F-PROCESSES

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1. Introduction

In [1] Doob associated with any sequence $\{X_n\}$ of random variables two new sequences $\{M_n\}$ and $\{W_n\}$ in such a way that $X_n = M_n + W_n$, $\{M_n\}$ is a martingale, and finally $\{W_n\}$ is a.s. monotone noncreasing in n if and only if $\{X_n\}$ is a supermartingale. Doob noted that an analogous decomposition in the continuous parameter case did not seem easy to obtain.

Recently, two interesting works have dealt with the continuous parameter problem. Consider the case where the parameter varies over a compact interval $[0, a]$. First P. A. Meyer [6], [7] showed that a right-continuous supermartingale $\{X_t\}$ will satisfy $X_t = M_t - A_t$ for some martingale $\{M_t\}$ and some process $\{A_t\}$ which has almost all sample functions right-continuous, monotone increasing, vanishing at $t = 0$ and which satisfies $E|A_a| < \infty$ if and only if $\{X_t\}$ satisfies a certain mild integrability condition. A process satisfying said integrability condition Meyer refers to as belonging to *class* D.

Then D. L. Fisk, following up ideas introduced by Herman Rubin in an invited address at the I.M.S. meetings at the University of Oregon in 1956, considered a class of processes, called F-processes below, and showed in [3] that an F-process with continuous sample paths could be decomposed as $X_t = M_t + W_t$, where $\{M_t\}$ is a martingale and $\{W_t\}$ has almost every sample function of bounded variation, the total variation even having a finite expectation, if and only if $\{X_t\}$ belongs to class D. Fisk's methods depend on the assumption of continuity; on the other hand, Meyer's methods depend on dealing with supermartingales, rather than just F-processes.

The present paper grew out of a desire to prove a decomposition theorem for F-processes without assuming continuity. Our main result is that for right-continuous F-processes the desired decomposition exists if and only if the process belongs to class D. The assumption of right-continuity is fairly harmless, as will be seen below. Our proof makes heavy use of ideas of both Meyer and Fisk.

We will say $X = \{X_t, \mathfrak{F}_t, \Gamma\}$ is a *stochastic process* if $\Gamma \subseteq [0, \infty)$, for each $t \in \Gamma$, X_t is a random variable, $\mathfrak{F}_t$ is a Borel field of events, $s < t$ and $s \in \Gamma$, $t \in \Gamma$ imply $\mathfrak{F}_s \subseteq \mathfrak{F}_t$, and finally X_t is measurable with respect to $\mathfrak{F}_t$.

Let $X = \{X_t, \mathfrak{F}_t, [0, a]\}$ be a stochastic process, and let a be a positive number. Then X is an *F-process* if $E|X_t| < \infty$ for all t, and there exists a number K such that for every partition $t_0 < t_1 < \cdots < t_n$ of $[0, a]$,

$$(1) \qquad E\left\{\sum_{j=0}^{n-1} |E\{(X_{t_{j+1}} - X_{t_j})|\mathfrak{F}_{t_j}\}|\right\} \le K.$$

A number K satisfying the indicated inequalities will be called an *F-bound* associated with X.

Still considering the parameter interval $[0, a]$, it is evident that every supermartingale and every submartingale is an F-process. From the analogy with functions of bounded variation one is prompted to ask whether every F-process is the difference of two supermartingales. We do not know the answer. If the process can be written as a sum of a martingale and a process which has sample functions of bounded variation, it is easy to see that it can be written as the difference of two supermartingales, so that our results here imply an affirmative solution to the problem for right-continuous F-processes of class D.

A number of remarks are appended at the end of the paper. The reader may find it useful to look at these before (or without) attacking the body of the paper.

2. F-processes

Our first lemma is a straightforward generalization of a submartingale inequality (see Doob [1], p. 314).

LEMMA 2.1. *Let $\{X_k, \mathfrak{F}_k, k = 1, \cdots, n\}$ be a stochastic process, $E|X_k| < \infty$ for each k and satisfying*

$$E \sum_{k=1}^{n-1} |E\{(X_{k+1} - X_k)|\mathfrak{F}_k\}| = K < \infty. \tag{2}$$

Then for $\lambda \geq 0$,

$$\lambda P[\max_k X_k \geq \lambda] \leq E|X_n| + K; \qquad \lambda P[\min_k X_k \leq -\lambda] \leq E|X_n| + K. \tag{3}$$

PROOF. It clearly suffices to prove one of the inequalities, say the first one. Set

$$\Lambda_k = [X_k \geq \lambda, X_{k-1} < \lambda, \cdots, X_1 < \lambda], \qquad \Lambda = \sum_{k=1}^{n} \Lambda_k. \tag{4}$$

