STATISTICAL QUESTIONS IN MESON THEORY

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The questions we are going to discuss have to do with the use of statistical methods in meson theory, when we perform certain calculations in an area in which the theory is most certainly wrong. In order, therefore, to lend some degree of reality to the discussion, it is probably well to begin with the experimental framework within which these problems arise.

It was found, about two decades ago, that the forces which act between nuclei, and between the individual components of a nucleus, exhibit the peculiarity that they are only operative when the particles involved are very close together (about 10^{-13} cm, or a ten billionth of the tip radius of a microgroove phonograph needle). At that time, the only forces which were known to act among particles of matter, the gravitational and electrical forces, both obeyed (and still obey) an inverse square law for the diminution of intensity with distance, and so were capable of extending their influence over large distances—as witness the stability of the solar system. Thus, the discovery of the short range forces posed a qualitatively new problem for physicists. This problem was considered by Yukawa who, guided by analogy with the electromagnetic field, concluded that the forces could arise from the spontaneous creation and exchange of elementary particles of finite mass. The argument, which is by now very well known, is that, if a particle is spontaneously created in such a situation, it can exist only for a time given by the uncertainty principle $\Delta E \Delta t \sim h$, or $\Delta t \sim h/\Delta E$. If, during this time, it travels with the speed of light c, it can go a distance equal to $hc/\Delta E$, which is then the range of the force it creates. In the case of light, $\Delta E = h\nu$, so that the range of the force is $c/\nu = \lambda$, the wave length of the light quantum exchanged. Since there is no upper limit to this, the range is quite large, and a more careful consideration leads to the inverse square law. Yukawa reasoned that, in order to limit the range, there must be a minimum energy for the exchanged particle, which is the case if it has a finite mass. Then the minimum energy is just the rest energy mc^2, and the range is h/mc. If this expression is compared with the known range of the forces, one finds that the mass of the particle responsible for the forces must be around 200 times that of an electron. Such particles had, in 1935, never been observed experimentally. Yukawa also predicted that these particles, if seen, would be radioactively unstable, since heavy nuclei were unstable. This last was a prediction that was gratuitous at the time.

These developments achieved sudden importance, shortly thereafter, when particles of mass around 200 were observed in the cosmic rays, and were, in fact, later shown to be radioactively unstable. There seemed little doubt at the time that these were Yukawa's particles. However, soon difficulties began to arise, the chief among them the fact that the mesons, as they came to be called, were never seen to interact with nuclei in any way—not even to collide with them. This was impossible to understand, and tended to cast doubt on the meson theory as a whole, though it should only have been interpreted to mean that these cosmic ray mesons were not Yukawa's particles. That this is true we now know from recent experiments in which a new type of meson, of mass 300, has been seen, both in cosmic rays and artificially produced, and has been observed to interact strongly with nuclei. There seems to be not much doubt that this is the Yukawa meson. The picture is, in fact, a good deal more complicated, but we need not look into these complications at the moment, except to mention the important fact that the heavier mesons are seen to turn into the lighter ones in about 1/100 of a microsecond. This accounts for the fact that it is almost exclusively the lighter mesons that are seen in the cosmic rays, the heavier ones having had time to decay before reaching sea level.

Now let us summarize our position. On theoretical grounds it seems reasonable that there are involved, in the forces between nuclei, particles of intermediate mass, which we call mesons. Particles of the same nature are found in cosmic rays, and are seen to decay into the lighter mesons that predominate at sea level, and it is not unreasonable to suppose that all of the lighter mesons arise from the heavier ones. One can also suppose that the latter are produced in the upper atmosphere by the cosmic ray primaries, which are thought to be largely protons, and fairly certainly are nuclei of some sort. Here, then, we come to the heart of our problem —the primary nuclei come into the Earth's atmosphere carrying their closely connected entourage of mesons, and, by the time they reach sea level, many of these mesons are liberated and come to us as free particles in their own right. How then, does this occur? This is the relation between cosmic rays and nuclear forces.

In this connection, one important fact can be uncovered by counting the number of primary particles at the top of the atmosphere, and comparing this with the number of mesons at sea level. This leads immediately to the interesting fact that, for every primary particle, there are about six mesons at sea level. The over all meson production is, then, multiple. It has, in fact, been possible to observe these meson production events directly, and to show that the individual events are themselves multiple.

