in A . Let us consider the following set of edges: $E_A = E \cap ((S \setminus T) \times A)$. The work of the attractor computation is only on edges with the start state in $(S \setminus T)$ and end state in A . That is the total work of attractor computation on edges is $O(m_A)$ where $m_A = |E_A|$. Also the counter initialization phase does not require to initialize counters for all states, but only initializes a counter for a state s , when some state $t \in E(s)$ gets included in A for the first time. This gives us the following lemma.

Lemma 1. *Given a game graph $G = (S, E)$ and a set $T \subseteq S$ of states let us denote by $A = Attr_\ell(T, G)$ the attractor for a player $\ell \in \{1, 2\}$ to the set T . The set A can be computed in time $O(|E_A|)$, where $E_A = E \cap ((S \setminus T) \times A)$.*

Algorithm 1 Classical algorithm for Weak-parity Objectives

Input : A 2-player game graph $G = ((S, E), (S_1, S_2))$ and priority function $p : S \rightarrow [d]$.
Output: A partition (W_1, W_2) of S .
1. $G^0 := G; W_1 := W_2 := \emptyset;$
2. **for** ($i := 0; i < d; i := i + 1$)
 2.1. $A_i = Attr_{(i \bmod 2+1)}(p^{-1}(i) \cap S^i, G^i);$
 2.2 $W_{(i \bmod 2+1)} = W_{(i \bmod 2+i)} \cup A_i;$
 2.3. $G^{i+1} = G^i \setminus A_i;$
end for
3. **return** $(W_1, W_2);$

3 The Classical Algorithm

We first present the classical algorithm for weak-parity games and present an improved analysis to show that the algorithm has a linear-time complexity. We first present an informal description of the algorithm; and a formal description of the algorithm is given as Algorithm 1.

Informal description of the classical algorithm. We will consider a priority function $p : S \rightarrow [d]$. The objective Φ for player 1 is the weak-parity objective $\text{WeakParityEven}(p)$ and the objective for player 2 is the complementary objective $\Psi = \text{WeakParityOdd}(p)$. The algorithm proceeds by computing attractors and removing the attractors from the game graph and proceeds on the subgame graph. At iteration i , we denote the game graph by G^i and the state space as S^i and the set of edges of G^i as E^i . At iteration i , the attractor set to the set of states of priority i in G^i (i.e., attractor to $p^{-1}(i) \cap S^i$) is computed. If i is even, the set is included in the winning set for player 1, and otherwise it is included in the winning set for player 2 and the set is removed from the game graph for the next iterations.

Correctness. The following theorem states the correctness of Algorithm 1.

Theorem 2 (Correctness). *Given a game graph $G = ((S, E), (S_1, S_2))$ and priority function $p : S \rightarrow [d]$ we have*

$$W_1 = W_1(\text{WeakParityEven}(p)); \quad S \setminus W_1 = W_2 = W_2(\text{WeakParityOdd}(p)),$$

where (W_1, W_2) is the output of Algorithm 1.

Proof. Observe that in the game graph G^i we have $S^i \subseteq \bigcup_{j \geq i} p^{-1}(j)$, i.e., the priorities in G^i are at least i . Let us denote by W_1^i and W_2^i the sets W_1 and W_2 at the end of iteration $i - 1$ of Algorithm 1. Then for all $s \in S^i \cap S_1$ we have $E(s) \subseteq S^i \cup W_2^i$ and for all $s \in S^i \cap S_2$ we have $E(s) \subseteq S^i \cup W_1^i$. We prove by induction that the following two conditions hold

$$W_1^i \subseteq W_1^G(\text{WeakParityEven}(p) \cap \{\omega \mid \min(p(\text{Ocurr}(\omega))) < i\});$$

$$W_2^i \subseteq W_2^G(\text{WeakParityOdd}(p) \cap \{\omega \mid \min(p(\text{Ocurr}(\omega))) < i\}).$$

