Electronics Research Laboratory
University of California
Berkeley, California
Internal Technical Memorandum M-77

A NOTE ON THE EVALUATION OF THE TOTAL SQUARE INTEGRAL*

by

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*This work was supported by Air Force Office of Scientific Research Grant No. AF-AFOSR-292-64.

June 13, 1964

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The purpose of this note is (a) to show the general formulation of the total square integrals in discrete systems and (b) to show for such a formulation we need to expand only (n-1)-order determinant.

Recently a tabulation of the total square integrals which arise in discrete systems have been presented.[1] This tabulation which was carried out for systems up to fourth order is based on the evaluation of the following determinants:

$$I_n = \frac{|\Omega|}{a_0|\Omega|} = \frac{1}{2\pi j} \oint_{\substack{\text{unit} \\ \text{circle}}} \mathcal{F}(z)\, \mathcal{F}(z^{-1})\, z^{-1} dz$$

where Ω is the following matrix†:

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots\cdots & a_n \\ a_1 & a_0+a_2 & a_1+a_3 & a_3+a_4 & \cdots\cdots & a_{n-1} \\ a_2 & a_3 & a_0+a_4 & a_1+a_5 & \cdots\cdots & a_{n-2} \\ \cdot & & & & & \\ \cdot & & & & & \\ a_n & 0 & 0 & 0 & & a_0 \end{bmatrix}$$

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†It should be noted that when all the poles of $\mathcal{F}(z)$ are inside the unit circle, this determinant never vanishes.

and Ω_1 is the matrix formed from Ω by replacing the first column by

$$\begin{bmatrix} \sum_{i=0}^{n} b_i^2 \\ 2 \sum b_i b_{i+1} \\ 2 \sum b_i b_{i+2} \\ \cdot \\ \cdot \\ 2 \sum b_i b_{i+n-1} \\ 2 b_0 b_n \end{bmatrix} .$$

The function $\mathcal{F}(z)$ is given by

$$\mathcal{F}(z) = \frac{B(z)}{A(z)} ,$$

where

$$A(z) = \sum_{r=0}^{n} a_r z^{n-r}, \quad a_0 \neq 0$$

$$B(z) = \sum_{r=0}^{n} b_r z^{n-r} .$$

To show the procedure of evaluation, a fifth-degree polynomial is first discussed and then the general results for the n-th degree is stated. For a fifth-degree polynomial the total square integral is given by:

$$I_5 = \frac{1}{2\pi j} \oint_{\substack{\text{unit} \\ \text{circle}}} \frac{B(z)}{A(z)} \quad \frac{B(z^{-1})}{A(z^{-1})} \quad z^{-1} \, dz$$

where

$$\frac{B(z)}{A(z)} = \frac{b_0 z^5 + b_1 z^4 + b_2 z^3 + b_3 z^2 + b_4 z + b_5}{a_0 z^5 + a_1 z^4 + a_2 z^3 + a_3 z^2 + a_4 z + a_5}$$

and

$$I_5 = \frac{|\Omega_1|}{a_0 |\Omega|}$$

The matrices Ω and Ω_1 are given by:

$$\Omega^{\dagger} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_1 & \underbrace{a_0+a_2}_{e_1} & \underbrace{a_1+a_3}_{e_2} & \underbrace{a_2+a_4}_{e_3} & \underbrace{a_3+a_5}_{e_4} & a_4 \\ a_2 & a_3 & \underbrace{a_0+a_4}_{f_1} & \underbrace{a_1+a_5}_{f_2} & a_2 & a_3 \\ a_3 & a_4 & a_5 & a_0 & a_1 & a_2 \\ a_4 & a_5 & 0 & 0 & a_0 & a_1 \\ a_5 & 0 & 0 & 0 & 0 & a_0 \end{bmatrix}$$

† For ease of calculation and to identify certain terms certain entries in the matrix have been labeled as indicated above.