Then

$$\int_\Lambda X_n dP = \sum_{k=1}^{n} \int_{\Lambda_k} X_n dP = \sum_{k=1}^{n} \int_{\Lambda_k} [X_k + (X_n - X_k)] dP \geq \sum_{k=1}^{n} \lambda P(\Lambda_k) + \sum_{k=1}^{n-1} \int_{\Lambda_k} (X_n - X_k)\, dP \tag{5}$$

$$\begin{aligned}\sum_{k=1}^{n-1} \int_{\Lambda_k} (X_n - X_k)\, dP &= \sum_{k=1}^{n-1} \int_{\Lambda_k} \sum_{j=k}^{n-1} (X_{j+1} - X_j)\, dP \\ &= \sum_{j=1}^{n-1} \int_{\bigcup_{k=1}^{j} \Lambda_k} (X_{j+1} - X_j)\, dP \\ &= \sum_{j=1}^{n-1} \int_{\bigcup_{k=1}^{j} \Lambda_k} E\{(X_{j+1} - X_j)|\mathfrak{F}_j\}\, dP.\end{aligned} \tag{6}$$

Hence,

$$\left|\sum_{k=1}^{n-1} \int_{\Lambda_k} (X_n - X_k)\, dP\right| \leq \sum_{j=1}^{n-1} \int_{\bigcup_{k=1}^{j} \Lambda_k} |E\,(X_{j+1} - X_j)|\mathfrak{F}_j\}|dP \leq K. \tag{7}$$

From (5) and (7) we obtain

$$\lambda P(\lambda) \leq \int_\Lambda X_n \, dP + K \leq E|X_n| + K. \tag{8}$$

We obtain at once our version of the Kolmogorov-Doob inequality.

THEOREM 2.1. *Let* $\{X_t, \mathfrak{F}_t, [0, a]\}$ *be a separable F-process, where* $0 < a < \infty$, *and let K be an F-bound. Then for* $\lambda \geq 0$,

$$\lambda P[\sup_{0 \leq s \leq a} X_s \geq \lambda] \leq E|X_a| + K; \qquad \lambda P[\inf_{0 \leq s \leq a} X_s \leq -\lambda] \leq E|X_a| + K. \tag{9}$$

PROOF. The theorem follows from lemma 2.1 as in the supermartingale case (see [1], p. 353).

From theorem 2.1 we know that the sample functions of X_t are a.s. bounded functions on $[0, a]$. To obtain further regularity properties of the sample functions, one can proceed as one does for supermartingales. We state an appropriate "up-crossing lemma."

LEMMA 2.2. *Let* $\{X_k, \mathfrak{F}_k, k = 1, \cdots, n\}$ *be as in lemma* 2.1. *If a, b are real numbers,* $a < b$, *then the expected number of up-crossings of* $[a, b]$ *by the sequence* $X_1, X_2, \cdots, X_m$ *is bounded by* $[E(X_n - a)^+ + K]/(b - a)$.

PROOF. Parallel the proof given in ([5], p. 392).

The previous lemma allows one to infer that the sample functions of an F-process are well behaved (see [1], p. 361).

THEOREM 2.2. *Let* $\{X_t, \mathfrak{F}_t, [0, a]\}$ *be a separable F-process, where* $0 < a < \infty$. *Then almost every sample function has finite left and right limits at every* $t \in (0, a)$, *the right (left) limit existing when* $t = 0 (t = a)$.

As is well known (see, for instance, [1]) the conclusion of theorem 2.2 implies that almost all sample functions are continuous, except for a denumerable number of discontinuities at most. In particular, the process can have at most a denumerable number of fixed discontinuities.

We conclude with a theorem which extends to F-processes an important property of supermartingales, but which requires a proof different from those usually given in the supermartingale case (see [1], p. 311).

THEOREM 2.3. *Let* $\{X_t, \mathfrak{F}_t, [0, a]\}$ *be an F-process,* $\{t_k\}$ *a decreasing sequence of elements in* $[0, a]$. *Then the* X_{t_k} *are uniformly integrable.*

PROOF. Assume the $\{X_{t_k}\}$ are not uniformly integrable. Note that one of the following three cases must hold.

Case a: there exists an $\epsilon > 0$ and a sequence $\{\lambda_k\}$ increasing to infinity, so that

$$\int_{\Lambda_k} X_{t_k} \, dP > \epsilon, \qquad k = 1, 2, \cdots, \qquad \text{where} \quad \Lambda_k = [X_{t_k} > \lambda_k]; \tag{10}$$

Case b: some subsequence of $\{X_{t_k}\}$ satisfies case a;

Case c: the sequence $\{-X_{t_k}\}$ satisfies case b.

Clearly, then, we may suppose that we are in case a, the other cases easily reducing to it. Define $k_1, k_2, \cdots$ inductively as follows: $k_1 = 1$. Suppose k_n has been defined. By theorem 2.1 $P[\Lambda_k] \to 0$ as $k \to \infty$. Thus there exists a $j^* > k_n$ such that

$$\int_{\Lambda_j} X_{t_{k_n}}\, dP < \epsilon/2 \tag{11}$$

for all $j \geq j^*$; the least such j^* is to be k_{n+1}. Note that

$$\begin{aligned}\epsilon/2 \geq \int_{\Lambda_{k_{n+1}}} X_{t_{k_n}}\, dP &= \int_{\Lambda_{k_{n+1}}} [X_{t_{k_{n+1}}} + E\{X_{t_{k_n}} - X_{t_{k_{n+1}}}|\mathfrak{F}_{t_{k_{n+1}}}\}]\, dP \\ &> \epsilon + \int_{\Lambda_{k_{n+1}}} E\{(X_{t_{k_n}} - X_{t_{k_{n+1}}})|\mathfrak{F}_{t_{k_n}}\}\, dP\end{aligned} \tag{12}$$

so that

$$E|E\{(X_{t_{k_n}} - X_{t_{k_{n+1}}})|\mathfrak{F}_{t_{k_n}}\}| \geq -\int_{\Lambda_{n+1}} E\{(X_{t_{k_n}} - X_{t_{k_{n+1}}})|\mathfrak{F}_{t_{k_{n+1}}}\} \geq \epsilon/2. \tag{13}$$