Here one has to consider the possibility that, in fact, not more than one meson is supplied from the field of each nucleon in a complex air nucleus. It is not possible, at the moment, to exclude this possibility, though recent experiments comparing carbon and lead imply, though by no means conclusively, that this is not the case. It is still, however, likely that one may have several events in a single nucleus, with more than one meson created in each event. Our study will center on the multiple production in a single event, and our problem will be to understand, within the framework we have mentioned, how this can take place.

First, however, we must look a little more closely into the structure of the meson field around a proton, for which purpose we will now adopt explicitly the so called pseudoscalar meson theory. In this theory the number of mesons in a range of momentum $dp = dp_x dp_y dp_z$, around a stationary nucleon, is proportional to $p_z^2 dp/E^3$, if we designate the direction of the intrinsic spin angular momentum of the nucleon as the z direction. If the nucleon is moving in the z direction with a velocity v, this becomes $(p_z - vE/c^2)^2 dp/[(E - vp_z)^2 E]$, as can be seen by simply applying a Lorentz transformation to the original expression. The constant of proportionality is of order of magnitude unity. (If we had used the so called scalar meson theory, the factor p_z^2 or $(p_z - vE/c^2)^2$ would simply be absent in these expressions, with no other change.)

There is immediately apparent one major difficulty with these expressions, most obvious in the expression involving a nucleon at rest. If one were to ask for the total number of mesons in the field, the answer would be obtained by integrating over the entire momentum space, yielding $\int_0^\infty p^4(m^2c^2 + p^2)^{-3/2}dp$, which is quadratically divergent and therefore quite unpleasant. In other words, there are an infinite number of mesons in the field surrounding a single nucleon—an assertion that offends one's sense of physical reasonableness. (We have here used a nonrelativistic theory of the nucleon—a relativistic treatment would lower the order of the divergence by one.) This difficulty is closely connected with the well known selfenergy problem, and it is at this time still insurmountable, despite some recent progress. We will see that this does not seriously impede us in our progress, though it does indeed raise questions concerning the validity of our results.

We notice that, if the velocity of the nucleon is very large, $v \sim c$, then the binomial expressions above just cancel each other, so that the density of mesons is just given by dp/E, a simple expression, and one which is invariant under Lorentz transformations of the coordinate system, as long as the condition that $v \sim c$ is preserved by the transformation.

We must now consider, in some detail, what happens when this nucleon meson system collides with another nucleon (which may or may not be a constituent of an air nucleus). In this collision, we must presume that the nucleon (both of them, in fact), is upset in some way and tends to free some of its mesons. In this connection the calculations which have been performed are most subject to criticism, and there seems to be no good way to estimate the extent to which this tendency is operative. The assumption that seems to be most indicated by the calculations that do exist is that the nucleon does tend to emit mesons appreciably, and that it has no prejudice concerning which mesons it prefers to release, provided some dubious approximations are used.

We might interject, at this point, a few words about the nature of the statistical problem with which we will have to deal. It will be clear that the forms of statistical analysis involved are indeed rather elementary, once the rules of the game have been properly formulated, the point at which many mathematical problems begin. Our task is, on the other hand, to study the physics of the question sufficiently closely to be able to formulate the rules of the game. Once this is done we will have no serious difficulties, but to do it involves an excursion into the most questionable areas of present day meson theory, and it is little more than hope that what we say may have some validity. However, we will try to state the statistical problem in a sufficiently precise way to be able to judge from the experimental data, as they become available, just where our considerations have to be modified.