$$\Omega_1{}^{\dagger} = \begin{bmatrix} \underset{B_0}{b_0{}^2+b_1{}^2+b_2{}^2+b_3{}^2+b_4{}^2+b_5{}^2} & a_1 & a_2 & a_3 & a_4 & a_5 \\ \underbrace{2(b_0b_1+b_1b_2+b_2b_3+b_3b_4+b_4b_5)}_{B_1} & \underbrace{a_0+a_2}_{e_1} & \underbrace{a_1+a_3}_{e_2} & \underbrace{a_2+a_4}_{e_3} & \underbrace{a_3+a_5}_{e_4} & a_4 \\ \underbrace{2(b_0b_2+b_1b_3+b_2b_4+b_3b_5)}_{B_2} & a_3 & \underbrace{a_0+a_4}_{f_1} & \underbrace{a_1+a_5}_{f_2} & a_2 & a_3 \\ \underbrace{2(b_0b_3+b_1b_4+b_2b_5)}_{B_3} & a_4 & a_5 & a_0 & a_1 & a_2 \\ \underbrace{2(b_0b_4+b_1b_5)}_{B_4} & a_5 & 0 & 0 & a_0 & a_1 \\ \underbrace{2b_0b_5}_{B_5} & 0 & 0 & 0 & 0 & a_0 \end{bmatrix}$$

The numerator determinant is given by:

$$|\Omega_1| = a_0B_0Q_0 - a_0B_1Q_1 + a_0B_2Q_2 - a_0B_3Q_3 + a_0B_4Q_4 - B_5Q_5$$

where the Q_r's are given as follow:

$$Q_0 = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ a_3 & f_1 & f_2 & a_2 \\ a_4 & a_5 & a_0 & a_1 \\ a_5 & 0 & 0 & a_0 \end{vmatrix}, \qquad Q_1 = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_3 & f_1 & f_2 & a_2 \\ a_4 & a_5 & a_0 & a_1 \\ a_5 & 0 & 0 & a_0 \end{vmatrix},$$

† See note page 3.

$$Q_2 = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ e_1 & e_2 & e_3 & e_4 \\ a_4 & a_5 & a_0 & a_1 \\ a_5 & 0 & 0 & a_0 \end{vmatrix} \qquad Q_3 = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ e_1 & e_2 & e_3 & e_4 \\ a_3 & f_1 & f_2 & a_2 \\ a_5 & 0 & 0 & a_0 \end{vmatrix}$$

$$Q_4 = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ e_1 & e_2 & e_3 & e_4 \\ a_3 & f_1 & f_2 & a_2 \\ a_4 & a_5 & a_0 & a_1 \end{vmatrix} \qquad Q_5 = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ e_1 & e_2 & e_3 & e_4 & a_4 \\ a_3 & f_1 & f_2 & a_2 & a_3 \\ a_4 & a_5 & a_0 & a_1 & a_2 \\ a_5 & 0 & 0 & a_0 & a_1 \end{vmatrix}$$

It is noticed from Q_5 that one has to expand a fifth-order determinant; however, the following relationship exists which reduces the order. Expanding Q_5 along the last column,

$$Q_5 = a_5 \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ a_3 & f_1 & f_1 & a_2 \\ a_4 & a_5 & a_0 & a_1 \\ a_5 & 0 & 0 & a_0 \end{vmatrix} - a_4 \begin{vmatrix} \quad \end{vmatrix} + \ldots \text{ so on.}$$

one obtains

$$Q_5{}^{\dagger} = a_5Q_0 - a_4Q_1 + a_3Q_2 - a_2Q_3 + a_1Q_4$$

and hence,

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† The author is grateful to Dr. Paul LeFevre of Paris, France, for pointing this relationship for the fourth and third order cases, which can be generalized to any order. For numerical calculation the above relationship is quite useful.

$$|\Omega| = (a_0B_0 - a_5B_5)\,Q_0 - (a_0B_1 - a_4B_5)\,Q_1$$

$$+ (a_0B_0 - a_3B_5)\,Q_2 - (a_0B_3 - a_2B_5)\,Q_3$$

$$+ (a_0B_4 - a_1B_5)\,Q_4$$

by replacing B's by a's in $|\Omega|$, one obtains

$$|\Omega| = (a_0^{\,2} - a_5^{\,2})\,Q_0 - (a_0a_1 - a_4a_5)\,Q_1 + (a_0a_2 - a_3a_5)\,Q_2$$

$$- (a_0a_3 - a_2a_5)\,Q_3 + (a_0a_4 - a_1a_5)\,Q_4\,.$$

Hence,

$$I_5 = \frac{(a_0B_0 - a_5B_5)\,Q_0 - (a_0B_1 - a_4B_5)\,Q_1 + (a_0B_2 - a_3B_5)\,Q_2 - (a_0B_3 - a_2B_5)\,Q_3 + (a_0B_4 - a_1B_5)\,Q_4}{a_0\left[(a_0^{\,2} - a_5^{\,2})\,Q_0 - (a_0a_1 - a_4a_5)\,Q_1 + (a_0a_2 - a_3a_5)\,Q_2 - (a_0a_3 - a_2a_5)\,Q_3 + (a_0a_4 - a_1a_5)\,Q_4\right]}\,.$$