Thus

$$E \sum_{n=1}^{m} |E\{(X_{t_{k_n}} - X_{t_{k_{n+1}}})|\mathfrak{F}_{t_{k_{n+1}}}\}| \geq m\epsilon/2, \tag{14}$$

and because this tends to infinity with m, we obtain a contradiction, since the process was assumed to be an F-process.

3. The bounded case

Let $X = \{X_t, \mathfrak{F}_t, [0, a]\}$ be a separable stochastic process, and let a be a positive real number. Let S be a denumerable separating set dense in $[0, a]$, $S = \{t_0, t_1, \cdots\}$, and assume $t_0 = 0$, $t_1 = a$. Let $(t_0^n, t_1^n, \cdots, t_n^n)$ be the elements $\{t_0, t_1, \cdots, t_n\}$ arranged in increasing order. We set

$$\Delta_j^n(X) = (X_{t_{j+1}^n} - X_{t_j^n}); \qquad D_j^n(X) = E\{\Delta_j^n(X)|\mathfrak{F}_{t_j^n}\}, \tag{15}$$

and we define

$$X_t^{(n)} = X_{t_j^n} \quad \text{for} \quad t_j^n \leq t < t_{j+1}^n; \qquad \mathfrak{F}_j^n = \mathfrak{F}_{t_j^n} \quad \text{for} \quad t_j^n \leq t < t_{j+1}^n. \tag{16}$$

We use the notation $\sum_{j\in(n,t)}$ to indicate summation over $\{j\colon t_{j+1}^n \leq t\}$. When the variable of summation is j, we may write simply $\sum_{(n,t)}$ for $\sum_{j\in(n,t)}$.

A process $W^n(X) = \{W^n(X), \mathfrak{F}_j^n, [0, a]\}$ is defined by the relation

$$[W^n(X)]_t = \sum_{(n,t)} D_j^n(X) \tag{17}$$

for each nonnegative integer n.

THEOREM 3.1. *Let $W^n = W_t^n(X)$, and set $M_t^n = X_t^{(n)} - W_t^n$. Then $(M_t^n, \mathfrak{F}_t^n)$ is a martingale.*

PROOF. The proof is obvious.

The following lemma appears in [3], and in [6] a similar result is used.

LEMMA 3.1 (Fisk). *Let $X = (X_t, \mathfrak{F}_t, [0, a])$ be a separable F-process which is bounded, say $\sup_s |X_s| \leq c$, a.s. Then for each t, $[W^n(X)]_t$ is integrable uniformly in n.*

PROOF. Write W_t^n for $[W^n(X)]_t$, D_j^n for $D_j^n(X)$. Let $q(n)$ be the biggest integer i such that $t_{i+1}^{(n)} \leq t$. Then

$$E\{(W_t^n)^2\} = E\{(\sum_{(n,t)} D_j^n)^2\} = E\{\sum_{(n,t)} (D_j^n)^2 + 2 \sum_{(n,t)} D_j^n \sum_{j<k\leq q(n)} D_k^n\}$$
$$\leq E\{ \sum_{(n,t)} |D_j^n|\,|\Delta_j^n| + 2 \sum_{(n,t)} |D_j^n| E\{ \sum_{j<k\leq q(n)} D_k^n | \mathfrak{F}_{t_j^n}\}\}$$
$$\leq E\{ \sum_{(n,t)} |D_j^n|\,|\Delta_j^n| + 2 \sum_{(n,t)} |D_j^n|\,|X_{t^n_{q(n)+1}} - X_{t^n_{j+1}}| \leq 6cK, \tag{18}$$

where K is an F-bound of X.

THEOREM 3.2. *Let $X = \{X_t, \mathfrak{F}_t, [0, a]\}$ be a separable F-process which is bounded, say $\sup_s |X_s| \leq c$, a.s. Then there exists a decomposition $X = M + W$, such that $M = \{M_t, \mathfrak{F}_t, [0, a]\}$ is a martingale and $W = \{W_t, \mathfrak{F}_t, [0, a]\}$ is a process having almost all sample functions of bounded variation, and such that the total variation of a sample function is a random variable with finite expectation.*

PROOF. We use the notation introduced at the beginning of this section. Since X has at most denumerably many fixed points of discontinuity, we assume also that the separating set S includes all these points. We now apply theorem 3.1 to write $X^{(n)} = M^n + W^n$. For $t \in S$, $X_t^n = X_t$ for all ω, provided n is big enough; hence, $X_t^n \to X_t$ both a.s. and L_1 as $n \to \infty$.

By lemma 3.1 W_t^n is integrable uniformly in n for fixed t. It follows (see [2], p. 294) that for fixed t some subsequence of $\{W_t^n\}$ converges in the weak L_1-topology. (If Z_n are integrable random variables, they converge to Z in the weak L_1-topology if and only if $E\{YZ_n\} \to E\{YZ\}$ for every bounded random variable Y).