The base case is trivial and we now prove the inductive case. For i even, for a state $s \in A_i$, the attractor strategy σ for player 1 in G^i to reach $p^{-1}(i) \cap S^i$ and then choosing edges in S^i , ensures that for all strategies π for player 2 we have

$$\omega(s, \sigma, \pi) \in (\text{WeakParityEven}(p) \cap \{\omega \mid \min(p(\text{Ocurr}(\omega))) \leq i\}) \cup \text{Reach}(W_1^i).$$

By the inductive hypothesis it follows that

$$A_i \subseteq W_1^G(\text{WeakParityEven}(p) \cap \{\omega \mid \min(p(\text{Ocurr}(\omega))) < i + 1\}).$$

Similarly, it follows for i odd that $A_i \subseteq W_2^G(\text{WeakParityOdd}(p) \cap \{\omega \mid \min(p(\text{Ocurr}(\omega))) < i + 1\})$. The desired result follows. ■

Running time analysis. In the running time analysis we will denote by n the number of states, and by m the number of edges in the game graph. The naive analysis of the running time of Algorithm 1 yields a $O(d \cdot m)$ running time analysis. This is because the loop of step 2 runs for d times, and each iteration can be computed in $O(m)$ time. Since d can be $O(n)$, the worst case bound of the naive analysis is $O(n \cdot m)$, which is quadratic. We will now present a linear-time analysis of the algorithm. The two key issues in the running time analysis of the algorithm is to analyze the computation of the attractors (step 2.1 of the algorithm) and obtaining the target sets $p^{-1}(i) \cap S^i$ in the attractor computation. We now analyze the running time of the algorithm addressing the two above issues.

The attractor computations. We first argue that the attractor computation over all iterations can be done in $O(m)$ time. To prove this claim we observe that the sets A_i computed at step 2.1 of the algorithm satisfies that $A_i \cap A_j = \emptyset$ for $i \neq j$, (since the set A_i once computed is removed from the game graph for further iterations). Let us consider the set $E_{A_i} = E^i \cap (S^i \times A_i)$ of edges. Then for $i \neq j$ we have $E_{A_i} \cap E_{A_j} = \emptyset$. By Lemma 1 it follows that the i -th iteration of the attractor can be computed in $O(|E_{A_i}|)$ time. Hence the total time for attractor computations over all iterations is

$$\sum_{i=0}^{d-1} O(|E_{A_i}|) = O(|E|) = O(m),$$

where the first equality follows since the edges E_{A_i} and E_{A_j} are disjoint for $i \neq j$.

Obtaining the target sets. We will now argue that the target sets $p^{-1}(i) \cap S^i$ can be computed in $O(n)$ time over all iterations. Without loss of generality we assume that the set of states S are numbered $0, 1, \dots, n - 1$ and the priority function $p : S \rightarrow [d]$ is given as an array $P[0..n - 1]$ of integers such that $P[i] = p(i)$. The procedure for obtaining the target sets will involve several steps. We present the steps below.

1. *Renaming phase.* First, we construct a renaming of the states such that states in $p^{-1}(i)$ are numbered lower than $p^{-1}(j)$ for $i < j$. Here is a $O(n)$ time procedure for renaming.
 - (a) Consider an array of counters $\text{ct}[0..d-1]$ all initialized to 0, and arrays $A[0], A[1], \dots, A[d-1]$ (each $A[i]$ is an array and will contain states of priority i).
 - (b) The first step is as follows.

```

for ( $i := 0; i < n; i := i + 1$ )
  {
     $k = P[i]; j = \text{ct}[k];$ 
     $A[k][j] = i;$ 
     $\text{ct}[k] = \text{ct}[k] + 1;$ 
  }
```

This step assigns to the array in $A[i]$ the set of states with priority i (in the same relative order) and also works in $O(n)$ time. The counter $\text{ct}[i]$ is the number of states with priority i .