For evaluating I_5, it appears that one has to evaluate the determinants Q_0, Q_1, Q_2, Q_3, and Q_4. However by using the following artifice, one need only to evaluate the last one, i.e., Q_4. All the others can be readily obtained by a certain substitution as follows:

1. Expand Q_4 by labeling its entries as follows.

$$Q'_4 = \begin{vmatrix} b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_3 & f_1 & f_2 & d_2 \\ k_4 & k_5 & k_0 & k_1 \end{vmatrix}$$

From Q'_4, one obtains Q'_3 by using the following substitution as noticed from Q_3,

$$k_4 = a_5, \quad k_5 = 0, \quad k_0 = 0, \quad k_1 = a_0 .$$

From Q'_3, one obtains Q'_2 as follows:

let, $d_3 = a_4$, $f_1 = a_5$, $f_2 = a_0$, $d_2 = a_1$.

Similarly Q'_1 is obtained from Q'_2 as follows,

$$c_1 = a_3, \quad c_2 = f_1, \quad c_3 = f_2, \quad c_4 = a_2 .$$

Finally, Q_0 is obtained from Q_1' by letting

$$b_1 = e_1, \ b_2 = e_2, \ b_3 = e_3, \ b_4 = e_4 .$$

2. By relabeling the entries of Q'_r to coincide with Q_r one obtains all the required Q_r's.

This process can be readily generalized for any order system which requires the evaluation of only one (n-1)-order determinant. Generalizing the above procedure, to obtain

$$a_0 I_n = \frac{\left[(a_0B_0 - a_nB_n) Q_0 - (a_0B_1 - a_{n-1}B_n) Q_1 + \cdots\cdots + (-1)^{n-1} (a_0B_{n-1} - a_1B_n) Q_{n-1}\right]}{\left[(a_0^2 - a_n^2) Q_0 - (a_0a_1 - a_{n-1}a_n) Q_1 + (a_0a_2 - a_{n-2}a_n) + \cdots + (-n)^{n-1} (a_0a_{n-1} - a_1a_n) Q_{n-1}\right]}$$

where

$$B_0 = \sum_{i=0}^{n} b_i^2$$

and

$$B_r = 2 \sum_{i=0}^{n} b_i b_{i+r} \qquad r = 1, 2, \ldots\ldots n.$$

Furthermore, the Q_r's for $r = 0, \ldots, (n-1)$ are $(n-1)$ by $(n-1)$ determinants, obtained as follows:

$$\left|\begin{array}{c|cccccc|c}
a_0 & a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} & a_n \\
a_1 & a_2+a_0 & a_3+a_1 & a_4+a_2 & \cdots & a_{n-1}+a_{n-3} & a_n+a_{n-2} & a_{n-1} \\
a_2 & a_3 & a_4+a_0 & a_5+a_1 & \cdots & a_n+a_{n-4} & 0+a_{n-3} & a_{n-2} \\
\hline
a_r & a_{r+1} & \cdots & & & & a_{n-r-1} & a_{n-r} \\
\hline
a_{n-1} & a_n & 0 & \cdots & & 0 & 0+a_0 & a_1 \\
a_n & 0 & 0 & \cdots & & 0 & 0 & a_0
\end{array}\right|$$

By deleting the r-th row and the n-th row and by deleting the first and n-th columns, the remaining rows and columns give (n-1)-order, Q_r, determinant.

As shown for the fifth-order case, we can obtain all the Q_r's by expanding only the matrix of the Q_{n-1}. Therefore, for obtaining the value of I_n we need to expand only one (n-1)-order determinant. If the coefficients are given in numerical value, then we have to calculate all the Q_r's.

REFERENCE

1. E. I. Jury, <u>Theory and Application of the z-transform method, John Wiley and Sons, Inc., New York, 1964., Ch. 4 and Table III.</u>