By applying a diagonal argument we obtain a sequence $n' \to \infty$ such that $W_t^{n'}$ converges in the weak L_1-sense to a limit W_t as $n \to \infty$ for every $t \in S$. One verifies easily that $\{W_t, \mathfrak{F}_t, t \in S\}$ is a stochastic process, and we claim that almost all its sample functions are of bounded variation. The definition of bounded variation is the usual one, even though the sample functions are defined only on a denumerable set S. Since for every finite partition of $[0, a] \cap S$ there exists an n such that $(t_0^n, t_1^n, \cdots, t_n^n)$ is a refinement of that partition, we need only show that the limit as $n \to \infty$ of $\sum_{(n,a)} |W_{t_{j+1}^n} - W_{t_j^n}|$ is a.s. finite.

The last-named sum is monotone nondecreasing in n, and so, by the monotone convergence theorem it will suffice to show that the expected value of the sum is bounded uniformly in n; in fact, this will show that the total variation of W has finite expectation. Observe now that if $Z_n \to Z$ in the weak L_1-sense, then $E|Z| \leq \liminf E|Z_n|$ as $n \to \infty$. Writing $\sum_{(m,s,t)}$ for the sum over $\{j\colon s < t_{j+1}^m \leq t\}$, we have

$$W_{t_{k+1}^n} - W_{t_k^n} = \text{weak } L_1\text{-}\lim_{m'\to\infty} \sum_{(m', t_k^n, t_{k+1}^n)} D_k^{m'}(X), \tag{19}$$

and so it follows that

$$E|W_{t_{k+1}^n} - W_{t_k^n}| \leq \liminf_{m'\to\infty} E|\sum_{(m', t_k^n, t_{k+1}^n)} D_j^{m'}(X)|$$
$$\leq \liminf_{m'\to\infty} E\{\sum_{(m', t_k^n, t_{k+1}^n)} |D_j^{m'}|\}. \tag{20}$$

Again writing K for an F-bound of X, we obtain

$$\sum_{k\in(n,a)} E|W_{t^n_{k+1}} - W_{t^n_k}| \leq \sum_{k\in(n,a)} \liminf_{m'\to\infty} E\{\sum_{(m',t^n_k,t^n_{k+1})} |D^{m'}_j|\} \tag{21}$$
$$\leq \liminf_{m'\to\infty} E \sum_{(m',a)} |D^{m'}_j| \leq K,$$

as desired.

Now set $M_t = X_t - W_t$ for $t \in S$. Then $\{M_t, \mathfrak{F}_t, t \in S\}$ is a martingale. This follows easily from the facts that $\{M^n_t, \mathfrak{F}^n_t, [0, a]\}$ is a martingale (theorem 3.1) and that $M^{n'}_t \to M_t$ in the weak L_1-sense, since the corresponding convergence assertion holds for both $X^{(n)}$ and W^n. For almost all ω it is true for each of the processes $\{M_t, \mathfrak{F}_t, t \in S\}$ and $\{W_t, \mathfrak{F}_t, t \in S\}$ that for every $t \in [0, a]$, the sample functions have a limit as s converges down to t through values in S. (For M this follows from ([1], p. 358); for W it is clear by the bounded variation property.) For $t \in [0, a] - S$ we now define W_t by right-continuity; on the exceptional ω-set, where these limits fail to exist, set $W_t(\omega) = 0$.

Now define M for all t by setting $M_t = X_t - W_t$. Since $(M_t, \mathfrak{F}_t, S)$ is a martingale, and since for $t \in [0, a] - SM_t$ is a.s. equal to the limit of M_s as s converges down to t through S, fixed discontinuity points of X all being in S, we verify easily that $\{M_t, \mathfrak{F}_t, [0, a]\}$ is a martingale. Since each sample function of $\{W_t, \mathfrak{F}_t, [0, a]\}$ has the same variation as the corresponding sample function of $\{W_t, \mathfrak{F}_t, S\}$, we still have that the sample functions of W have finite total variation, the expected value of the total variation being bounded by K.

4. Unbounded case

We again let $X = \{X_t, \mathfrak{F}_t, [0, a]\}$, where a is to be a positive real number. As in the previous section, we take X to be a separable F-process, but now we also impose the hypothesis of *right-continuity*, by which we mean (i) almost all sample functions of X are right-continuous, and (ii) for $t \in [0, a]$, $\mathfrak{F}_t = \bigcap_{s>t}\mathfrak{F}_s$. (For a discussion of condition (ii), see [6].)

We make some definitions. For $N \geq 0$, we define the stopping time $T_N = \inf\{t: |X_t| \geq N\}$, and on the set where T_N is otherwise undefined, we set it equal to ∞. We define the *truncated process* X^N by setting $X^N_t = X_t$ for $0 \leq t \leq T_N \leq a$, and $X^N_t = X_{T_N}$ for $T_N \leq t \leq a$. We shall also need $\overline{X}^N$ defined by $\overline{X}^N_t = X_t$ for $0 \leq t < T_N \leq a$, and $\overline{X}^N_t = 0$ for $T_N \leq t \leq a$.