- (c) *The renaming step.* We now construct arrays B and C in $O(n)$ time to store renaming and the inverse renaming. For simplicity let us assume $\text{ct}[-1] = 0$ and the procedure is as follows.

```

for ( $i := 0; i < d; i := i + 1$ )
  for ( $j := 0; j < \text{ct}[i]; j := j + 1$ )
    {
       $B[\text{ct}[i - 1] + j] = A[i][j];$ 
       $C[A[i][j]] = \text{ct}[i - 1] + j;$ 
    }

```

This creates the renaming such that for $B[0..\text{ct}[0] - 1]$ are states of priority 0, then we have states of priority 1 for $B[\text{ct}[0]..\text{ct}[1] - 1]$, and so on. The array C stores the inverse of the renaming, i.e., if $B[i] = j$, then $C[j] = i$. Moreover, though it is a nested loop, since $\sum_{i=1}^{d-1} \text{ct}[i] = n$ this procedure also works in $O(n)$ time.

2. In the renaming phase we have obtained in $O(n)$ time a renaming in the array B and the inverse renaming in the array C . Since renaming and its inverse, for a given state, can be obtained in constant time¹ we can move back and forth the renaming without increasing the time complexity other than in constants. We now obtain the set of states as targets required for the attractor computation of step 2.1 of Algorithm 1 in total $O(n)$ time across the whole computation. First, we create a copy of B as an array D , and keep a global counter called g initialized to 0. We modify the attractor computation in step 2.1 such that in the attractor computation when a state j is removed from the game graph, then $D[k]$ is set to -1 such that $D[k] = j$, (the entry of the array D that represent state j is set to -1). This is simply done as follows $D[C[j]] = -1$. This is a constant work for a state and hence the extra work in the attractor computation of step 2.1 across the whole computation is $O(n)$. The computation to obtain the target for priority i (i.e., $p^{-1}(i) \cap S^i$), denoted as procedure *ObtainTargets*, is described below. The procedure *ObtainTargets* is called by Algorithm 1 with parameter i in step 2.1 to obtain $p^{-1}(i) \cap S^i$.

- (a) We have the global counter $g := 0$ (initialized to 0) and the value of the global counter persists across calls to the procedure *ObtainTargets*. We present the pseudocode for the procedure *ObtainTargets* to obtain in an array T the set $p^{-1}(i) \cap S^i$ of states. The procedure assumes that when *ObtainTargets*(i) is invoked we have $g = 0$, if $i = 0$, and for $i > 0$, we have $g = \sum_{j=0}^{i-1} \text{ct}[j]$. Also, for all $j \in S \setminus S^i$ we have $D[C[j]] = -1$ (the set of states in $S \setminus S^i$ is set to -1 in the attractor computation). The procedure invoked with i returns T as an array with states in $p^{-1}(i) \cap S^i$, and sets $g = \sum_{j=0}^i \text{ct}[j]$.

```

ObtainTargets( $i$ )
   $k := 0;$ 
  for ( $j := 0; j < \text{ct}[i] - 1; j := j + 1$ )
    {
      if ( $D[j + g] \neq -1$ )
        {  $T[k] = D[j + g]; k := k + 1;$  }
    }
   $g := g + \text{ct}[i];$ 
  return  $T$ .

```

¹ We assume the random access model, and an element in the arrays B and C can be accessed in constant time.

The work for a given i is $O(\text{ct}[i])$ and since $\sum_{i=0}^{d-1} \text{ct}_i = n$, the total work to get the target sets over all iterations is $O(n)$.

This completes the $O(n + m) = O(m)$ running time analysis for Algorithm 1. This yields the following result.

Theorem 3 (Running time). *Given a game graph $G = ((S, E), (S_1, S_2))$ and priority function $p : S \rightarrow [d]$, the sets $W_1(\text{WeakParityEven}(p))$ and $W_2(\text{WeakParityOdd}(p))$ can be computed in $O(m)$ time, where $m = |E|$.*

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