One further word is now appropriate with regard to the distribution of mesons which we found above, proportional to dp/E, for a fast moving nucleon. There is a serious difficulty connected with this, in that it is spherically symmetric. Offhand, this would not be a matter for concern, except that it is spherically symmetric for *every* velocity of the nucleon, and, if we look back, even for a nucleon at rest. Only when a nucleon is not at rest, but not moving too rapidly, is it not completely isotropic. That this is not a selfconsistent state of affairs can be seen by considering what happens to a spherical distribution of mesons around a stationary nucleon, when that nucleon starts to move rapidly in (say) the z-direction. Then each of the mesons, as it is carried along, takes on an additional z-component of velocity, and lo! the distribution is no longer isotropic. For sufficiently large nucleon velocities, this additional component of meson velocity will dominate, and the mesons will tend to be concentrated around the z-axis in momentum space, more and more as the velocity increases. But our results indicate that the meson distribution, initially isotropic, goes through a region of anisotropy, and then becomes more and more spherically symmetric as the velocity of the nucleon increases. What then is the difficulty with the preceding argument? The answer is that there is nothing wrong with it, but that our original calculation is at fault, and that, in fact, the continued spherical symmetry is a direct consequence of the infinite number of mesons in our field—a feature which we are not fully prepared to believe. This is clear since it is just the feature that dp/E is an invariant under Lorentz transformation that caused our difficulty. This is the case because, although a transformation or velocity in the z-direction adds a z-component of momentum to each meson in the field, it brings into the field at a given energy mesons which previously had a large negative component of velocity in the z-direction. If this supply were not unlimited, the isotropy would not persist. So we must connect the spherical symmetry with just the dubious feature of the theory that gives rise to the infinite number of mesons. We will, however, use this property of the field, keeping tongue in cheek.

In keeping, now, with our assumptions that, in a collision, a nucleon is strongly impelled to emit mesons, and that it has no *a priori* preference concerning which mesons it would like to release, the most primitive assumption we might make is that the probability of emitting N mesons into the N regions of momentum space $dp_1 dp_2 \ldots dp_N$, is proportional to

$$\prod_{n=1}^{N} \frac{dp_n}{E_n} \tag{1}$$

where the constant of proportionality must be experimentally determined. We will assume that the multiplicative constant is of the form K^N, which one would obtain by attaching a factor of K to the meson distribution around the nucleon. K is the so called coupling constant. Whether the constant is, indeed, of this simple form, is a question that is quite difficult, and the assumption that it is is simply a consistent application of the hypothesis of *a priori* statistical independence of the emissions.

If we were now to assume, as a start, that the mesons are *completely* uncorrelated, the total probability of emitting N mesons in the collision would be obtained by integrating this expression over the whole 3 N-dimensional momentum space, yielding a high order indeed of infinity. This, in itself, might not be too disturbing in the present state of meson theory, but far worse is the fact that the ratio of the probability of emitting $N+1$ mesons to that of emitting N is just

$$\frac{P_{N+1}}{P_N} \sim \int \frac{dp}{E} \tag{2}$$

which is, itself, infinite. In other words, this model yields the result that the more mesons the better *ad infinitum*—an unreasonable result.

We know, however, that certain simple correlations are, in fact, operative, and tend to make the result more reasonable. We know, for example, that the number of mesons emitted is certainly limited by the conservation of energy, which, itself, would suffice to make the most probable number a finite one. There are also other conservation laws which impose correlations, and we have not explored the question of whether the primitive law for the emission ought to embody some correlations.

If we take into account now, only the conservation of energy, then we must do the integral over the $3N$-dimensional momentum space, subject to the condition that the sum of the N energies is equal to ϵ, the energy loss of the nucleons involved, in the coordinate system under consideration. (Since the imposition of energy conservation, without momentum conservation, is not a relativistically invariant procedure, the result will depend upon the coordinate system we use. Reasonable results are obtained by using the center of mass coordinate system, for a two nucleon collision.) The integral we must do is, then,

$$\int \cdots \int \prod_{n=1}^{N} \frac{dp_n}{E_n}\, \delta\left(\epsilon - \sum_{n=1}^{N} E_n\right) \tag{3}$$

where δ represents Dirac's δ-function, a singular function defined by $\delta(x) = 0$ for $x \neq 0$, and $\int_{-\infty}^{\infty} \delta(x)dx = 1$. Any expression containing the δ-function, which depends upon properties other than these two, is meaningless. This integral is similar to one encountered in the statistical mechanics of an ideal gas, but is not quite so simple, since, in this relativistic case, the δ-function does not isolate a spherical shell in the momentum space.