Our first aim is to show that if X is an F-process, then the X^N are F-processes with an F-bound K' independent of N. We need an estimate of Fisk (see [3], p. 24.)

LEMMA 4.1 (Fisk). *The following inequality holds*:

$$E \sum_{n,a} |D^n_j(\overline{X}^N)| \leq E \sum_{(n,a)} |D^n_j(X)| + \int_{T_N \leq a} |X_a| \, dP. \tag{22}$$

PROOF. Let $A_N(n, j) = [T_N > t^n_j]$, $Q_N(n, j) = [t^n_j < T_N \leq t^n_{j+1}]$, and let A^c denote the complement of A. Evidently,

$$(23)\qquad E\sum_{(n,a)}|D_j^n(\overline{X}^n)| = E\{\sum_{(n,a)}|D_j^n(X)|\} + \sum_{(n,a)}\int_{A_N(n,j)}(|D_j^n(\overline{X}^N)| - |D_j^n(X)|)\,dP - \sum_{(n,a)}\int_{A_N^c(n,j)}|D_j^n(X)|\,dP,$$

and estimates for the two sums on the right will now be given:

$$(24)\qquad \sum_{(n,a)}\int_{A_N(n,j)}(|D_j^n(\overline{X}^N)| - |D_j^n(X)|)\,dP$$
$$\leq \sum_{(n,a)}\int_{A_N(n,j)}(|D_j^n(\overline{X}^N) - D_j^n(X)|)\,dP$$
$$\leq \sum_{(n,a)}\int_{A_N(n,j)}|\overline{X}^N_{t^n_{j+1}} - X_{t^n_{j+1}}|\,dP = \sum_{(n,a)}\int_{Q_N(n,j)}|X_{t^n_{j+1}}|;$$

$$(25)\qquad \sum_{(n,a)}\int_{A_N^c(n,j)}|D_j^n(X)|\,dP \geq \sum_{j\in(n,a)}\sum_{k<j}\int_{Q_N(n,k)}|D_j^n(X)|\,dP$$
$$= \sum_{k\in(n,a)}\int_{Q_N(n,k)}\Big(\sum_{\{j:j>k,j\in(n,a)\}}|D_j^n(X)|\Big)\,dP$$
$$\geq \sum_{k\in(n,a)}\int_{Q_N(n,k)}\Big(\sum_{\{j:j>k,j\in(n,a)\}}E\{(-\operatorname{sgn}(X_{t^n_{k+1}})\Delta_j^n(X))|\mathfrak{F}_{t^n_j}\}\Big)\,dP$$
$$= \sum_{k\in(n,a)}\int_{Q_N(n,k)}\Big(\sum_{\{j:j>k,j\in(n,a)\}} - \operatorname{sgn}(X_{t^n_{k+1}})\Delta_j^n(X)\Big)\,dP$$
$$\geq \sum_{k\in(n,a)}\int_{Q_N(n,k)}(|X_{t^n_{k+1}}| - |X_a|)\,dP$$
$$= \sum_{k\in(n,a)}\int_{Q_N(n,k)}(|X_{t^n_{k+1}}|\,dP - \int_{T_N\leq a}|X_a|)\,dP.$$

Obviously (23)–(25) give the desired conclusion.

The next lemma is required to get us from $\overline{X}^N$ to X^N. By a *stopping time* will be meant a random variable T with values in $[0, a]$ such that $[T \leq t] \in \mathfrak{F}_t$ for all t; since T need not be defined for all ω, we may take $T = \infty$ on the set where T is not otherwise defined.

Lemma 4.2. *Let X be a right-continuous F-process with F-bound, K, and T a stopping time. Then $\int_{T\leq a}|X_T|\,dP \leq \int_{T\leq a}X_a\,dP + K \leq E|X_a| + K$.*

Proof. First assume that there exist numbers $\lambda_1 < \lambda_2, \cdots < \lambda_n = a$ such that $[T \leq a] = \bigcup_{k=1}^n [T = \lambda_k]$. Let $\Lambda_k = [T = \lambda_k]$. The desired estimate is obtained by proceeding as in lemma 1.1. The case of an arbitrary stopping time T is obtained by approximating by a sequence of stopping times $T^{(n)}$ of the kind just considered such that $T^{(n)} \downarrow T$ a.s. as $n \to \infty$ and by using the right-continuity of X.

Lemma 4.3. *Let X be a right-continuous F-process. Then for $N > 0$, X^N is also a right-continuous F-process, and there exists a number K' independent of N serving as F-bound for all X^N. Further, $X^N = M^N + W^N$ where $M^N = (M_t^N, \mathfrak{F}_t, [0, a])$ is a right-continuous martingale and $W^N = \{W_t^N, \mathfrak{F}_t, [0, a]\}$ has sample functions of bounded variation and finite expected total variation.*

PROOF. Write $X^N = \overline{X}^N + (X^N - \overline{X}^N)$. The second term on the right is a process with sample functions which vanish for $t < T_N$ and equal X_{T_N} for $t \geq T_N$. So applying lemma 4.2 to this term and lemma 4.1 to the first term on the right, we obtain the first desired conclusion. Since $|\overline{X}^N| \leq N$, the final assertion follows from theorem 3.2 and lemma 4.2.

At this point we see that X^N has a quasi-martingale decomposition $X^N = M^N + W^N$. To carry the argument further along the ideas of Fisk one needs a suitable kind of canonical decomposition. Fisk was able to restrict himself to continuous decompositions, because he assumed X to be continuous. In our present context we have available the important concept of *natural* decomposition discovered by Meyer [7].

To explain the notion, consider a process $W = \{W_t, \mathfrak{F}_t, [0, a]\}$ which is right-continuous, has almost all sample functions of bounded variation, and is such that the expectation of the total variation is finite. Let $Y = \{Y_t, \mathfrak{F}_t, [0, a]\}$ be a right-continuous martingale which is bounded, that is $\sup_t |Y_t| < c < \infty$ a.s. Then for $\epsilon > 0$, $s \in [0, a]$,

$$\sum_{t \leq s, |Y_t - Y_{t-}| > \epsilon} [(W_t - W_{t-})(Y_t - Y_{t-})] \tag{26}$$

is a.s. a finite sum, and thus well defined. As $\epsilon \downarrow 0$, the sum converges a.s., and in the L_1-sense to a limit denoted by

$$\sum_{t \leq s} (W_t - W_{t-})(Y_t - Y_{t-}). \tag{27}$$

The process W is *natural* if $E\{\sum_{t \leq s} (W_t - W_{t-})(Y_t - Y_{t-})\} = 0$, for all $s \in [0, a]$ and all bounded right-continuous martingales Y.

LEMMA 4.4. *Suppose $X = (X_t, \mathfrak{F}_t, [0, a])$ is right-continuous and has a decomposition $X = M' + W'$, where $M' = (M'_t, \mathfrak{F}_t, [0, a])$ is a right-continuous martingale and $W' = (W'_t, \mathfrak{F}_t, [0, a])$ has almost all sample functions of bounded variation and the expected total variation of W' is finite. Then X has a decomposition of the same kind $X = M + W$ such that W is natural. There exists only one such decomposition, and W is given by*

$$W_t = \text{weak } L_1\text{-}\lim_{k \downarrow 0} \frac{1}{h} \int_0^t E\{X_{s+k} - X_s | \mathfrak{F}_s\}\, ds. \tag{28}$$

PROOF. Under the additional assumption that X is a supermartingale, this result is in Meyer [6], [7]. Writing W' as the difference between its positive and negative variation, we see that X can be written as the difference of two supermartingales, $X = X' - X''$, where each of X' and X'' satisfy all the hypotheses of the present lemma. Therefore, the results of Meyer apply to X' and X'', and we deduce that $X = M + W$, with W given by the formula of the theorem, is a decomposition of X, with W natural. To prove that the decomposition is uniquely determined by the requirement that W be natural, one can proceed as in Meyer ([7], p. 4).

We continue to use the notations X^N, $\overline{X}^N$, T_N in the manner in which they were introduced at the beginning of this section. We are now in a position to use Fisk's idea for showing in the decompositions $X^N = M^N + W^N$, that the W^N converge a.s., provided one uses the canonical decomposition. If X is a right-continuous process, lemma 4.2 shows that X^N satisfies the hypotheses of lemma 4.4.

LEMMA 4.5. *Let X be a right-continuous F-process, $N_1 < N_2$, and let*

$$X^{N_1} = M^{N_1} + W^{N_1}, \qquad X^{N_2} = M^{N_2} + W^{N_2} \tag{29}$$

be the decompositions guaranteed by lemma 4.4. Then

$$W_t^{N_1} = W_t^{N_2} \quad \text{for} \quad t \leq T_{N_1} \qquad \text{a.s.} \tag{30}$$

PROOF. Let

$$M_t^* = M_t^{N_2} \quad \text{for} \quad 0 \leq t < T_{N_1} \leq a, \tag{31}$$

$$M_t^* = M_{T_{N_1}}^{N_2} \quad \text{for} \quad T_{N_1} \leq t \leq a, \tag{32}$$

and

$$W_t^* = W_t^{N_2} \quad \text{for} \quad 0 \leq t < T_{N_1} \leq a, \tag{33}$$

$$W_t^* = W_{T_{N_1}}^{N_2} \quad \text{for} \quad T_{N_1} \leq t \leq a. \tag{34}$$

One verifies at once that

$$X^{N_1} = M^{N_1} + W^{N_1} = M^* + W^*, \tag{35}$$

and that the second decomposition is also a "natural decomposition." In order to see that W^* is natural, note that if Y is a bounded right-continuous martingale and

$$\left.\begin{aligned} Y_t^* &= Y_t \quad \text{for} \quad t < T_{N_1} \leq a, \\ Y_t^* &= Y_{T_{N_1}} \quad \text{for} \quad T_{N_1} \leq t \leq a, \end{aligned}\right\}, \tag{36}$$

then

$$E\{\sum_{t \leq s} (W_t^* - W_{t-}^*)(Y_t - Y_{t-})\} = E\{\sum_{t \leq s} (W_t - W_{t-})(Y_t^* - Y_{t-}^*)\} = 0. \tag{37}$$

The fact that Y^* and M^* are martingales follows, of course, from the optimal stopping theorem. So the uniqueness result of lemma 4.4 applies, giving $W^{N_1} = W^*$, and hence the desired conclusion.