If we use the representation of the δ-function as an integral $\delta(x) = (2\pi)^{-1}\int_{-\infty}^{\infty} \exp(ixy)dy$, we find for the total probability of emitting N mesons, with energy conservation

$$\begin{aligned} \frac{K^N}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda\epsilon} \int \cdots \int e^{i\lambda \sum_1^N E_n} \prod_1^N \frac{dp_n}{E_n} &= \frac{K^N}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda\epsilon}\left[\int \frac{dp}{E}\, e^{i\lambda E}\right]^N \\ &= \frac{(-2i\pi^2 K)^N}{2\pi}\int_{-\infty}^{\infty} \frac{d\lambda}{\lambda^N}\, e^{-i\lambda\epsilon}\, [H_1^{(1)}(\lambda)]^N \end{aligned} \tag{4}$$

where $H_1^{(1)}(\lambda)$ is the Hankel function of the first kind and first order. The path of integration must pass above the singularity at the origin, and the rest energy of the meson has been set equal to unity. Thus, for $N > \epsilon$, the asymptotic form of the Hankel function permits us to close the contour in the upper half of the complex plane, giving rigorously zero for the value of the integral. This is encouraging, since it means that a number of mesons cannot be emitted whose rest energy alone exceeds the total amount of energy available. If, however, $\epsilon > N$, the integral has a finite value, since the two ends of the contour may be bent downward into the lower half of the complex plane, where the integrand is nicely convergent.

One can now obtain an estimate of the value of this integral by noting that, with the folded contour, and for sufficiently large ϵ, the major contributions come from small values of λ. Thus, one can expand the Hankel function in powers of λ, keeping only the leading term. This reduces the singularity at the origin to a pole (the logarithmic terms not appearing in the leading power of λ) whose residue gives the required estimate. We obtain

$$P_N \sim (4\pi K)^N \frac{\epsilon^{2N-1}}{(2N-1)!} \tag{5}$$

which is, in effect, the leading term in an expansion in powers of ϵ. From this, we find for the mean number of mesons

$$\begin{aligned} \bar{N} &= \tfrac{1}{2}\left[1 + \epsilon\sqrt{4\pi K}\coth\left(\epsilon\sqrt{4\pi K}\right)\right] \\ &\approx \epsilon\sqrt{\pi K} \end{aligned} \tag{6}$$

where the latter expression applies if it is much greater than one. From the expression given above for P_N, we also find that large ϵ are favored, so that, most likely, all of the energy available tends to go into meson production.

For large energies, an available energy ϵ in the center of mass coordinate system implies an incident energy in the coordinate system in which one of the nucleons is at rest (the laboratory system) of $E_0 = \epsilon^2/2M$, where M is the mass energy of the nucleon. So $\bar{N} \sim \sqrt{2\pi K M E_0}$, where nothing is known *a priori* about the value of K. It is clear that these results do, as predicted, depend upon the choice of coordinate system in which the energy loss is ϵ, since that choice determines the relation between ϵ and E_0.

In order to lift this self-imposed difficulty, we must look a little more closely into the correlations that arise from momentum conservation, as well as energy conservation. The results should then be relativistically invariant. Thus, we must not only require that the total energy of the mesons be equal to E, the energy loss of the nucleons, but that their momenta add up to $\boldsymbol{P}$, the loss of momentum of the nucleons in the collision. We have to consider

$$K^N \int \dots \int \delta\left(E - \sum_1^N E_n\right) \delta\left(\boldsymbol{P} - \sum_1^N \boldsymbol{p}_n\right) \prod_{n=1}^N \frac{d\boldsymbol{p}_n}{E_n} \tag{7}$$

which is a somewhat more difficult integral to reduce to a manageable form.

By a transformation similar to that used above, we can write this

$$\frac{2(4\pi K)^N}{(2\pi)^3 P} \int_{-\infty}^{\infty} \int_0^{\infty} \mu^{1-N} \sin \mu P e^{-i\lambda E} d\mu\, d\lambda \left[\int_1^{\infty} d\tau\, e^{i\lambda\tau} \sin \mu (\tau^2 - 1)^{1/2}\right]^N \tag{8}$$

where P is the absolute magnitude of $\mathbf{P}$. The expression in the bracket can then be integrated by expanding the sine in power series (assuming that λ has a small positive imaginary part, to insure convergence), yielding, after some obvious transformations,

$$(9)\quad \frac{2(-2i\pi^2K)^N}{(2\pi)^3P}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{\mu\sin\mu P}{\lambda^N}e^{-i\lambda E}d\mu\, d\lambda\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\mu^2}{2\lambda}\right)^n H_{n+1}^{(1)}(\lambda)\right]^N$$

where, as a result of the convergence requirement above, the λ integral must be passed above the real axis. The convergence of the series to an analytic function of λ and μ is easily demonstrated as is, then, the convergence of the double integral, with the appropriate contours.

The sum can, in fact, be evaluated by means of the recurrence formula for the Hankel functions, and yields

$$(10)\quad \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\mu^2}{2\lambda}\right)^n H_{n+1}^{(1)}(\lambda) = \frac{\lambda}{\sqrt{\lambda^2-\mu^2}}H_1^{(1)}(\sqrt{\lambda^2-\mu^2}).$$

That this is true for $\mu > \lambda$ requires further investigation for its establishment, as does the branch of the square root that we must choose. Referring back to the integral that originally gave rise to the series, we see that this representation is correct if we choose the branch which has a positive imaginary part for $\mu > \lambda$. Thus, over the entire product space, we have

$$(11)\quad \frac{(-2i\pi^2K)^N}{(2\pi)^3P}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mu\sin\mu P}{(\lambda^2-\mu^2)^{N/2}}e^{-i\lambda E}d\mu\, d\lambda\,[H_1^{(1)}(\sqrt{\lambda^2-\mu^2})]^N$$

where, in virtue of the evenness of the integrand with regard to μ, the μ integral has been extended to $-\infty$. A similar argument to that used before enables us to use only the first term in the expansion of the Hankel function, obtaining, after an integration by parts,

$$(12)\quad \frac{(4\pi K)^N}{2(N-1)(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\cos\mu P e^{-i\lambda E}}{(\mu^2-\lambda^2)^{N-1}}d\mu\, d\lambda.$$

This integral is easily evaluated by transforming to $\mu+\lambda$ and $\mu-\lambda$ as variables, remembering that λ has a small positive imaginary part, and we find

$$(13)\quad \frac{2}{\pi}\frac{(\pi K)^N}{(N-1)!(N-2)!}[E^2-P^2]^{N-2}$$

in which the energy and momentum appear in the expected invariant combination. From this, we find

$$(14)\quad \bar{N} = 1+\sqrt{\pi K(E^2-P^2)}\,\frac{I_0[\sqrt{4\pi K(E^2-P^2)}]}{I_1[\sqrt{4\pi K(E^2-P^2)}]}$$

where the I_n are the modified Bessel functions of the first kind. (The reason that this expression predicts a finite number of mesons, even for $K = 0$, is that we have ignored the statistical weights of the nucleons. For very small numbers of mesons, these would dominate the problem.)

For sufficiently large argument, the ratio of I_0 to I_1 approaches unity, so that we find, if $\bar{N}$ is large.

$$(14')\quad \bar{N}\sim\sqrt{\pi K(E^2-P^2)},$$

which is precisely the same as was obtained using only energy conservation, except that E appears now in an invariant combination. It is also true, from (13), that large energy loss and small momentum loss tend to be favored for any N. What this entails is most easily recognized in the center of mass coordinate system of the two nucleons involved in a collision, in which they have, initially, equal and opposite momenta. Maximum energy, and minimum momentum loss then imply that, after the collision, the nucleons are effectively stopped, converting all their energy into mesons. (This result would be, of course, slightly modified, if one were to take into account the final nucleons, as mentioned above.) The corresponding consequences, in other coordinate systems, can be obtained by transformation.

There is one further statistical feature of these problems that we would like to take into account here, which arises from the indistinguishability of the various mesons from each other. This has the consequence that we have overestimated our total momentum space, since a final state in which two mesons have momenta p_1 and p_2 is, in fact, no different from one in which these two momenta are interchanged. However, in our method of summing, we have counted them each separately, and must now correct for this error.