LEMMA 4.6. *Let $X = \{X_t, \mathfrak{F}_t, [0, a]\}$ be a right-continuous process. Then*

$$\sup_h E \int_0^a \frac{1}{h} |E\{(X_{s+h} - X_s)|\mathfrak{F}_s\}| \, ds \leq \sup E\{\sum_{j=0}^{n-1} |E\{(X_{t_{j+1}} - X_{t_j})|\mathfrak{F}_{t_j}\}|\} \tag{38}$$

the supremum extending over all partitions $(t_0, t_1, \cdots, t_n)$ of $[0, a]$.

PROOF. The lemma needs proof only under the assumption that the supremum on the right-hand side is some finite number K, so that X is an F-process. We rewrite the left side by interchanging expectation and integral. Hence θ, so we must consider an integral of the form $\int_0^a E(1/h)|\cdots| \, ds$. If this exists as a Riemann integral, we can show that it is bounded by K by examining Riemann sums formed by taking equal intervals of width h/m, where m is a big integer, and comparing them with the expression on the right side of the desired inequality. We shall show Riemann integrability by proving that the integrand is right-continuous. That is, it must be shown that as $s \downarrow s_0$, the L_1-norm of $E\{X_{s+h} - X_s|\mathfrak{F}_s\}$ tends to that of

$$E\{X_{s_0+h} - X_{s_0}|\mathfrak{F}_{s_0}\}. \tag{39}$$

We easily see more, namely

$$E\{X_{s+h} - X_s|\mathfrak{F}_s\} \rightarrow E\{X_{s_0+h} - X_{s_0}|\mathfrak{F}_{s_0}\} \tag{40}$$

in the L_1-sense. For

$$E\{X_s|\mathfrak{F}_s\} = X_s \rightarrow X_{s_0} \tag{41}$$

in L_1-sense by right-continuity, which is assumed, and uniform integrability as guaranteed by theorem 2.3. Also

$$\begin{aligned} E|\{X_{s+h}|\mathfrak{F}_s\} - E\{X_{s_0+h}|\mathfrak{F}_{s_0}\}| \leq\ & E|E\{X_{s+h}|\mathfrak{F}_s\} - E\{X_{s_0+h}|\mathfrak{F}_s\}| \\ & + E|E\{X_{s_0+h}|\mathfrak{F}_s\} - E\{X_{s_0+h}|\mathfrak{F}_{s_0}\}|, \end{aligned} \tag{42}$$

and the last term vanishes when $s \leq s_0 + h$. As for the first term on the right, the assumption of right-continuity and theorem 2.3 again guarantee convergence to zero as $s \downarrow s_0$.

LEMMA 4.7. *Let X be a right-continuous F-process, and let $X^N = M^N + W^N$ be the canonical decomposition guaranteed by lemma 4.4. There exists a constant K', not depending on N, such that the expected total variation of W^N is bounded by K'.*

PROOF. Since W^N is right-continuous, there exists a sequence of partitions $\{t_j^n\}$ such that

$$\sum_j |W^N_{t^n_{j+1}} - W^N_{t^n_j}| \tag{43}$$

converges to the total variation of W^N a.s. Since the approximating sum is a monotone function of n, it suffices to show that it has an expectation bounded by K'. Now,

$$\begin{aligned} E \sum_{(n,a)} (W^N_{t^n_{j+1}} - W^N_{t^n_j}) & \\ = E\Big\{\sum_{(n,a)}\Big| & \text{weak } L_1\text{-}\lim_{h\downarrow 0}\Big[\frac{1}{h}\int_{t^n_j}^{t^n_{j+1}} E\{X^N_{s+h} - X^N_s|\mathfrak{F}_s\}\,ds\Big]\Big|\Big\} \\ \leq \sum_{(n,a)} \liminf_{h\downarrow 0} & E\Big|\Big\{\frac{1}{h}\int_{t^n_j}^{t^n_{j+1}} E\{(X^N_{s+h} - X^N_s)|\mathfrak{F}_s\}\,ds\Big|\Big\} \\ \leq \liminf_{h\downarrow 0} & \Big[\frac{1}{h}\int_0^a |E\{(X^N_{s+h} - X^N_s)|\mathfrak{F}_s\}|\,ds \\ \leq K', & \end{aligned} \tag{44}$$

by lemmas 4.3 and 4.6.

THEOREM 4.1. *Let $X = \{X_t, \mathfrak{F}_t, [0, a]\}$ be a right-continuous F-process. Then X has a decomposition $X = M + W$, where $M = \{M_t, \mathfrak{F}_t, [0, a]\}$ is a right-*

continuous martingale and $W = \{W_t, \mathfrak{F}_t, [0, a]\}$ *is a process which has almost all sample functions of bounded variation with expected total variation of the sample functions being finite, if and only if for every* $t \in [0, a]$, $X_t^N \to X_t$ *in the* L_1*-sense as* $N \to \infty$.