This would be particularly simple if there were only one kind of meson, since it would then result in a factor of $N!$ in the denomination of (13), and would have no other consequences. Thus, the mean number of mesons, for a high enough multiplicity, would become

$$\bar{N} \sim [\pi K(E^2 - P^2)]^{1/3} \tag{15}$$

or, in terms of the incident energy E_0 in the laboratory coordinate system, assuming that all the energy in the center of mass system is converted into mesons,

$$\bar{N} \sim (2\pi KME_0)^{1/3}, \tag{15'}$$

so that the multiplicity now increases with the cube root of the primary energy in the laboratory system.

However, we must take into account the fact that there are probably at least three different kinds of mesons, positive, negative, and neutral, which are indistinguishable within a class, but distinguishable from each other. Thus we ought really to divide (13) by $N_1!N_2!N_3!$, where N_1, N_2, and N_3 are the number of positive, negative, and neutral mesons, respectively, and $N_1 + N_2 + N_3 = N$. If we were now to ask for the total probability of emitting N mesons of any type, we would sum this over all distributions of the N mesons among the three types, and would find, in addition to the factor of $N!$ in the denominator of (13), a factor of 3^N in the numerator, so that the average number of mesons emitted is increased by a factor of the cube root of 3 over the case of complete indistinguishability. Unfortunately, though, the case is not quite so simple, because of charge conservation, whose role in the problem raises a physical question that is, at present, unanswerable, but which we must discuss.

Consider, as an example, the collision of two protons, the end results of the collision being a certain number of positive, negative, and neutral mesons, and two nucleons. (We will continue to ignore the possibility of pair production of nucleons.) Ordinarily, we would think of the nucleons as either protons or neutrons, these being the only types presently known experimentally. However, certain forms of meson theory (the so called strong coupling theories) predict that it is possible to form nucleons with an arbitrary number of positive or negative charges, provided that one can add a sufficient amount of excitation energy. These are called nucleon isobars, and the energy of excitation is called the isobar energy. Now, if the isobar energy were zero, an extreme case, the nucleons could emerge from the collision with any amount of charge, positive or negative, with energies quite independent of their charge status. Then the relative numbers of positive and negative mesons would be completely uninhibited by questions of charge, and the result would be just that mentioned in the last paragraph, namely

$$\bar{N} \sim [3\pi K(E^2 - P^2)]^{1/3} \sim [6\pi KME_0]^{1/3}. \tag{16}$$

On the other extreme, when the isobar energy is very large, one can suppose that only protons and neutrons can appear in the final state. In this case, for our two proton collision, the number of positive mesons must equal or exceed the number of negatives, but cannot exceed the latter by more than two. Thus N_1 and N_2 are correlated, and the sum over N_1, N_2, and N_3 has to be carried out under the condition that $0 \leqq N_1 - N_2 \leqq 2$, in addition to $N_1 + N_2 + N_3 = N$. Here one need not carry out the sum, since it is certainly less than $3^N/N!$, the unrestricted sum, and certainly larger than its largest term, which is approximately $3^N/(N+1)!$, so that the expression (16) still obtains. This is of course also true in the intermediate case when the isobar energy is neither zero nor very large, but finite. Then the energy available when the nucleons come from the collision in isobar states is decreased, along with the available momentum space, which makes the calculation complicated. However, the results must rigorously lie between the two extreme cases, so that $\bar{N}$ remains the same. Thus the charge conservation does not introduce any significant correlations into the problem.

To summarize our discussion we might say that we have found that, if we assume the unbelievable feature of meson theory (pseudoscalar) that a nucleon carries with it a cornucopia of mesons, then plentiful meson emissions are expected to occur in a collision. These emissions are expected to be statistically independent, and limited only by the various conservation theorems one expects to be operative in such events, and which are alone responsible for the finite results. Energy and momentum conservation turn out to be most important, and charge conservation, while it cannot easily be taken into account in the present state of the theory, can be safely said not to play a significant role. The average number of mesons produced by a primary nucleon of energy E_0 is given by $(6\pi KME_0)^{1/3}$, where K is an unspecified coupling constant. If K is the same for positive, negative, and neutral mesons, we expect them to be emitted in equal numbers, on the average.