PROOF. First suppose the desired decomposition exists. Write

$$E|X_t - X_t^N| = \int_{T_N<t} |X_t - X_{T_N}|\, dP \leq \int_{T_N<t} |X_t|\, dP + \int_{T_N<t} |X_{T_N}|\, dP, \tag{45}$$

and note that as $N \to \infty$, the first integral in the last member tends to zero. Using the decomposition, the last integral becomes

$$\int_{T_N<t} |(M_{T_N} + W_{T_N})|\, dP \leq \int_{T_N<t} |M_{T_N}|\, dP + \int_{T_N<t} |W_{T_N}|\, dP. \tag{46}$$

Since M is a martingale, $|M|$ is a submartingale; hence, by the optional stopping theorem,

$$\int_{T_N<t} |M_{T_N}|\, dP \leq \int_{T_N<t} |M_a|\, dP, \tag{47}$$

and the last integral converges to zero as $N \to \infty$. The integral

$$\int_{T_N<t} |W_{T_N}|\, dP \tag{48}$$

is handled by writing W as the difference of two monotone functions.

Now to prove the theorem in the other direction, let

$$X^N = M^N + W^N \tag{49}$$

be the canonical decomposition of lemma 4.4. From lemma 4.5 it follows that as $N \to \infty$ through the integers, W^N converges to a limit W, a.s., and indeed

$$\begin{aligned} W_t^N(\omega) &= W_t(\omega) && \text{for } 0 \leq t \leq T_N \leq a; \\ W_t^N(\omega) &= W_{T_N}(\omega) && \text{for } T_N(\omega) \leq t \leq a. \end{aligned} \tag{50}$$

Let $\operatorname{var} f(\cdot)$ stand for the total variation of $f(\cdot)$; the interval over which the total variation is considered will be indicated unless it is $[0, a]$. Observe first that

$$\operatorname{var} W_{\cdot}^N(\omega) = \operatorname*{var}_{0 \leq t \leq T_N} W_t^N(\omega) \to \operatorname{var} W_{\cdot}(\omega), \qquad \text{a.s.,} \tag{51}$$

the convergence being monotone. The monotone convergence theorem together with lemma 4.7 gives

$$E\{\operatorname{var} W_{\cdot}\} = \lim_{N\to\infty} E\{\operatorname{var} W_{\cdot}^N\} \leq K'. \tag{52}$$

Next note that

$$|W_t^N - W_t| \leq \operatorname*{var}_{T_N \leq s \leq t} W_s \leq \operatorname*{var}_{T_N \leq s \leq a} W_s, \tag{53}$$

hence

$$E|W_t^N - W_t| \leq \int_{[T_N \leq a]} (\operatorname{var} W_{\cdot})\, dP \to 0 \tag{54}$$

as $N \to \infty$. We see that as $N \to \infty$, $W_t^N \to W_t$ in the L_1-sense, even uniformly in t.

Using the hypothesis that $X_t^N \to X_t$ in the L_1-sense as $N \to \infty$, it follows that also $M_t^N \to M_t$, say, as $N \to \infty$, and hence, $M = \{M_t, \mathfrak{F}_t, [0, a]\}$ is a martingale. This concludes the proof of the theorem.

REMARKS. (i) According to Meyer, a stochastic process $\{X_t\}$ belongs to *class* D if the random variables $\{X_T\}$ are uniformly integrable, T ranging over all stopping times. Consider an F-process with compact parameter interval $[0, a]$. Obviously, if the process is of class D, it will satisfy the conditions of our theorem 4.1. The converse implication follows from the fact that the existence of the decomposition obtained in the theorem is easily seen to imply that the process belongs to class D.

(ii) Fisk used the following condition D' instead of "class D": $P[\sup_{0 \leq t \leq a} |X_t| \geq N] = o(1/N)$. For continuous F-process it turns out that D' is equivalent to "class D." In the supermartingale case, this was first pointed out by Johnson and Helms in [4]. It is interesting, in view of lemma 2.1, that one always has $O(1/N)$, so that for continuous F-processes it is the difference between O and o that decides whether or not one is in class D.

(iii) Meyer works on the interval $[0, \infty)$. However, the necessary and sufficient condition for the right-continuous supermartingale $X = \{X_t, \mathfrak{F}_t, [0, \infty)\}$ to have the desired decomposition is that $\{X_t, \mathfrak{F}_t, [0, a]\}$ be of class D for every finite a. It is, of course, clear that we could also work with processes on $[0, \infty)$ which are F-processes when restricted to compact parameter intervals.

(iv) Johnson and Helms [4] give a most instructive example of a supermartingale (nonnegative, with continuous sample functions vanishing at ∞) $X = \{X_t, \mathfrak{F}_t, [0, \infty)\}$ such that the X_t are uniformly integrable but X is not of class D. It follows from ([6], proposition 1) that $\{X_t, \mathfrak{F}_t, [0, a]\}$ also fails to be of class D for some finite a.

(v) Let $X = \{X_t, \mathfrak{F}_t, [0, a]\}$ be right-continuous. Our proof makes it natural to consider such a process satisfying the additional requirement

$$E \int_0^a |E\{X_{t+h} - X_t | \mathfrak{F}_t\}| \, dt = O(h). \tag{55}$$

We know this relation will hold if X is an F-process, by lemma 4.6. What about the converse implication? It is interesting to look at the discussion in ([8], p. 372), of an analogous problem, the relation between function of bounded variation and those satisfying $\int_0^a |f(x + h) - f(x)| \, dx = O(h)$.

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